Catoids as a Basis for Algebras of Programs

Georg Struth

University of Sheffield, UK
& Collegium de Lyon

I've worked on algebras of programs for some years
(semirings, Kleene algebras, quantales, relation algebras, . . .)

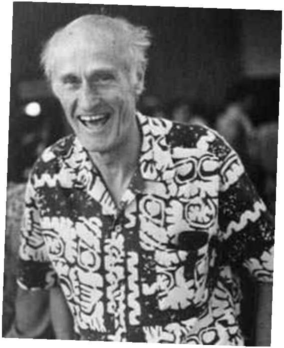
developed variants such as modal/concurrent Kleene algebras
and studied their models/properties

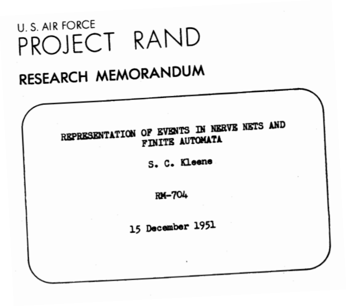
formalised algebra/models with proof assistants
and built program verification tools based on them

formalising models felt like playing variations on a theme

but which theme?

Kleene's Quest





U. S. AIR FORCE
PROJECT RAND

RESEARCH MEMORANDUM

REPRESENTATION OF EVENTS IN NERVE NETS AND
FINITE AUTOMATA

S. C. Kleene

RM-704

15 December 1951

Kleene Algebra

regular expressions $t ::= 0 \mid 1 \mid a \in \Sigma \mid t + t \mid tt \mid t^*$

languages $X \subseteq \Sigma^*$

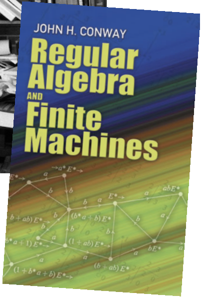
interpretation map $L : \mathsf{RegExp}_\Sigma \to \mathcal{P}\Sigma^*$ defines regular languages

task: axiomatise congruence $s \approx t \;\Leftrightarrow\; L(s) = L(t)$

find algebra KA with signature $(+, \cdot, 0, 1, {}^*)$

prove $\mathsf{KA} \vdash s = t \Leftrightarrow L(s) = L(t)$

Conway's Visions



Kleene Algebra Axioms

$$(K, +, \cdot, 0, 1, ^*)$$

$$x + (y + z) = (x + y) + z \qquad x + y = y + x \qquad x + 0 = x \qquad x + x = x$$

$$x(yz) = (xy)z \qquad x1 = x \qquad 1x = x$$

$$x(y + z) = xy + xz \qquad (x + y)z = xz + yz$$

$$x0 = 0 \qquad 0x = 0$$

$$1 + xx^* = x^* \qquad z + xy \leq y \Rightarrow x^*z \leq y$$

$$1 + x^*x = x^* \qquad z + yx \leq y \Rightarrow zx^* \leq y$$

where $x \leq y \Leftrightarrow x + y = y$

and indeed $\mathsf{KA} \vdash s = t \Leftrightarrow L(s) = L(t)$

Language Kleene Algebras

soundness proof constructs language KA over free monoid Σ^*

$$(\mathcal{P}\Sigma^*, \cup, \cdot, \emptyset, \{\varepsilon\}, {}^*)$$

$$AB = \{vw \mid v \in A,\ w \in B\}$$
$$A^* = \bigcup_{i \geq 0} A^i \quad \text{for } A^0 = 1,\ A^{i+1} = AA^i$$

or just KA $\mathcal{P}M$ for any monoid M

regular languages are then sub-KAs generated by Σ

weighted languages $f : \Sigma^* \to K$ form convolution KAs

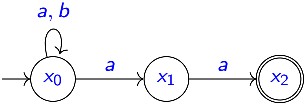
$$(K^{\Sigma^*}, +, *, 0, id, ^*)$$

$$\begin{aligned}
(f + g)(w) &= f(w) + g(w) \\
0(w) &= 0 \\
(f * g)(w) &= \sum_{w = u \cdot v} f(u) \cdot g(v) \\
id(w) &= \delta_\varepsilon(w) \\
f^*(\varepsilon) &= f(\varepsilon)^* \\
f^*(w) &= f^*(\varepsilon) \cdot \sum_{\substack{w = u \cdot v \\ u \neq 1}} f(u) \cdot f^*(v) \quad \text{for } x \neq 1
\end{aligned}$$

standard languages take weights in KA 2

Matrix Kleene Algebras

completeness proof formalises automata as K-valued matrices

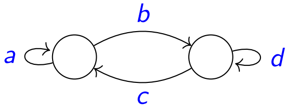


$$\left[\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}a+b & a & 0\\ 0 & 0 & a\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$

KAs are closed under matrix formation: for $m, n : I \times I \to K$

$$(m+n)_{ij} = f_{ij} + g_{ij} \qquad (m \cdot n)_{ij} = \sum_k f_{ik} \cdot g_{kj} \qquad 0_{ij} = 0 \qquad id_{ij} = \delta_{ij}$$

the star is somewhat tricky



$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad M^* = \begin{pmatrix} f^* & f^* b d^* \\ d^* c f^* & d^* + d^* c f^* b d^* \end{pmatrix} \quad \text{for } f = a + bd^* c$$

partition larger matrices into submatrices with squares along diagonal

Relation Kleene Algebras

binary relations are 2-valued matrices $X \times X \to 2$

and thus KAs

$$(\mathcal{P}(X \times X), \cup, ;, \emptyset, \Delta, ^*)$$

$$\begin{aligned} (RS)_{ab} &\Leftrightarrow \exists c.\ R_{ac} \wedge S_{cb} \\ \Delta_{ab} &\Leftrightarrow a = b \\ (R^*)_{ab} &\Leftrightarrow \exists k \geq 0.\ (R^k)_{ab} \end{aligned}$$

but we can't write $(RS)_{a,b} = \sum_c R_{a,c} \wedge R_{c,b}$ — sums may be infinite!

Quantales

quantale $(Q, \leq, \cdot, 1)$ consists of complete lattice $(Q, \leq)$ and monoid $(Q, \cdot, 1)$ such that

$$x(\bigvee Y) = \bigvee\{xy \mid y \in Y\} \qquad (\bigvee X)y = \{xy \mid x \in X\}$$

quantales are KAs with $x^* = \bigvee_{i \geq 0} x^i$

examples: $(\mathbb{R}_+^\infty, \geq, \max, 0)$ (Lawvere quantale) or $([0,1], \leq, \cdot, 1)$

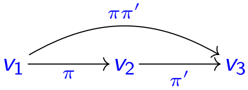
we can now construct quantale $Q^{X \times X}$ of Q-valued relations and convolution quantale Q^M for any monoid M

Path Quantales

automata are digraphs $s, t : E \to V$

paths are sequences $\pi : v_1 \to v_n = (v_1, e_1, v_2, \dots, v_{n-1}, e_{n-1}, v_n)$

we compose them on matching ends:



we define $AB = \{\pi\pi' \mid \pi \in A,\ \pi' \in B,\ t(\pi) = s(\pi')\}$ and $id = \{(v) \mid V\}$

this yields path KA/quantale . . . and we can add weights to edges

more generally, Q^C forms a category quantale for any (small) category C

Single-Set Categories?

categories. A category is a set C of arrows with two functions $s, t : C \to C$, called "source" and target", and a partially defined binary operation $\#$, called composition, all subject to the following axioms, for all x, y, and z in C:

The operation $x \# y$ is defined iff $s x = t y$ and then

$$s(x \# y) = s y\,, \qquad t(x \# y) = t x\,; \tag{1}$$

$$x \# s x = x\,, \qquad t x \# x = x\,; \tag{2}$$

$$(x \# y) \# z = x \# (y \# z) \quad \text{if either side is defined}\,; \tag{3}$$

$$s s x = s x = t s x\,;$$

$$t t x = t x = s t x\,. \tag{4}$$

Then x is an identity iff $x = s x$ or, equivalently, iff $x = t x$.

Shuffle Quantales

shuffle $\Sigma^* \times \Sigma^* \to \mathcal{P}\Sigma^*$ is defined, for $a, b \in \Sigma$ and $v, w \in \Sigma^*$ as

$$v \| \varepsilon = \{v\} = \varepsilon \| v \qquad (av)\|(bw) = a(v\|(bw)) \cup b((av)\|w)$$

we extend to $\| : \mathcal{P}\Sigma^* \times \mathcal{P}\Sigma^* \to \mathcal{P}\Sigma^*$

$$A\|B = \bigcup\{v\|w \mid v \in A,\ w \in B\}$$

we can construct shuffle KA/quantale — and convolution algebras with

$$(f\|g)(w) = \sum_{w \in u\|v} f(u) \cdot g(v)$$

words under $\|$ don't form category!

Catoids

a catoid $(X, \odot, s, t)$ equips set X with multioperation $\odot : X \times X \to \mathcal{P}X$ and source/target maps $s, t : X \to X$ that satisfy

$$\bigcup\{x \odot v \mid v \in y \odot z\} = \bigcup\{u \odot z \mid u \in x \odot y\}$$

$$x \odot y \neq \emptyset \Rightarrow t(x) = s(y) \qquad s(x) \odot x = \{x\} \qquad x \odot t(x) = \{x\}$$

if we extend to $\odot : \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X$

$$A \odot B = \bigcup_{x \in A, y \in B} x \odot y,$$

the first axiom becomes

$$x \odot (y \odot z) = (x \odot y) \odot z$$

a catoid morphism $f : X \to Y$ satisfies

$$f(x \odot_X y) \subseteq f(x) \odot_Y f(y) \qquad f \circ s_X = s_Y \circ f \qquad f \circ t_X = t_Y \circ f$$

it is bounded if $f(x) \in u \odot_Y v$ implies $x \in y \odot_X z$, $u = f(y)$, $v = f(z)$ for some $y, z \in X$

a catoid is functional if $x, x' \in y \odot z \Rightarrow x = x'$

and local if $t(x) = s(y) \Rightarrow x \odot y \neq \emptyset$

a single-set category is a local functional catoid

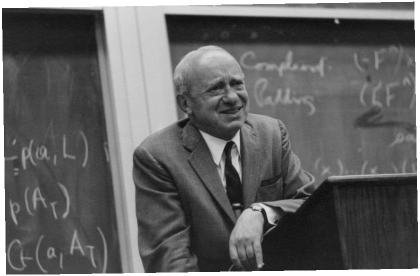
$X_s = \{x \mid s(x) = x\} = X_t$ determines objects of (small) category all structures considered so far are catoids

relations are constructed from the pair groupoid on $X \times X$

shuffle languages form the shuffle catoid
with $\|$ total and $s(w) = \varepsilon = t(w)$ for all $w \in \Sigma^*$

there are many other interesting examples

Jónsson-Tarski Duality



in boolean algebras with operators
n-ary modalities in B are dual to $n+1$-ary relations in X

we view $\cdot : \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X$ as binary modality
and $\odot : X \times X \to \mathcal{P}X$ as ternary relation

for powerset structures this duality is almost trivial

$$x \in y \odot z \Leftrightarrow \{x\} \subseteq \{y\} \cdot \{z\}$$

atoms in powerset structure Q define relational structure Q_+

relational structure X yields powerset structure X^+ with

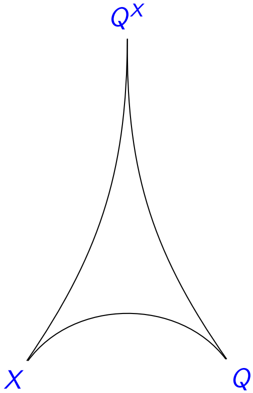
$$AB = \bigcup\{y \odot z \mid y \in A,\ z \in B\}$$

Jónsson/Tarski have shown that $(Q_+)^+ \cong Q$ and $(X^+)_+ \cong X$

in fact, the categories of powerset and relational structures are dually equivalent

Jónsson-Tarski duality yields modal correspondences translating identities between X and Q

more generally we can prove 2-out-of-3 correspondences in convolution algebras



$$(f * g)(x) = \bigvee_{x \in y \odot z} f(y) \cdot g(z)$$

$$id_{X_s}(x) = \begin{cases} 1 & \text{if } x \in X_s \\ 0 & \text{otherwise} \end{cases}$$

$$(\bigvee F)(x) = \bigvee \{ f(x) \mid f \in F \}$$

$$0(x) = 0$$

Basic Correspondences

theorem:

1. if X is catoid and Q quantale, then Q^X is quantale
2. if Q^X is quantale and Q supported quantale, then X is catoid
3. if Q^X is quantale and X supported catoid, then Q is quantale

"supported" means structures have enough elements for a construction (e.g., $0 \neq 1$ or some composable elements)

we get KA if X is finitely decomposable: $\{(y, z) \mid x \in y \odot z\}$ finite f.a. x

$$
\begin{aligned}
(f*(g*h))(x) &= \bigvee_{x\in u\odot y} f(u)\cdot\left(\bigvee_{y\in v\odot w} g(v)\cdot h(w)\right)\\
&= \bigvee_{x\in u\odot(v\odot w)} f(u)\cdot(g(v)\cdot h(w))\\
&= \bigvee_{x\in(u\odot v)\odot w} (f(u)\cdot g(v))\cdot h(w)\\
&= \bigvee_{x\in y\odot w}\left(\bigvee_{y\in u\odot w} f(u)\cdot g(v)\right)\cdot h(w)\\
&= ((f*g)*h)(x)
\end{aligned}
$$

$$
\begin{aligned}
x\in u\odot(v\odot w) &\Leftrightarrow (\delta_u*(\delta_v*\delta_w))(x)=1\\
&\Leftrightarrow ((\delta_u*\delta_v)*\delta_w)(x)=1\\
&\Leftrightarrow x\in(u\odot v)\odot w
\end{aligned}
$$

Catoids and Modal Quantales

a domain quantale equips a quantale with $dom : Q \to Q$ satisfying

$$dom(x)x = x \qquad dom(x + y) = dom(x) + dom(y)$$
$$dom(0) = 0 \qquad dom(x) \leq 1 \qquad dom(x\,dom(y)) = dom(xy)$$

a codomain quantale (Q, cod) is a domain quantale (Q^{op}, dom)

a modal quantale is a domain and codomain quantale such that

$$dom \circ cod = cod \qquad cod \circ dom = dom$$

in relation quantale $dom(R)_{aa} \Leftrightarrow \exists b.\ R_{ab}$ and $cod(R)_{aa} \Leftrightarrow \exists b.\ R_{ba}$

domain elements $Q_{dom} = \{x \mid dom(x) = x\}$ form distributive lattice and boolean algebra if Q is boolean

we define modal operators for $x \in Q$ and $p \in Q_{dom}$

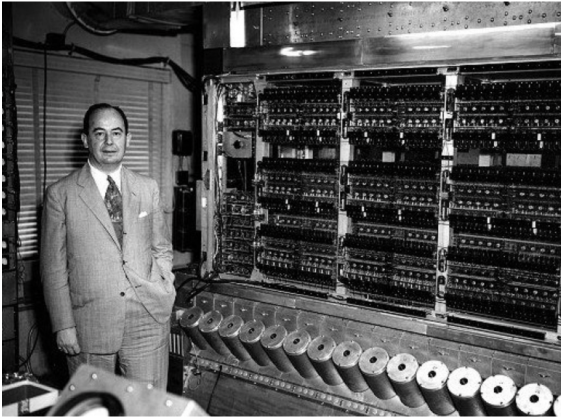
$$|x\rangle p = dom(xp) \qquad \langle x|p = cod(px)$$
$$|x]p = \bigvee\{q \mid |x\rangle q \leq p\} \qquad [x|p = \bigvee\{q \mid \langle x|q \leq p\}$$

this yields dynamic logics/algebras, predicate transformer algebras, boolean algebras with operators

in relation quantale

$$(|R\rangle P)_{aa} \Leftrightarrow \exists b.\ R_{ab} \wedge P_{bb} \qquad (|R]P)_{aa} \Leftrightarrow \forall b.\ R_{ab} \Rightarrow P_{bb}$$

Modal Quantales and Program Correctness



Modal Quantales and Program Correctness

we use relations over program store to verify programs

$x \in Q$ as programs, $+$ as nondeterministic choice, $\cdot$ as sequential composition, $(-)^*$ as finite iteration

in boolean quantale, for $x \in Q$, $p \in Q_{dom}$

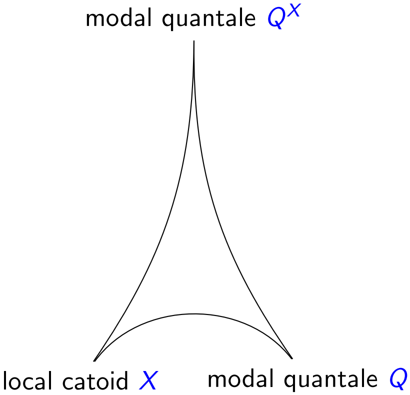
$$\text{if } p \text{ then } x \text{ else } y = px + \overline{p}y \qquad \text{while } p \text{ do } x = (px)^*\overline{p}$$

$|x]p$ calculates wlp of program x from postcondition q

program x is (partially) correct if $p \leq |x]q$

Local Catoids and Modal Quantales

theorem: we have 2-out-of-3 correspondences



$$dom(f) = \bigvee_{x \in X} dom(f(x))\delta_{s(x)}$$

$$cod(f) = \bigvee_{x \in X} cod(f(x))\delta_{t(x)}$$

for $Q = 2$

1. if X is local catoid, then $(\mathcal{P}X, \subseteq, \odot, X_s, \mathcal{P}s, \mathcal{P}t)$ is modal quantale
2. if $\mathcal{P}X$ is modal quantale, then X is local catoid

we derive $s(xs(y)) = s(xy)$ and $s \circ r = r$ in X and lift to *dom*-axioms in $\mathcal{P}X$ (other *dom*-axioms don't depend on identities in X)

$$\begin{aligned} dom(A \odot dom(B)) &= \bigcup \{s(x \odot s(y)) \mid x \in A,\ y \in B, t(x) = s(s(y))\} \\ &= \bigcup \{s(x \odot y) \mid x \in A,\ y \in B, t(x) = s(y)\} \\ &= dom(A \odot B) \end{aligned}$$

we can recover the catoid axioms from the atom structure in $\mathcal{P}X$

$$\begin{aligned} s(x \odot s(y)) &= dom(\{x\} \odot dom(\{y\})) \\ &= dom(\{x\} \odot \{y\}) \\ &= s(x \odot y) \end{aligned}$$

Models of Modal Quantales

if you want to build a modal convolution quantale, look for a catoid

the lifting is then generic

locality axiom $dom(x dom(y)) = dom(xy)$ is precisely the composition pattern of categories

absorption axiom $dom(x)x = x$ corresponds to left unit axiom of catoids

every category gives rise to modal quantale

Catoids and Concurrent Quantales

word concatenation interacts with shuffle via interchange law

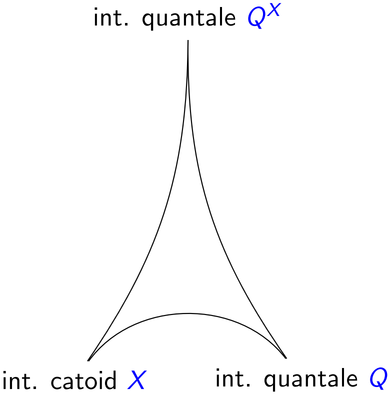
$$(v\|v') \cdot (w\|w') \subseteq (v \cdot w)\|(v' \cdot w')$$

we can lift it to $(A\|A') \cdot (B\|B') \subseteq (A \cdot B)\|(A'\|B')$

an interchange catoid $(X, \odot_0, s_0, t_0, \odot_1, s_1, t_1)$ consists of two catoids that interact via $(x \odot_1 x') \odot_0 (y \odot_1 y') \subseteq (x \odot_0 y) \odot_1 (x' \odot_0 y')$

an interchange quantale $(Q, \leq, \cdot_0, 1_0, \cdot_1, 1_1)$ consists of two quantales that interact via $(x \cdot_1 x') \cdot_0 (y \cdot_0 y') \leq (x \cdot_0 y) \cdot_1 (x' \cdot_0 y')$

theorem: we have 2-out-of-3 correspondences



it suffices to consider correspondences for interchange

Interleaving Concurrency

correspondences yield (weighted) shuffle languages with interchange laws

$\|$ is commutative, there's a general 2-out-of-3 for commutativity

the shuffle catoid has one single unit ε

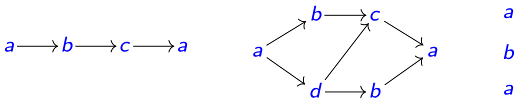
in interchange catoids/quantales with one single unit there's a collapse à la Eckmann-Hilton, small interchange laws are derivable

$x \cdot_0 y \leq x \cdot_1 y \qquad x \cdot_0 (y \cdot_1 z) \leq (x \cdot_0 y) \cdot_1 z \qquad (x \cdot_1 y) \cdot_0 z \leq x \cdot_1 (y \cdot_0 z)$

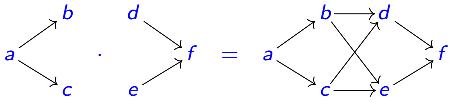
and commutative variants in catoid/quantale

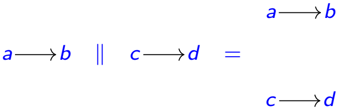
Non-Interleaving Concurrency

pomsets are a standard model of non-interleaving concurrency

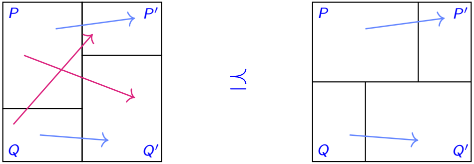


they are composed using serial/parallel composition





operations $\cdot$ and $\|$ share the empty pomset ε as their unit



pomset Q subsumes pomset P, $P \preceq Q$, if there exists pomset morphism $Q \to P$ that is bijective on points

$\preceq$ is partial order on pomsets

we get interchange catoid $(\mathrm{Pom}(\Sigma), \cdot, \Downarrow, \varepsilon)$ with $x \Downarrow y = \{z \mid z \preceq x \| y\}$

it lifts to a powerset interchange quantale,
the downclosed languages form subquantale

this generalises to convolution quantales (under technical restrictions)

Models of Conurrent Quantales

construction of interchange/concurrent quantales motivated this approach

correspondences for interchange catoids/quantales simplified discussions about potential models

Single-Set n-Categories

Similarly a 2-category can be considered to be a single set X considered as the set of 2-cells (e.g., of natural transformations). Then the previous 1-cells (the arrows) and the 0-cells (the objects) are just regarded as special "degenerate" 2-cells. On the set X of 2-cells there are then two category structures, the "horizontal" structure $(\#_0, s_0, t_0)$ and the "vertical" structure $(\#_1, s_1, t_1)$. Each satisfies the axioms above for a category structure and in addition

(i) Every identity for the 0-structure is an identity for the 1-structure;
(ii) The two category structures commute with each other.

Here, the condition (ii) means, of course, that

$$s_0\, s_1 = s_1\, s_0\ , \qquad s_0\, t_1 = t_1\, s_0\ , \qquad t_0\, s_1 = s_1\, t_0\ , \qquad t_0\, t_1 = t_1\, t_0 \tag{7}$$

and that, for $\alpha, \beta = 0$, 1 or 1, 0, and for all x, y, u, and v

$$(x \#_\alpha y) \#_\beta (u \#_\alpha v) \#_\alpha (y \#_\beta v)\ , \tag{8}$$

$$t_\alpha(x \#_\beta y) = (t_\alpha x) \#_\beta (t_\alpha y)\ ,$$

$$s_\alpha(x \#_\beta y) = (s_\alpha x) \#_\beta (s_\alpha y)\ ,$$

whenever both sides are defined.

Since $s_0 x$ and $t_0 x$ are identities for the 0-structure, they are also identities for the 1-structure by condition (i) above. Hence,

$$s_1\, s_0 = s_0\ , \qquad t_1\, s_0 = s_0\ , \qquad s_1\, t_0 = t_0\ , \qquad t_1\, t_0 = t_0\ . \tag{9}$$

With this preparation, we can now readily define a 3-category or more generally an n-category for any natural number n. The latter is a set X with n different category structures $(\#_i, s_i, t_i)$, for $i = 0, \ldots, n-1$, which commute with each other and are such that an identity for structure i is also an identity for structures j whenever $j > i$. Put differently, each pair $\#_i$ and $\#_j$ for $j > i$ constitute a 2-category. This readily leads to a definition of the useful notion of an ω-category: $i = 0, 1, 2, \ldots$.

n-Catoids

a (globular) n-catoid $(X, \odot_i, s_i, t_i)_{0 \leq i < n}$ consists of n-catoids $(X, \odot_i, s_i, t_i)$ that interact, for all $0 \leq i < j < n$, via

$$s_i \circ s_j = s_j \circ s_i \qquad s_i \circ t_j = t_j \circ s_i \qquad t_i \circ s_j = s_j \circ t_i \qquad t_i \circ t_j = t_j \circ t_i$$

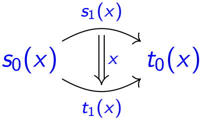
$$(w \odot_j x) \odot_i (y \odot_j z) \subseteq (w \odot_i y) \odot_j (x \odot_i z)$$

$$s_j(x \odot_i y) = s_j(x) \odot_i s_j(y) \qquad t_j(x \odot_i y) = t_j(x) \odot_i t_j(y)$$

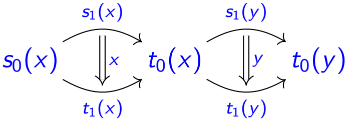
$$s_i(x \odot_j y) \subseteq s_i(x) \odot_j s_i(y) \qquad t_i(x \odot_j y) \subseteq t_i(x) \odot_j t_i(y)$$

$$s_j \circ s_i = s_i \qquad s_j \circ t_i = t_i \qquad t_j \circ s_i = s_i \qquad t_j \circ t_i = t_i$$

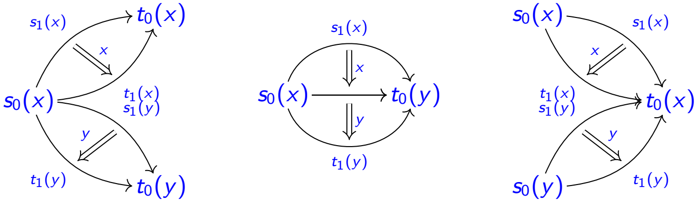
a single-set n-category is a local functional n-catoid



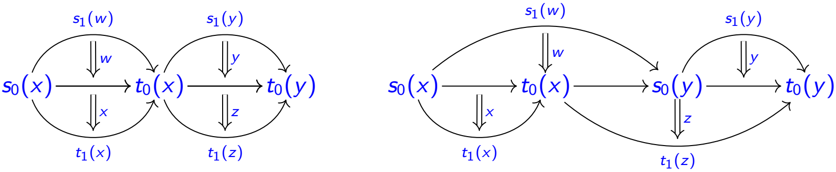
$s_1(x \odot_0 y) = s_1(x) \odot_0 s_1(y)$ and $t_1(x \odot_0 y) = t_1(x) \odot_0 t_1(y)$



$s_0(x \odot_1 y) \subseteq s_0(x) \odot_1 s_0(y)$ and $t_0(x \odot_1 y) \subseteq t_0(x) \odot_1 t_0(y)$



$$(w \odot_1 x) \odot_0 (y \odot_1 z) \subseteq (w \odot_0 y) \odot_1 (x \odot_0 z)$$



Reduced n-Catoid Axioms

the following axioms are irredundant and subsume the previous ones

$$(w \odot_j x) \odot_i (y \odot_j z) \subseteq (w \odot_i y) \odot_j (x \odot_i z)$$

$$s_j(x \odot_i y) = s_j(x) \odot_i s_j(y) \qquad t_j(x \odot_i y) = t_j(x) \odot_i t_j(y)$$

this streamlines correspondence proofs

n-Quantales

a *(globular)* n-*quantale* $(Q, \leq, \cdot_i, 1_i, dom_i, cod_i)_{0 \leq i < n}$ consists of n modal quantales $(Q, \leq, \cdot_i, 1_i, dom_i, cod_i)$ that interact, for all $0 \leq i < j < n$, via

$$(w \cdot_j x) \cdot_i (y \cdot_j z) \leq (w \cdot_i y) \cdot_j (x \cdot_i z)$$

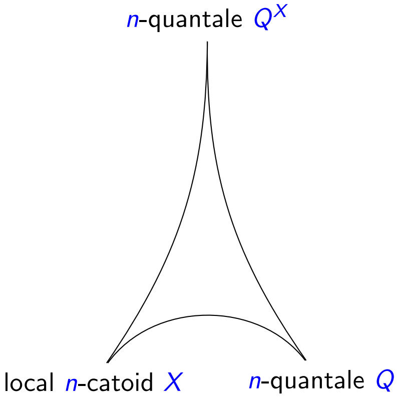
$$dom_j(x \cdot_i y) = dom_j(x) \cdot_i dom_j(y) \qquad cod_j(x \cdot_i y) = cod_j(x) \cdot_i cod_j(y)$$

$$dom_i(x \cdot_j y) \leq dom_i(x) \cdot_j dom_i(y) \qquad cod_i(x \cdot_j y) \leq cod_i(x) \cdot_j cod_i(y)$$

$$dom_j(dom_i(x)) = dom_i(x)$$

n-Catoids and n-Quantales

theorem: we have 2-out-of-3 correspondences



relative to previous correspondences it remains to check the globular ones

Higher Rewriting

(modal) Kleene algebras allow proving facts from abstract rewriting (Church-Rosser theorem, Newman's lemma, . . .)

n-Kleene algebras allow proving analogous fact from higher rewriting (using free (n, p)-categories constructed using polygraphs/computads)

our correspondences justify the axioms of n-Kleene algebra firmly in terms of (free) n-categories

we can justify those of (n, p)-Kleene algebras by integrating (single-set) groupoids

Jónnson-Tarski knew about correspondence between groupoids and relation algebras

single-set approach makes approach easily accessible to proof assistants and even SMT-solvers

Conclusion

catoids simplify the construction of models for algebras of programs

they often tell where axioms in algebras of programs come from

they provide a particular way of dealing with partiality
(in algebra or category theory)

they might allow formaling higher categories using automated theorem provers/SMT solvers . . . but this is speculation

Thanks

Cameron Calk, James Cranch, Simon Doherty, Brijesh Dongol,
Uli Fahrenberg, Éric Goubault, Ian Hayes, Christian Johansen,
Philippe Malbos, Damien Pous, Krzysztof Ziemiański

Papers

C. Calk, P. Malbos, G. Struth, D. Pous. Catoids and Globular Convolution Quantales (manuscript)

U. Fahrenberg, C. Johansen, G. Struth, K. Ziemiański. lr-Multisemigroups and Modal Convolution Algebras. CoRR abs/2105.00188, 2021

J. Cranch, S. Doherty, G. Struth. Convolution and Concurrency, *MSCS*, 2021

B. Dongol, I. J. Hayes, G. Struth. Convolution Algebras: Relational Convolution, Generalised Modalities and Incidence Algebras, *LMCS*, 2021

C. Calk, U. Fahrenberg, C. Johansen, G. Struth, K. Ziemiański. lr-Multisemigroups, Modal Quantales and the Origin of Locality. *RAMiCS 2021*

C. Calk, É. Goubault, P. Malbos, G. Struth. Algebraic Coherent Confluence and Higher-Dimensional Globular Kleene Algebra, CoRR abs/2006.16129, 2020

B. Dongol, I. J. Hayes, G. Struth. Convolution as a Unifying Concept: Applications in Separation Logic, Interval Calculi and Concurrency, *ACM TOCL*, 2016