Capturing Changes in Combinatorial Dynamical Systems via Persistent Homology

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Overview & Outline

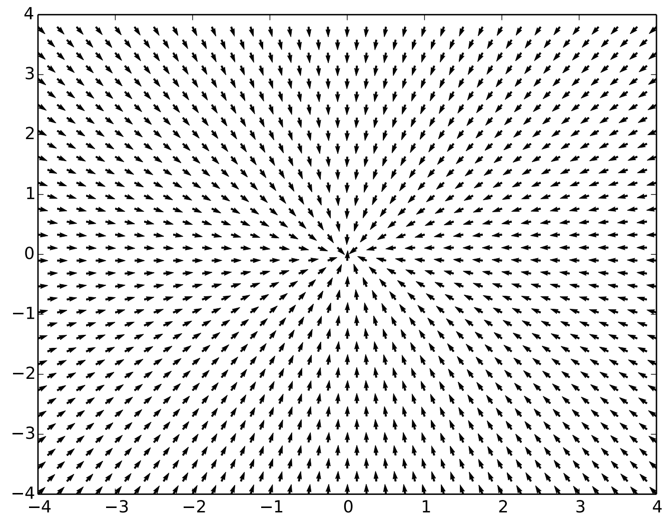
- Motivating Example and Persistence
- Combinatorial Dynamical Systems & Conley Index
- Conley Index Persistence
- Conley-Morse Graph Persistence

Motivating Example: Hopf Bifurcation

$$x' = -y + x(\lambda - x^2 - y^2)$$

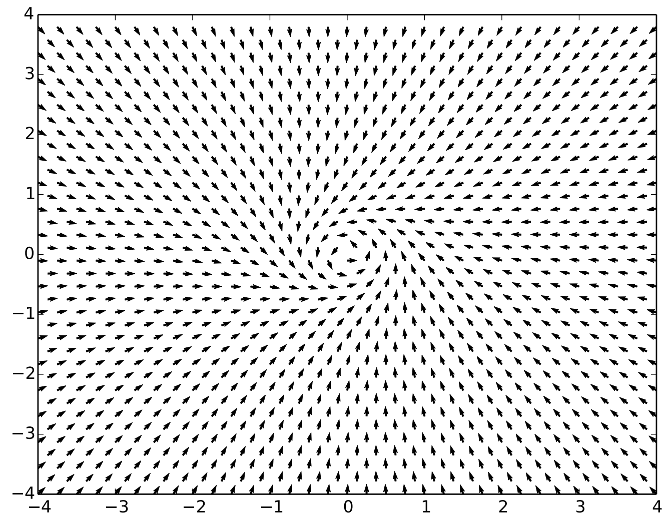
$$y' = x + y(\lambda - x^2 - y^2)$$

Motivating Example: Hopf Bifurcation



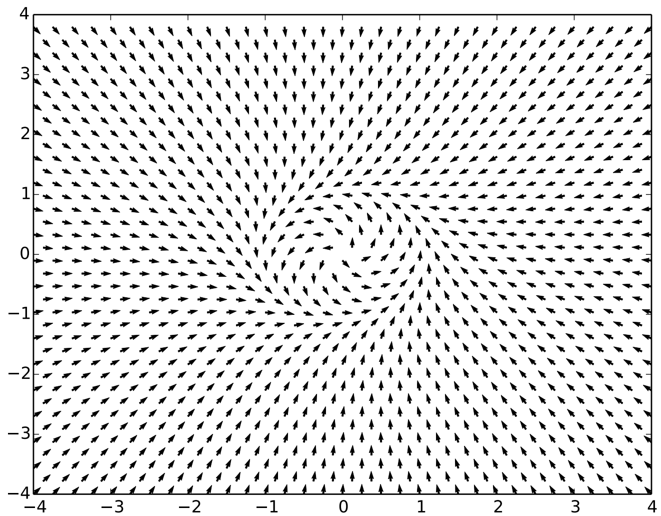
$$\lambda \ll 0$$

Motivating Example: Hopf Bifurcation



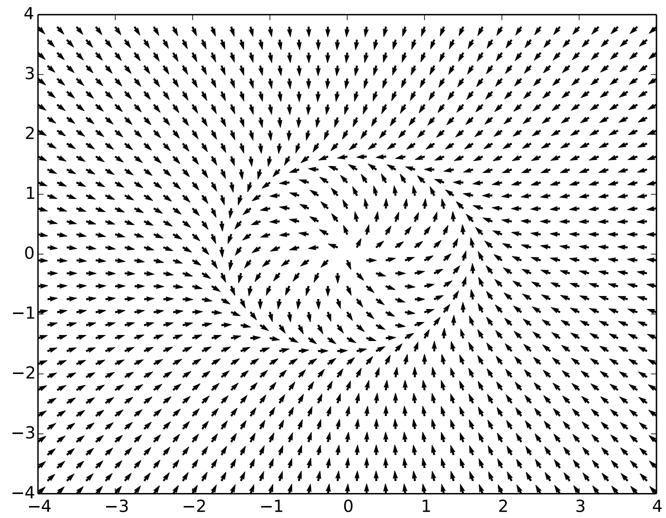
$$\lambda = 0$$

Motivating Example: Hopf Bifurcation



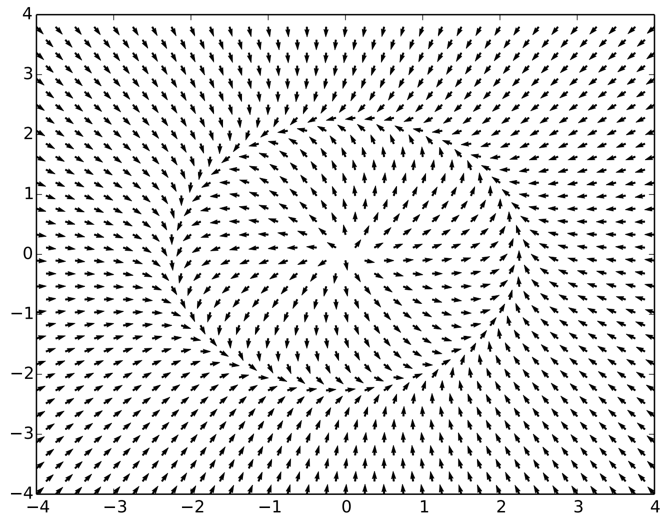
$$\lambda = 1$$

Motivating Example: Hopf Bifurcation



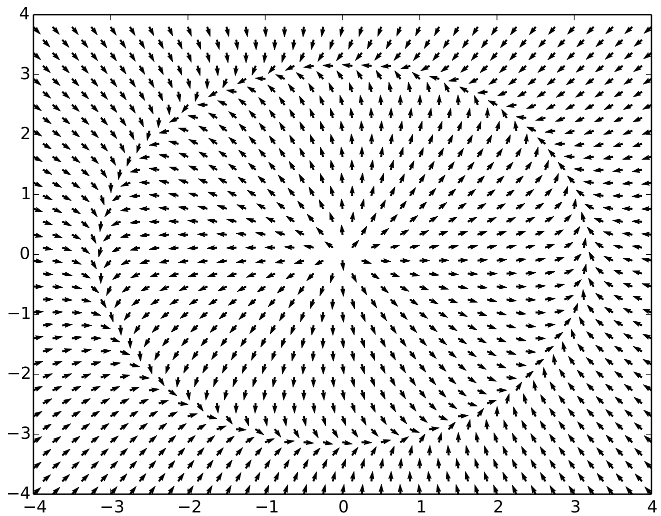
$\lambda = 2.5$

Motivating Example: Hopf Bifurcation



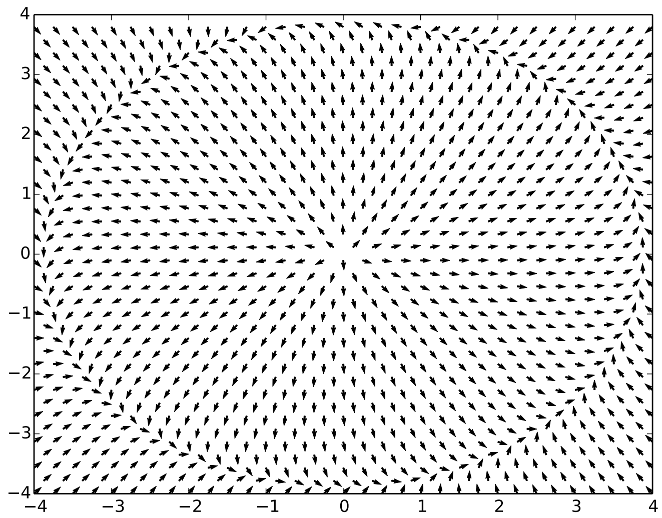
$$\lambda = 5$$

Motivating Example: Hopf Bifurcation



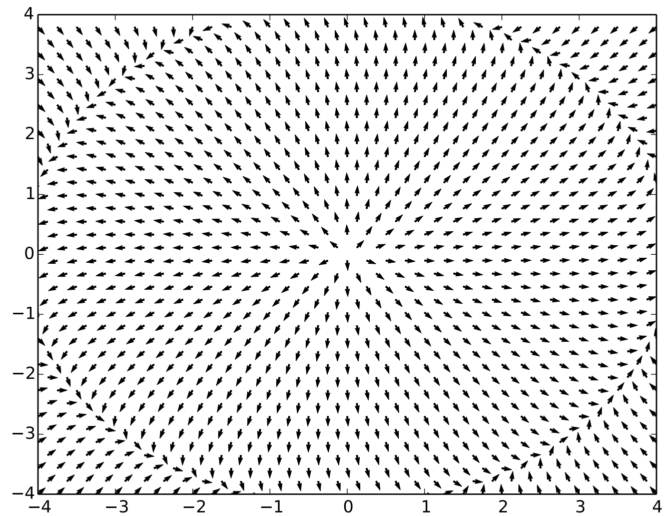
$$\lambda = 10$$

Motivating Example: Hopf Bifurcation



$$\lambda = 15$$

Motivating Example: Hopf Bifurcation



$$\lambda = 17.5$$

Motivating Example: Hopf Bifurcation

Note: attractor from $\lambda = -\infty$ to $\lambda = 16$

Repeller from $\lambda = 0$ to $\lambda = \infty$

Can we use computational topology to automatically detect these features?

Motivating Example: Hopf Bifurcation

Note: attractor from $\lambda = -\infty$ to $\lambda = 16$

Repeller from $\lambda = 0$ to $\lambda = \infty$

Can we use computational topology to automatically detect these features?

Yes, by using persistence

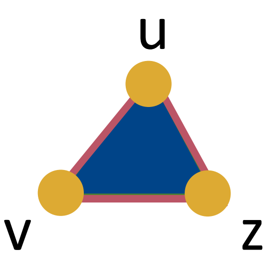
Persistent Homology

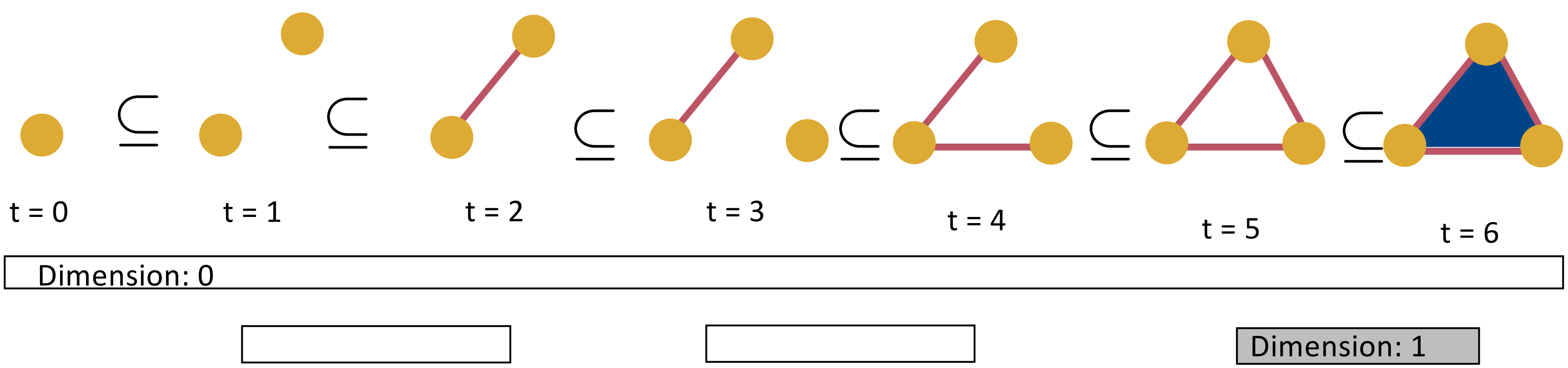
Summarizes changing homology of a filtration [ELZ00]

$$K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n = K$$

Persistence Example

K =



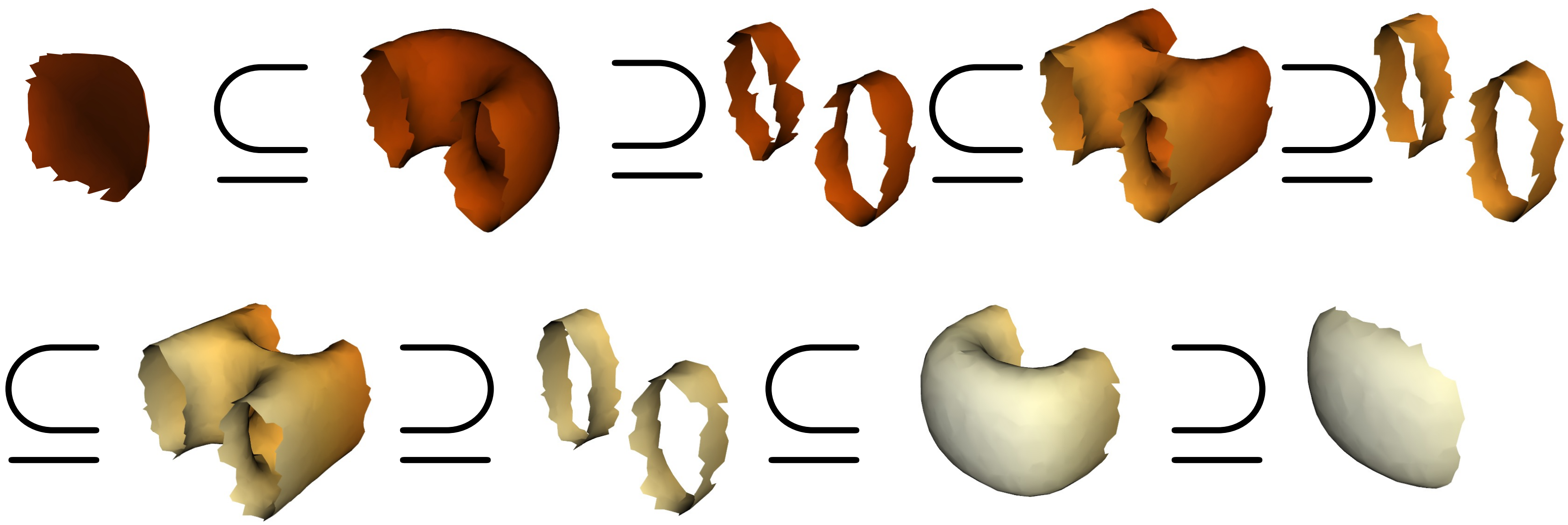


$$K_0 \subset K_1 \subset K_2 \subset K_3 \subset K_4 \subset K_5 \subset K_6$$

Zigzag Persistence

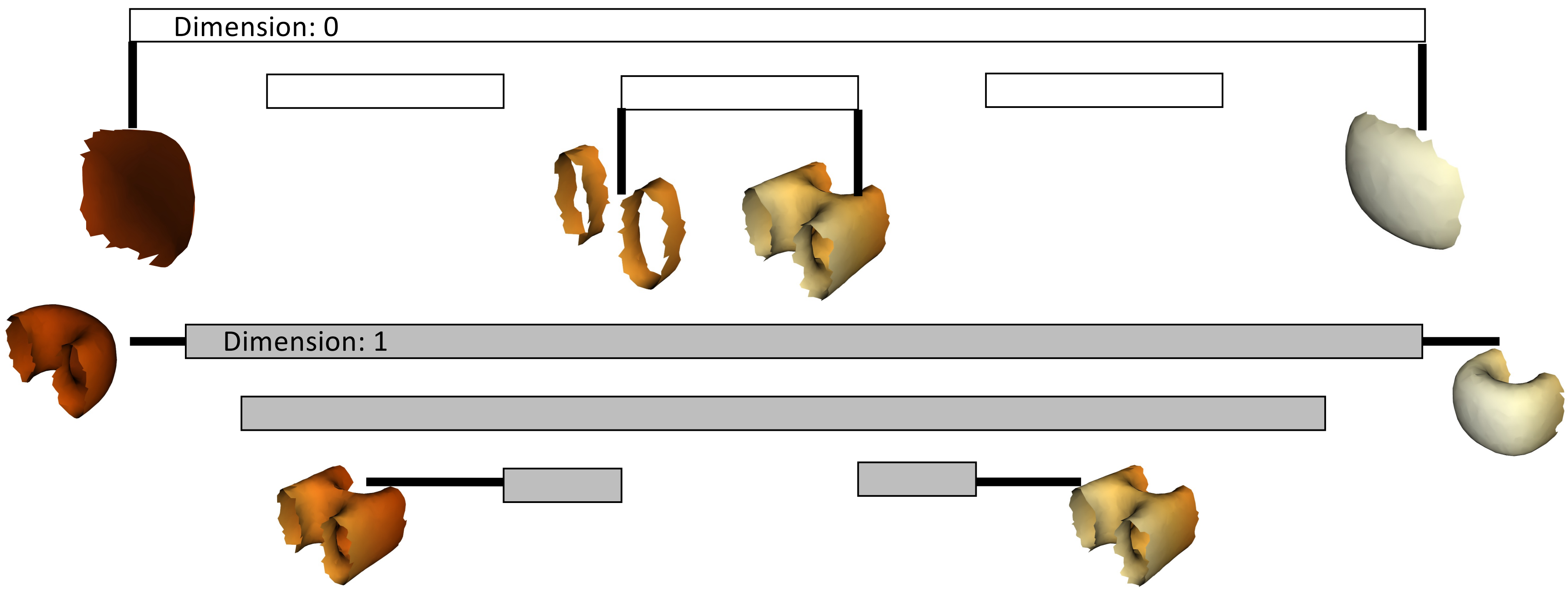
$$K_1 \subseteq K_2 \supseteq K_3 \subseteq \ldots \supseteq K_n$$

“Level Set” Persistence



[CDM09] [DW07]

Level Set Barcode



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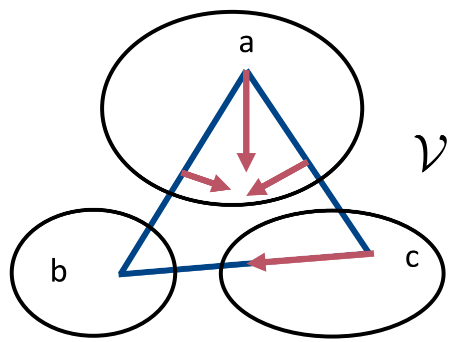
Multivectors

Let K denote a simplicial complex and $\leq$ denote the face relation.

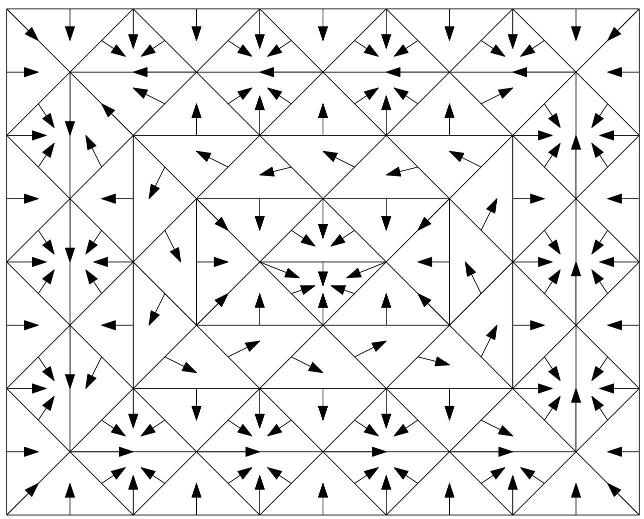
Definition: A multivector V is a convex subset of K with respect to $\leq$.

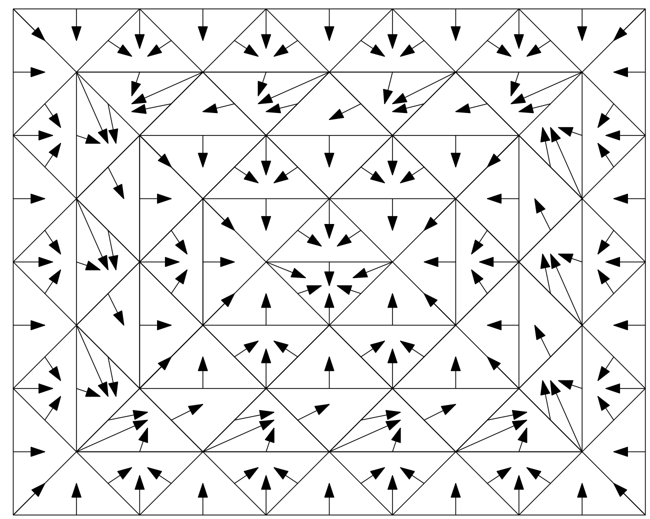
Definition: A multivector field $\mathcal{V}$ is a partition of K into multivectors.



$$\mathcal{V} = \{\{b\}, \{c, bc\}, \{a, ab, ac, abc\}\}$$

Multivector Fields





Multivector Fields as a Dynamical System

Let $\sigma \in K$. Then $\text{cl}(\sigma) = \{\tau \in K \mid \tau \leq \sigma\}$.

$[\sigma]_{\mathcal{V}}$ denotes the vector in $\mathcal{V}$ containing σ

Dynamics generator $F_{\mathcal{V}} : K \multimap K$ defined as:

$$F_{\mathcal{V}}(\sigma) = [\sigma]_{\mathcal{V}} \cup \text{cl}(\sigma)$$

Multivector Fields as a Dynamical System

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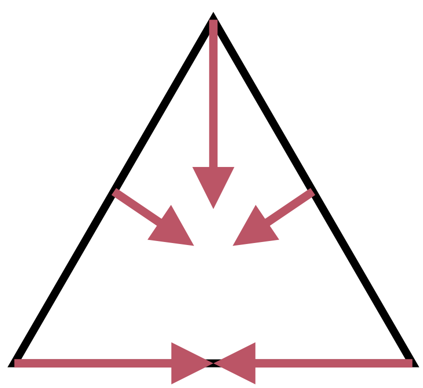


Multivector field

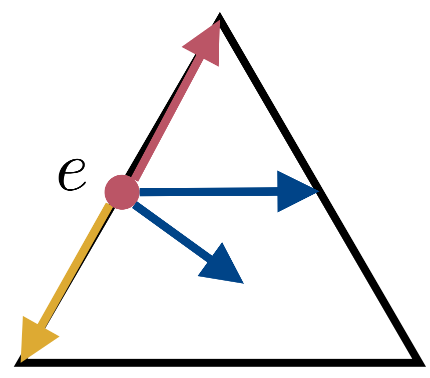


Image of marked edge e

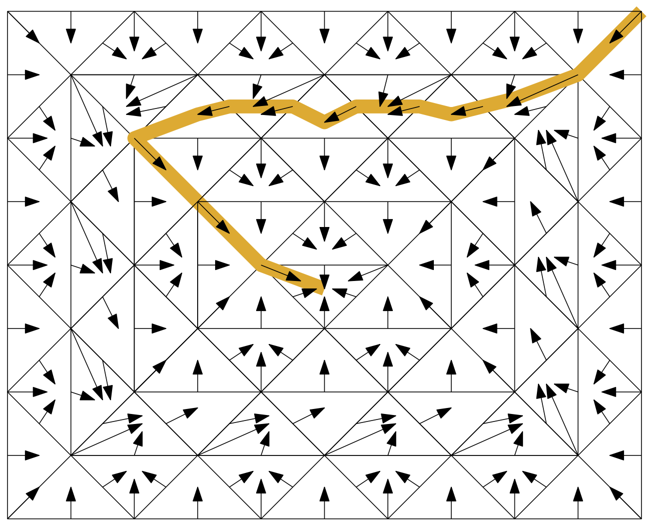
Blue simplices are in $[e]_{\mathcal{V}} \setminus \mathsf{cl}(e)$

Yellow simplices are in $\mathsf{cl}(e) \setminus [e]_{\mathcal{V}}$

Red simplices are in $[e]_{\mathcal{V}} \cap \mathsf{cl}(e)$

Paths

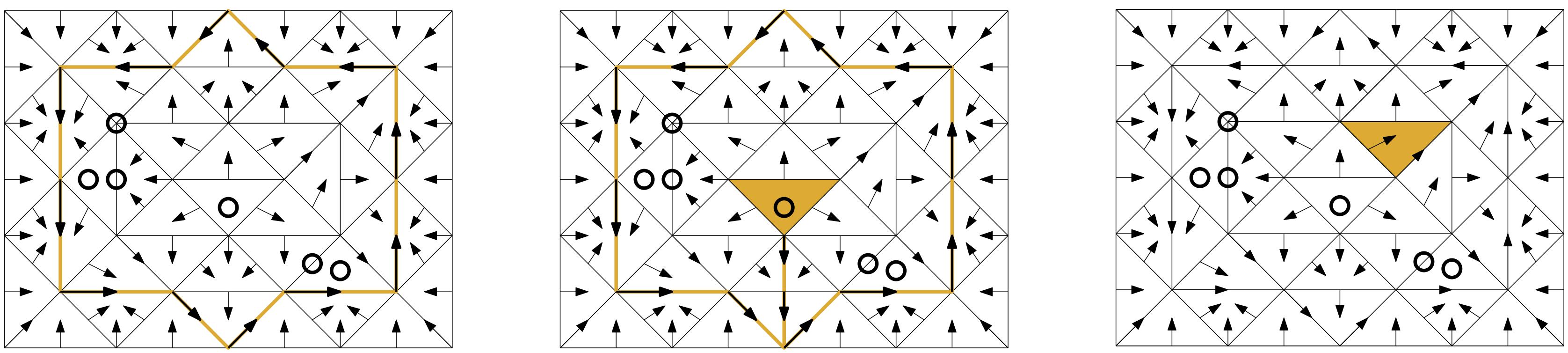
<u>Definition:</u> A path is a finite sequence of simplices $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that $\sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i)$



Solutions

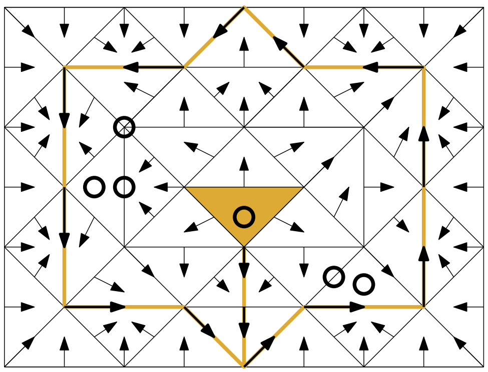
Definition: A solution is a bi-infinite sequence of simplices

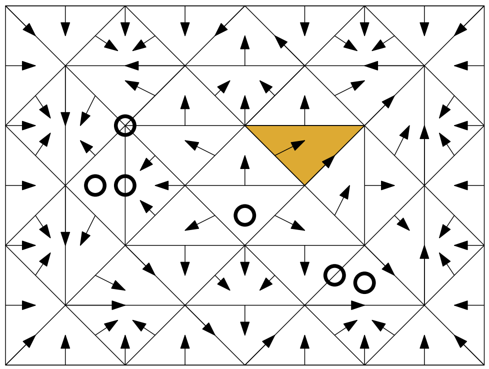
$\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ such that $\sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i)$



Solutions

Definition: A solution is a bi-infinite sequence of simplices $\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ such that $\sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i)$





But as $F_{\mathcal{V}}(\sigma) = [\sigma]_{\mathcal{V}} \cup \mathrm{cl}(\sigma)$, every simplex gives a solution!

Critical Multivectors

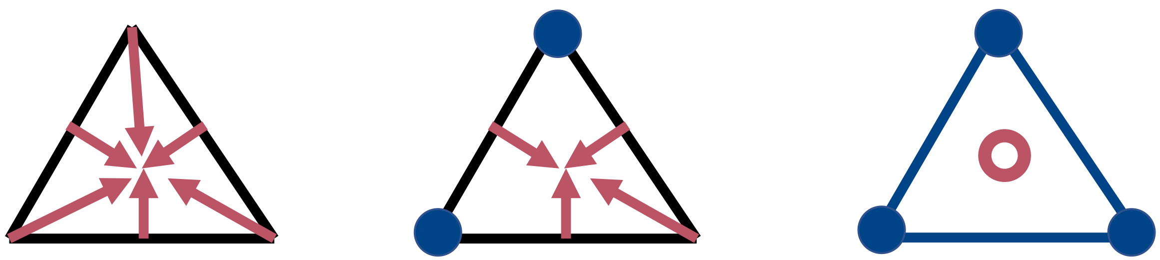
Definition: Let $A \subseteq K$. The mouth of A is defined as

$$\mathsf{mo}\,(A) := \mathsf{cl}\,(A) \setminus A$$

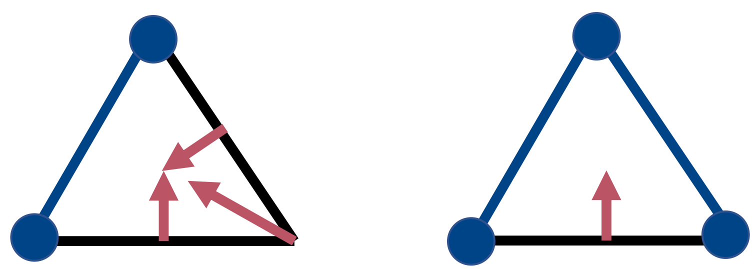
Definition: A multivector $[\sigma]_{\mathcal{V}}$ is critical if there exists a k such that $H_k(\mathsf{cl}\,([\sigma]_{\mathcal{V}})\,, \mathsf{mo}\,([\sigma]_{\mathcal{V}}))$ is nontrivial.

Critical Multivectors

Critical:

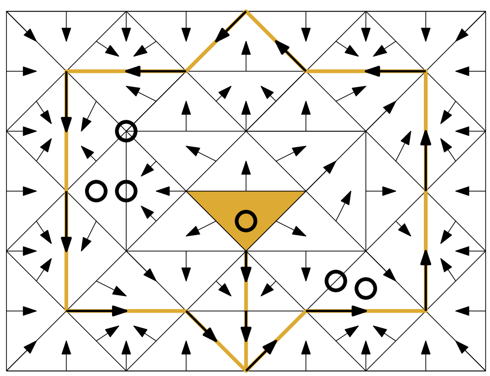


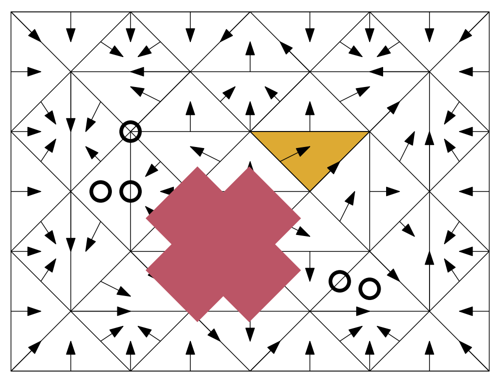
Regular:



Essential Solutions

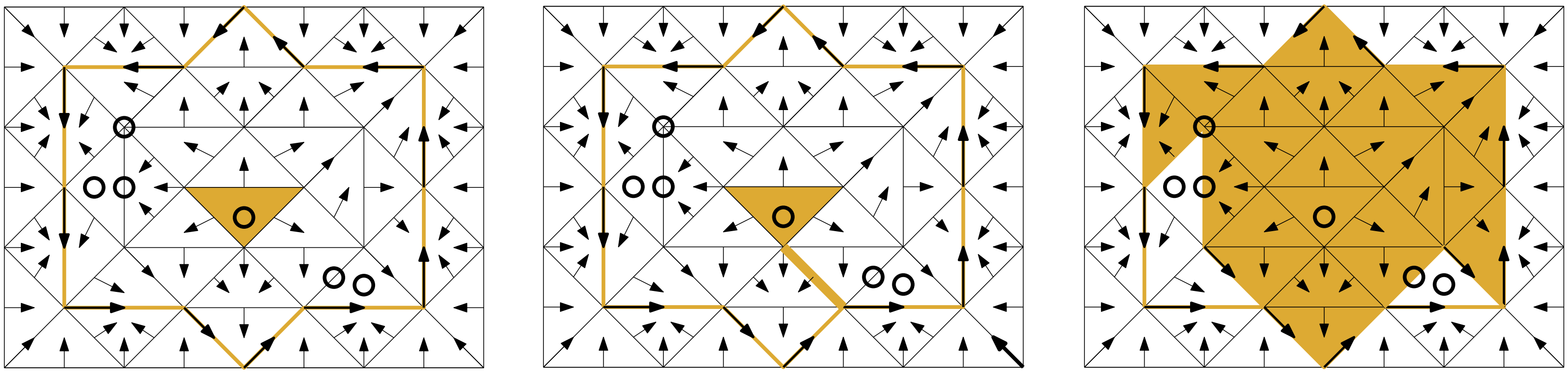
Definition: Let $\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ denote a solution. If for each σ_i where $[\sigma_i]_{\mathcal{V}}$ is noncritical, there exists a $j > i$ and $j' < i$ where $[\sigma_i]_{\mathcal{V}} \neq [\sigma_j]_{\mathcal{V}}$ and $[\sigma_i]_{\mathcal{V}} \neq [\sigma_{j'}]_{\mathcal{V}}$,
then $\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ is an essential solution.





Invariant Sets

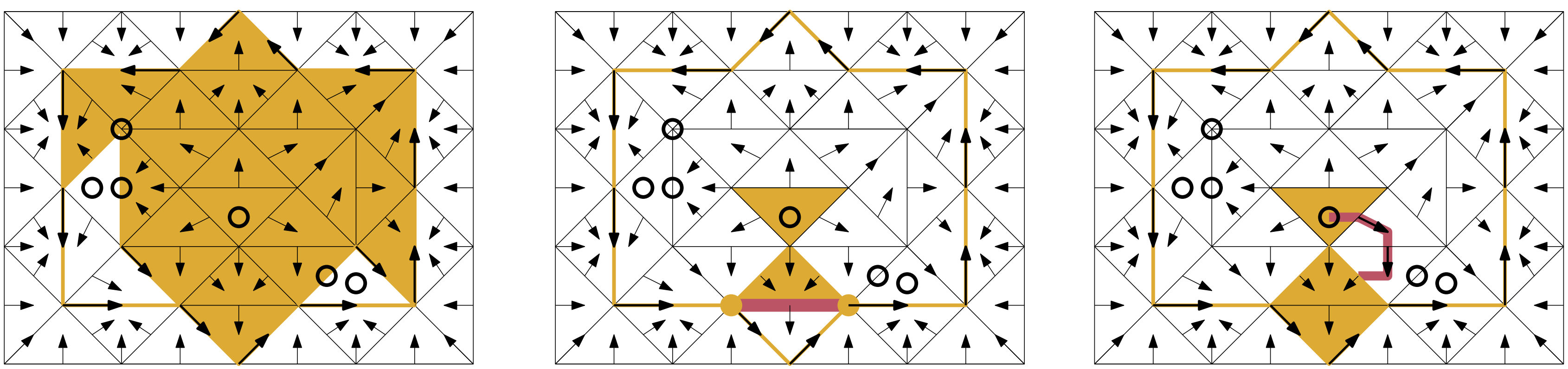
Definition: Let $A \subseteq K$. The invariant part of A, denoted $\text{Inv}(A)$, is the set of simplices in A which appear in an essential solution in A.



If $A = \text{Inv}(A)$, then A is an invariant set.

Isolated Invariant Sets

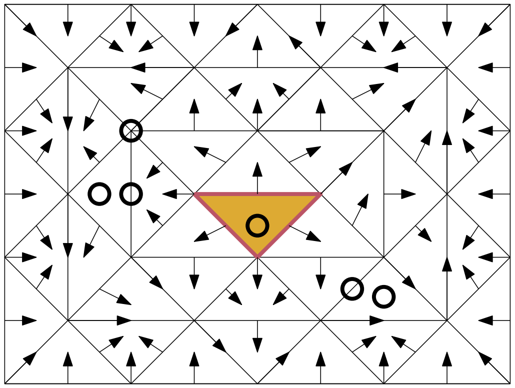
<u>Definition:</u> Let $A \subseteq N \subseteq K$, where A is an invariant set and N is closed (i.e. $N = \text{cl}(N)$). If every path in N with endpoints in A is contained in A, then A is an isolated invariant set, and N is an isolating neighborhood for A.



Index Pairs

<u>Definition:</u> Let A be an isolated invariant set, and E and P closed sets such that $E \subseteq P$. If:

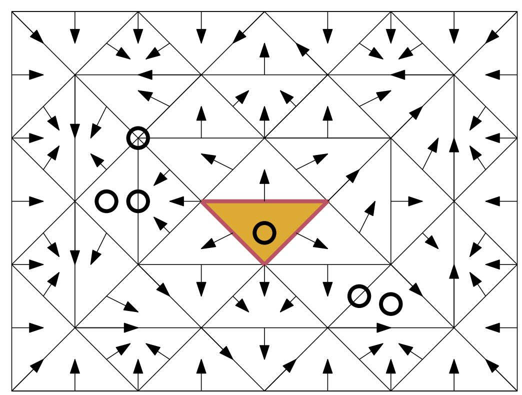
1. $F_{\mathcal{V}}(E) \cap P \subset E$,
2. $F_{\mathcal{V}}(P \setminus E) \subseteq P$, and
3. $A = \mathsf{Inv}(P \setminus E)$



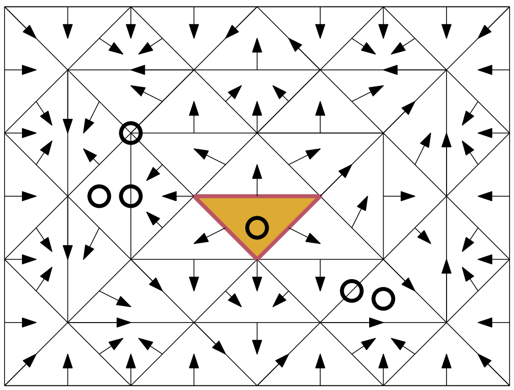
Then (P, E) is an index pair for A.

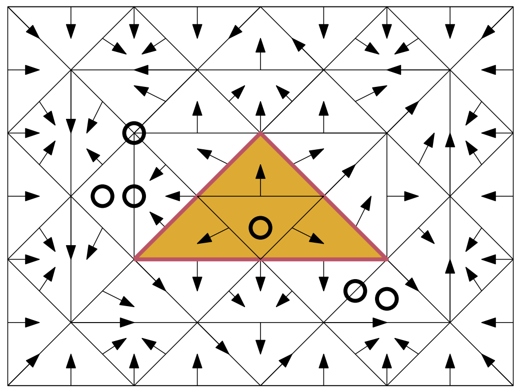
Conley Index

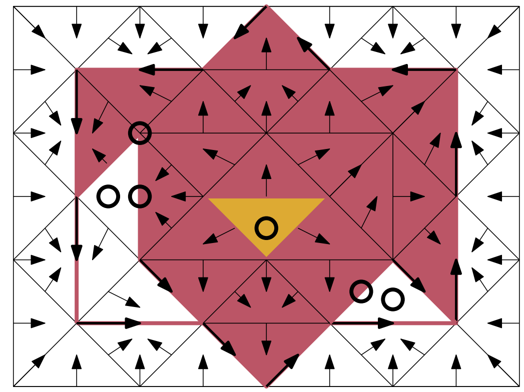
Theorem [LKMW2019]: Let A denote an isolated invariant set. The pair $(\text{cl}(A), \text{mo}(A))$ is an index pair for A.



Index Pairs are Not Unique





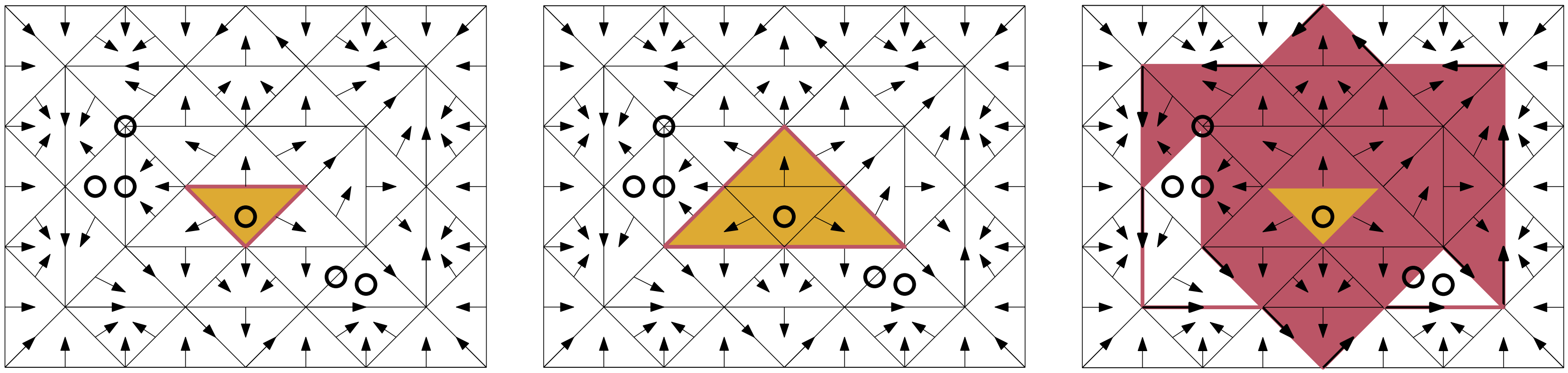


Conley Index

Definition: Let (P, E) be an index pair for A. Then the k-dimensional Conley Index is given by $H_k(P, E)$.

Theorem [LKMW 2019]: The k-dimensional Conley Index for A is well defined.

Conley Indices



$$H_2(R \cup Y, R) = \mathbb{Z}_2$$

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[DMS20] T. K. Dey, M. Mrozek, R. Slechta. "Persistence of the Conley Index in Combinatorial Dynamical Systems." SoCG 2020.

Conley Index Persistence

First attempt: for each $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$, compute an isolated invariant set, $A_1, A_2, \ldots, A_n$ and corresponding index pairs.

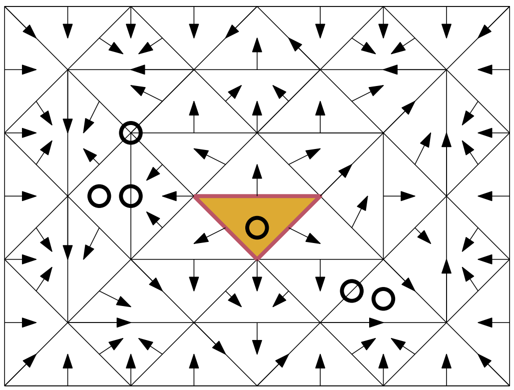
$$(\mathsf{cl}(A_1), \mathsf{mo}(A_1)), (\mathsf{cl}(A_2), \mathsf{mo}(A_2)), \ldots, (\mathsf{cl}(A_n), \mathsf{mo}(A_n))$$

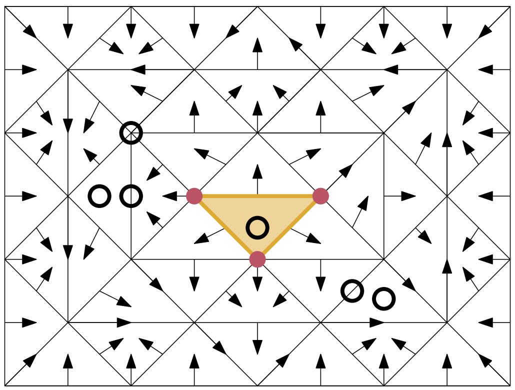
Gives a relative zigzag filtration:

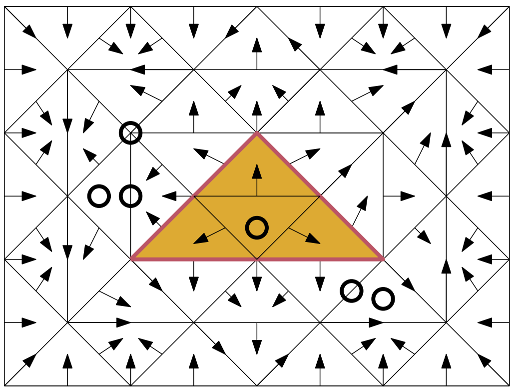
$$\ldots \subseteq (\mathsf{cl}(A_i), \mathsf{mo}(A_i)) \supseteq (\mathsf{cl}(A_i) \cap \mathsf{cl}(A_{i+1}), \mathsf{mo}(A_i) \cap \mathsf{mo}(A_{i+1})) \subseteq (\mathsf{cl}(A_{i+1}), \mathsf{mo}(A_{i+1})) \supseteq \ldots$$

Problem: $(\mathsf{cl}(A_i) \cap \mathsf{cl}(A_{i+1}), \mathsf{mo}(A_i) \cap \mathsf{mo}(A_{i+1}))$ generally not an index pair.

Intersection Example





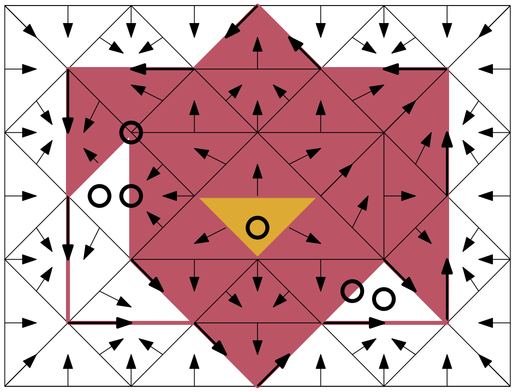


Index Pairs in an Isolating Neighborhood

Let $E \subset P \subseteq N$ for closed P, E, N, and $A \subseteq N$. If:

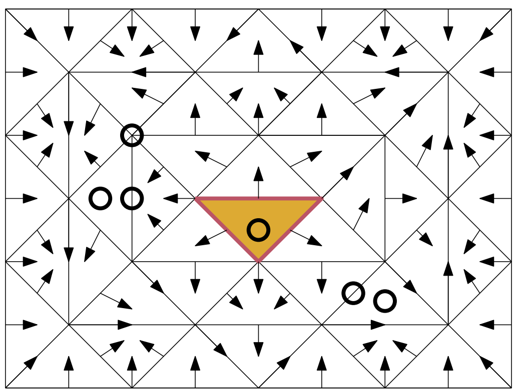
1. $F_{\mathcal{V}}(P) \cap N \subseteq P$,
2. $F_{\mathcal{V}}(E) \cap N \subseteq E$,
3. $F_{\mathcal{V}}(P \setminus E) \subseteq N$, and
4. $A = \mathrm{Inv}(P \setminus E)$

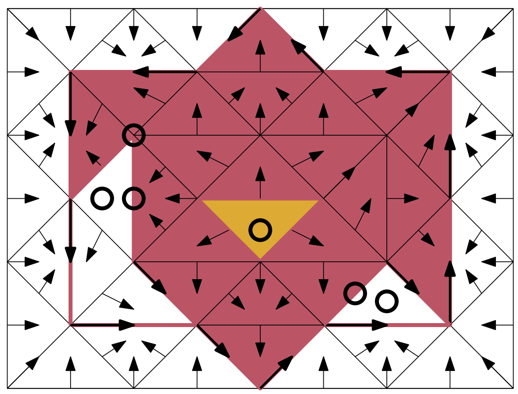
then (P, E) is an index pair in N.



Push Forward

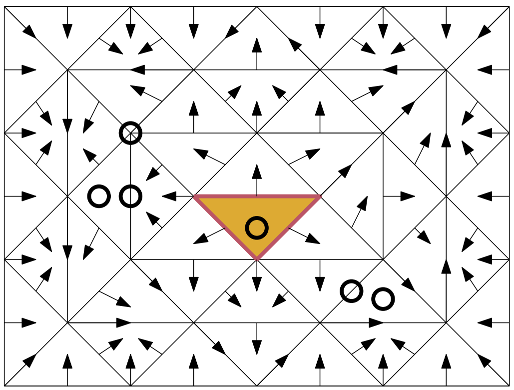
Let $A \subseteq K$ denote an arbitrary set in some closed N. Then the push forward of A in N is A together with all simplices in N which are reachable from paths originating in A and contained in N.

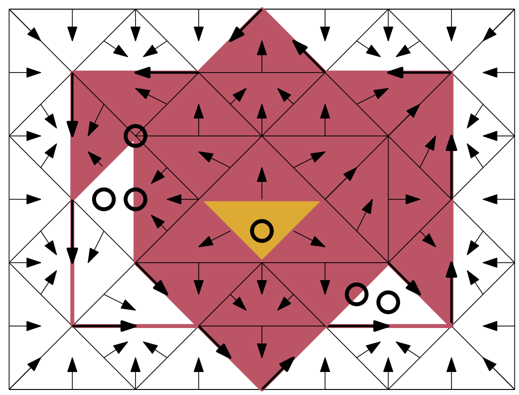




Finding Index Pairs in N

Theorem [DMS20]: Let A denote an isolated invariant set, and let N denote an isolating neighborhood for A. The pair $(\mathsf{pf}(\mathsf{cl}(A)), \mathsf{pf}(\mathsf{mo}(A)))$ is an index pair in N for A.





Index Pairs in an Isolating Neighborhood

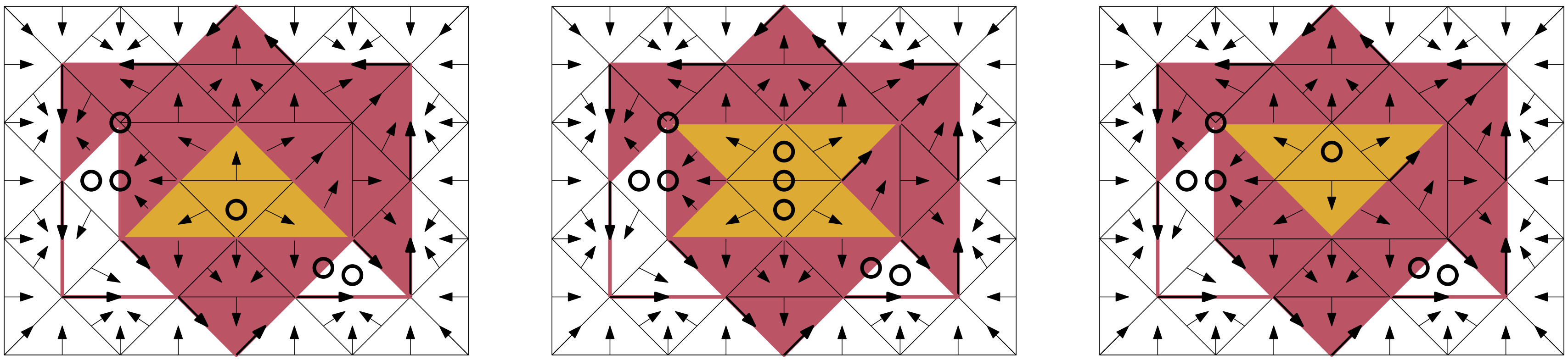
Theorem (DMS20): Index Pairs in N are index pairs.

Definition: Let $\mathcal{V}_1$, $\mathcal{V}_2$ denote multivector fields over K. The intersection of multivector fields is given by

$$\mathcal{V}_1 \overline{\cap} \mathcal{V}_2 = \{V_1 \cap V_2 \mid V_1 \in \mathcal{V}_1,\ V_2 \in \mathcal{V}_2\}$$

Theorem (DMS20): Let (P_1, E_1) , (P_2, E_2) denote index pairs in N under $\mathcal{V}_1, \mathcal{V}_2$. The pair $(P_1 \cap P_2, E_1 \cap E_2)$ is an index pair in N under $\mathcal{V}_1 \overline{\cap} \mathcal{V}_2$ for $\mathrm{Inv}((P_1 \cap P_2) \setminus (E_1 \cap E_2))$

Intersection Example



Dimension: 2

All simplices in N, Yellow union
Red is P, and Red is E

Conley Index Persistence: New Strategy

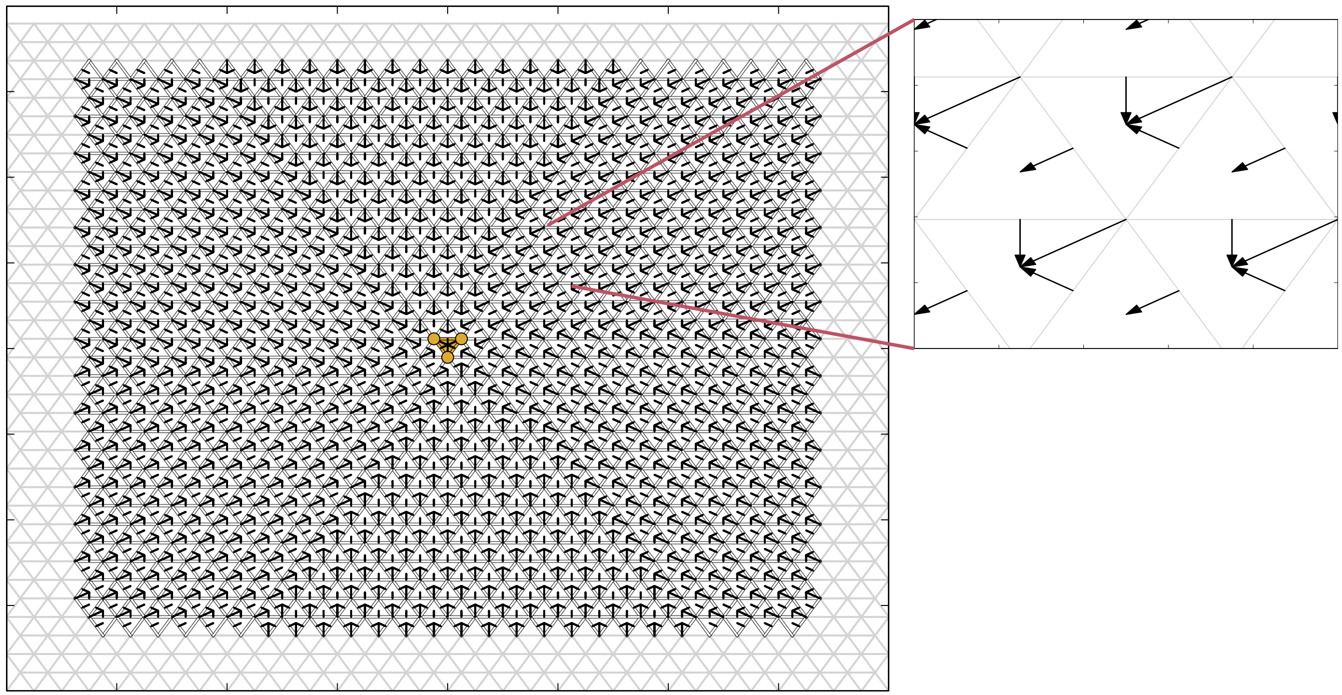
Fix N, and for each $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$, compute the maximal invariant set in N, denoted $A_1, A_2, \ldots, A_n$, and corresponding index pairs.

$$(\mathsf{cl}(A_1), \mathsf{mo}(A_1)), (\mathsf{cl}(A_2), \mathsf{mo}(A_2)), \ldots, (\mathsf{cl}(A_n), \mathsf{mo}(A_n))$$

Gives a relative zigzag filtration:

$$(\mathsf{pf}_N(\mathsf{cl}A_i), \mathsf{pf}_N(\mathsf{mo}A_i)) \supseteq (\mathsf{pf}_N(\mathsf{cl}A_i) \cap \mathsf{pf}_N(\mathsf{cl}A_{i+1}), \mathsf{pf}_N(\mathsf{mo}A_i) \cap \mathsf{pf}_N(\mathsf{mo}A_{i+1})) \subseteq (\mathsf{pf}_N(\mathsf{cl}A_{i+1}), \mathsf{pf}_N(\mathsf{mo}A_{i+1}))$$

Motivating Example: Hopf Bifurcation



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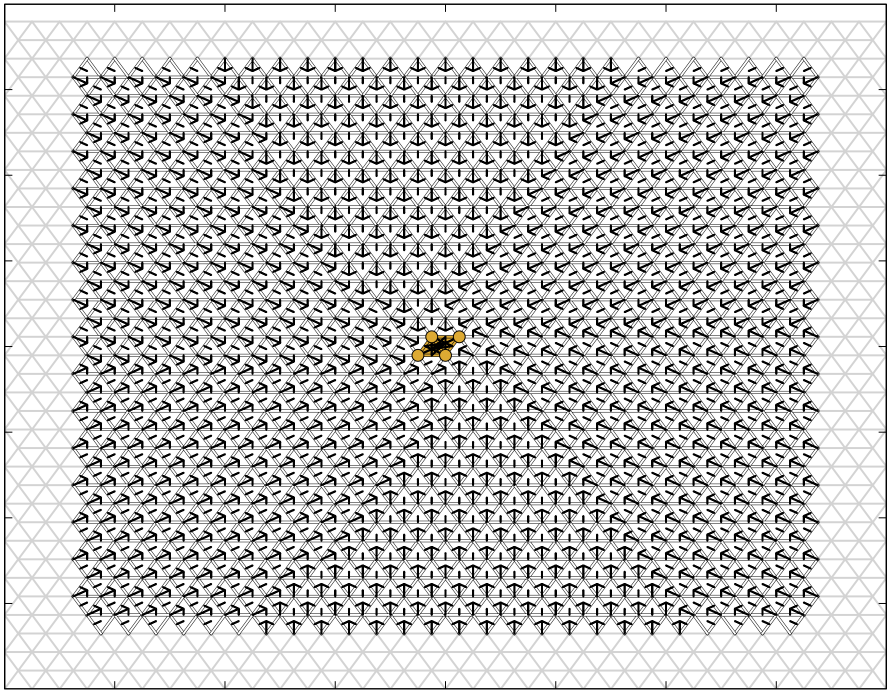

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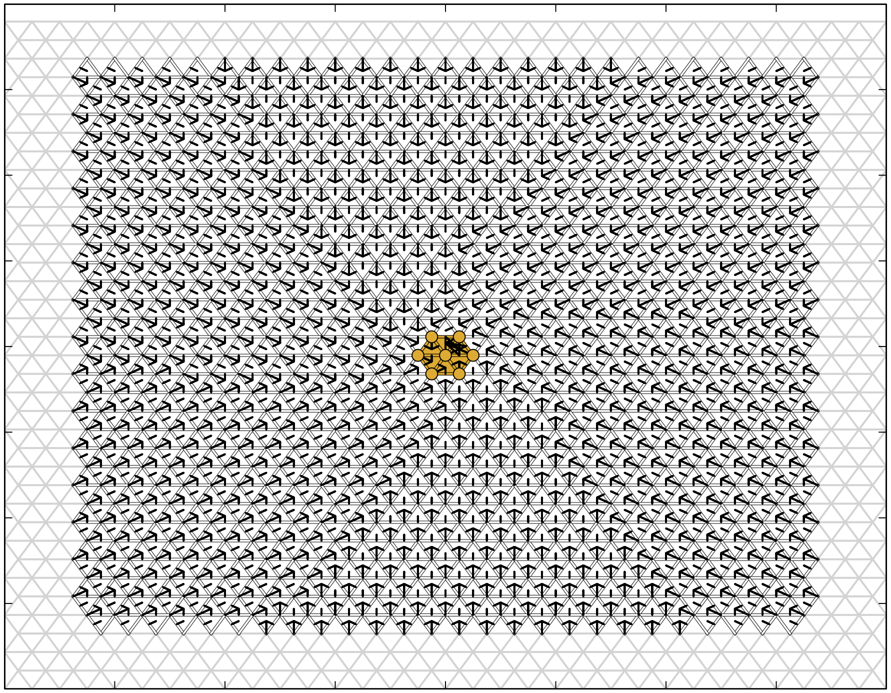

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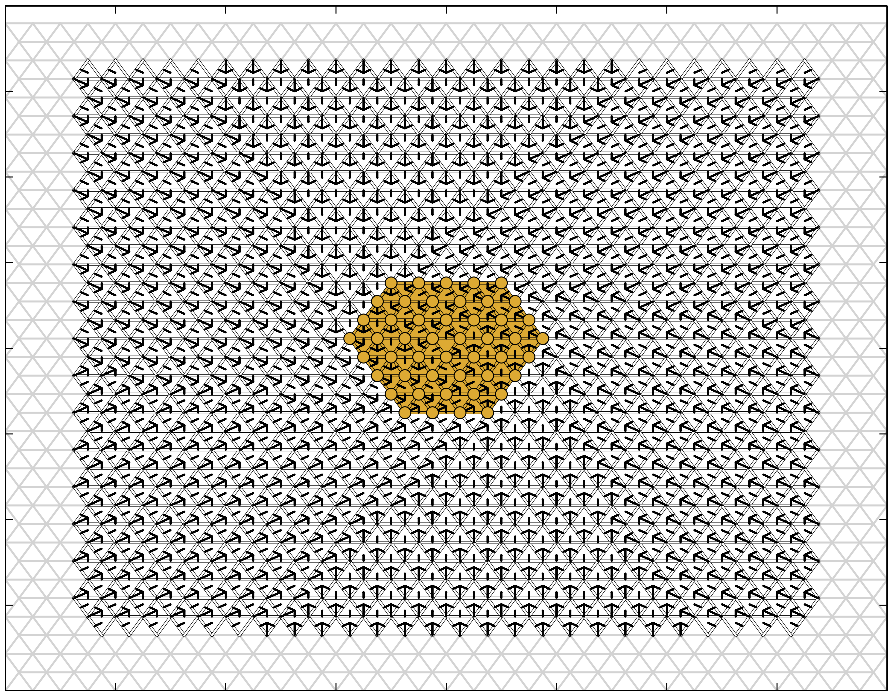

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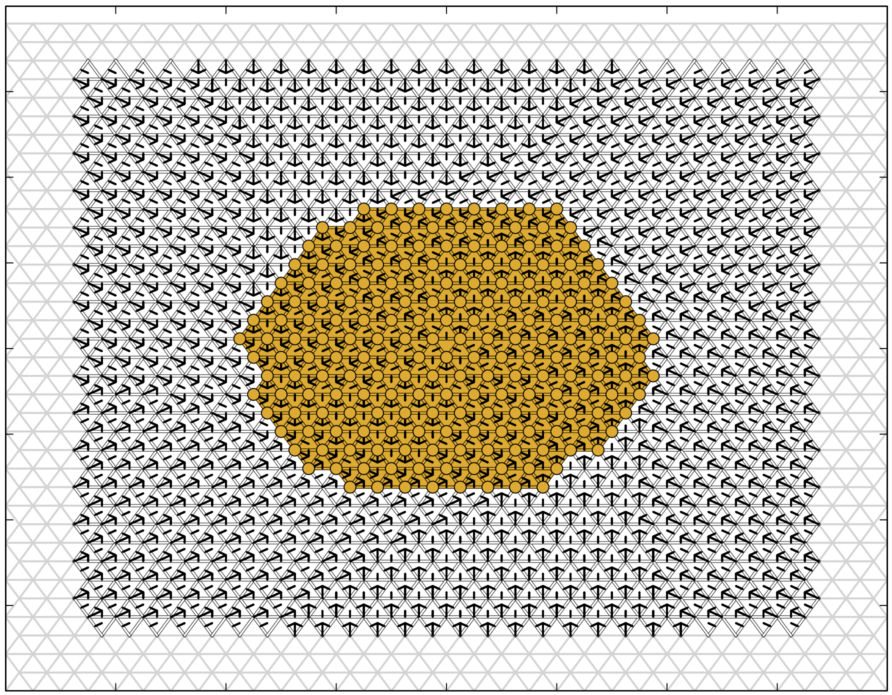

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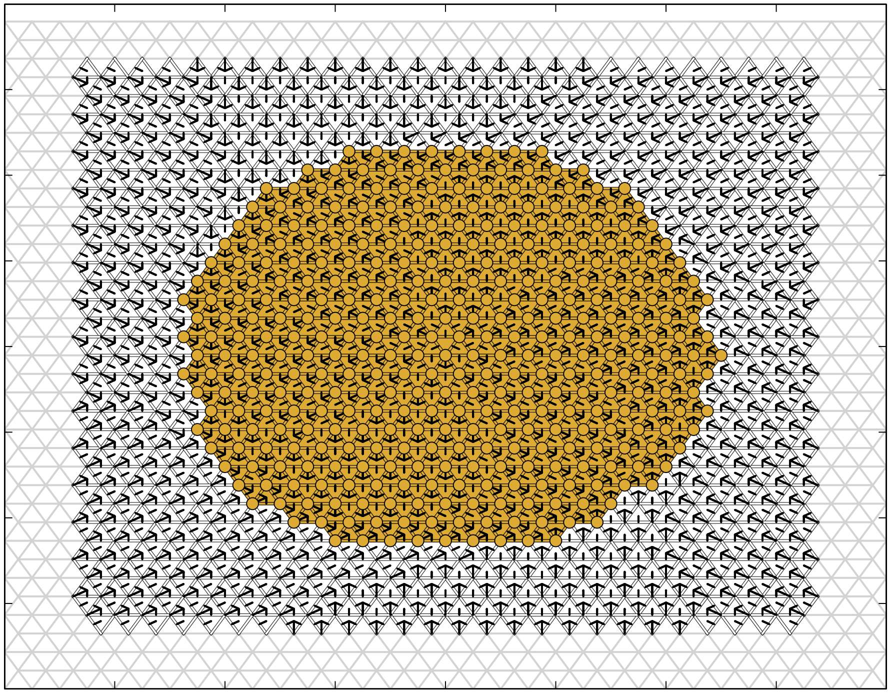
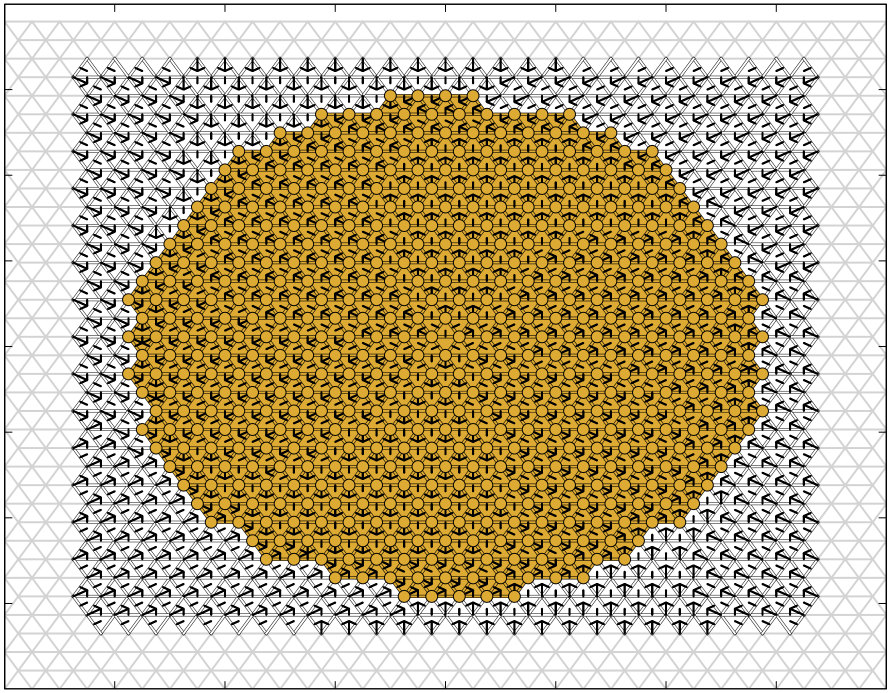
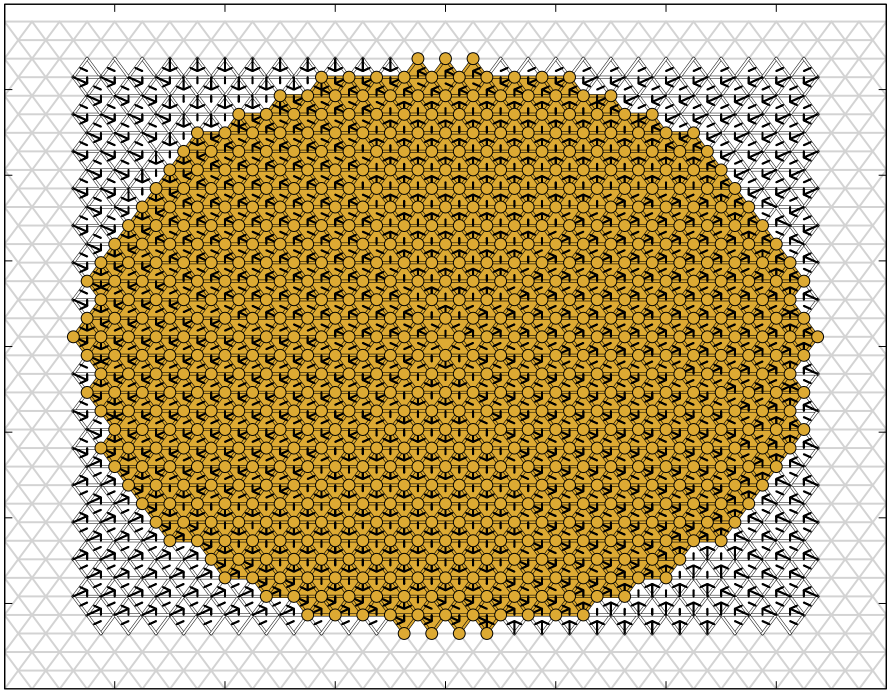


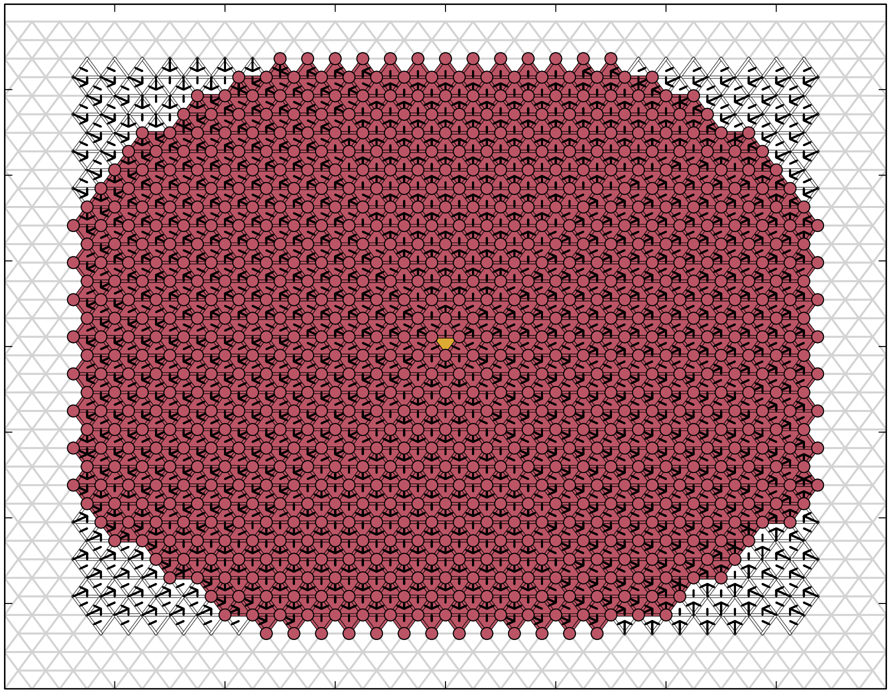
Motivating Example: Hopf Bifurcation



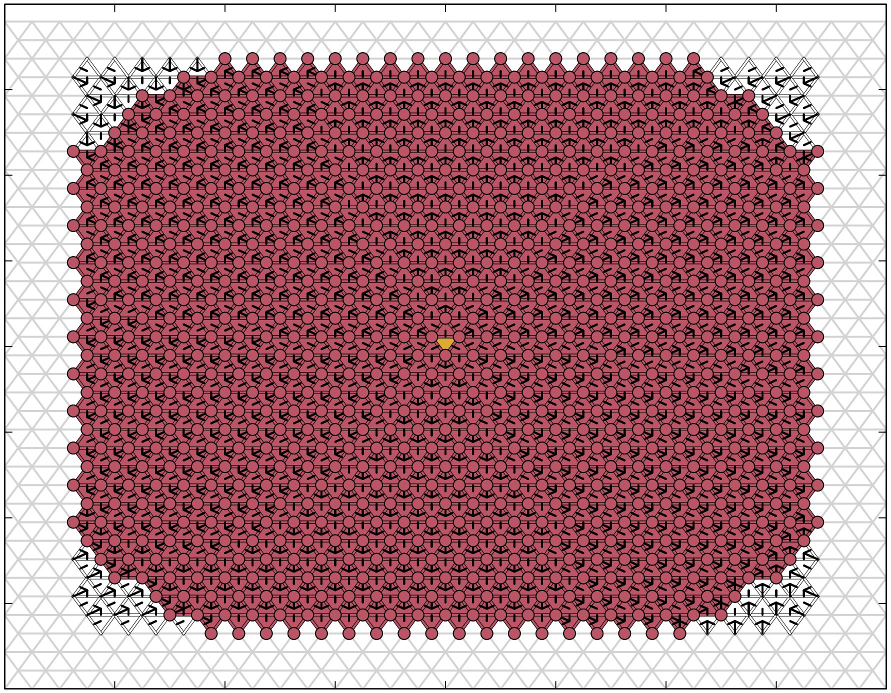
Motivating Example: Hopf Bifurcation



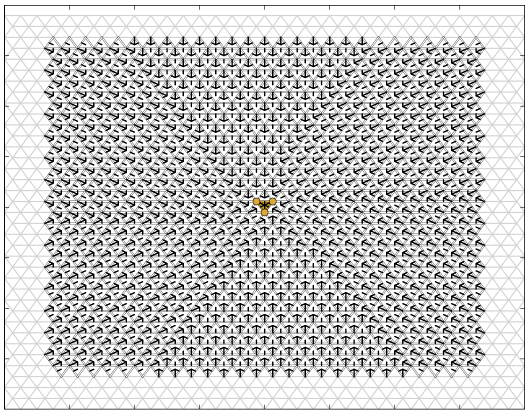
Motivating Example: Hopf Bifurcation

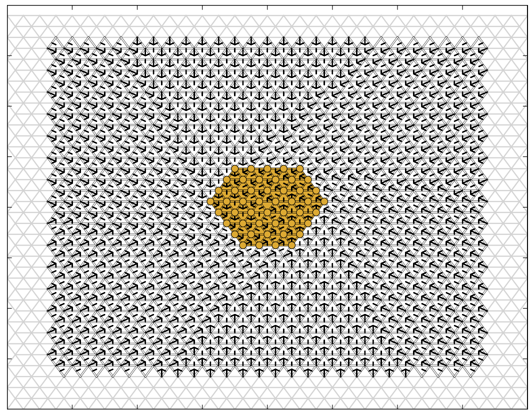


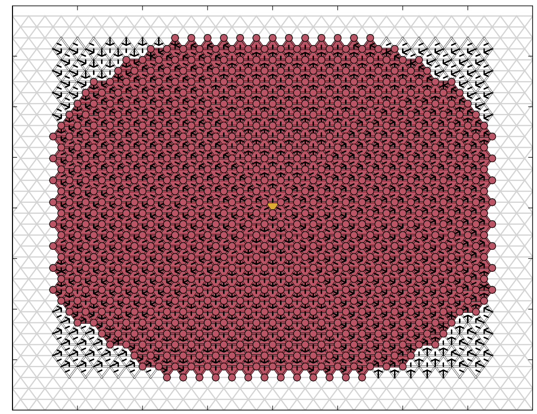
Motivating Example: Hopf Bifurcation



Conley Index Persistence







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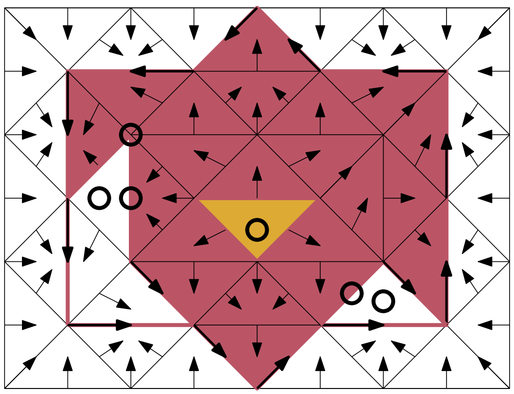
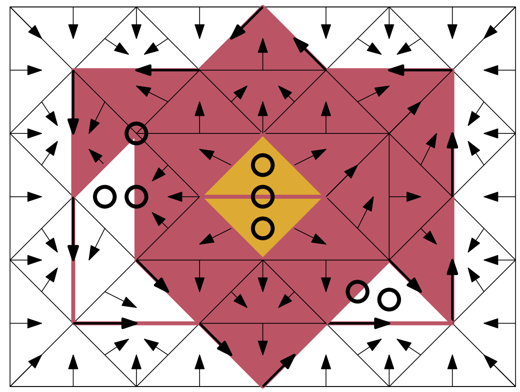
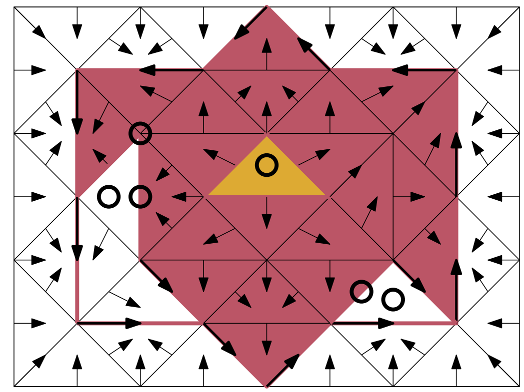
Dimension: 2

Problem: Noise Resilience





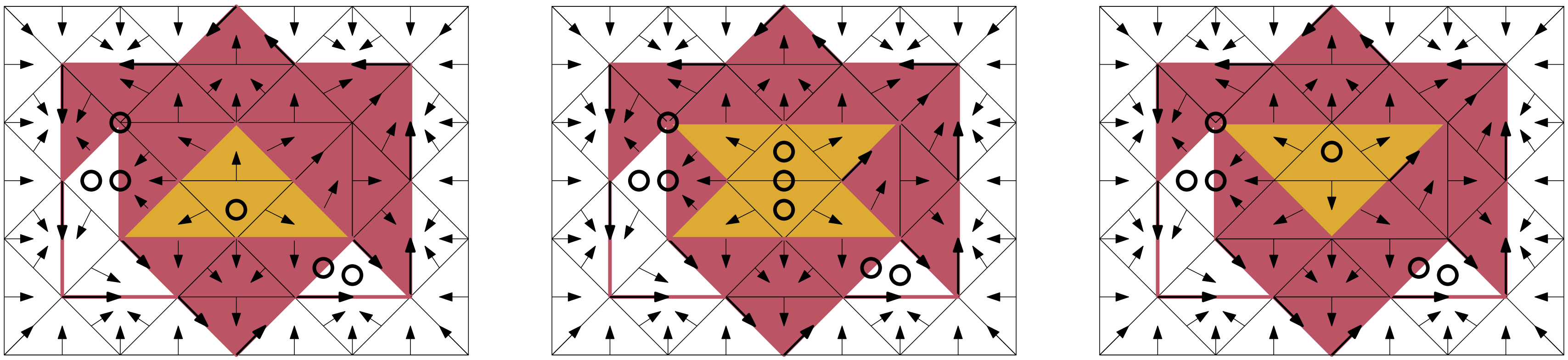


Dimension: 2

Dimension: 2

All simplices are in N,
Yellow union Red = P,
and Red = E

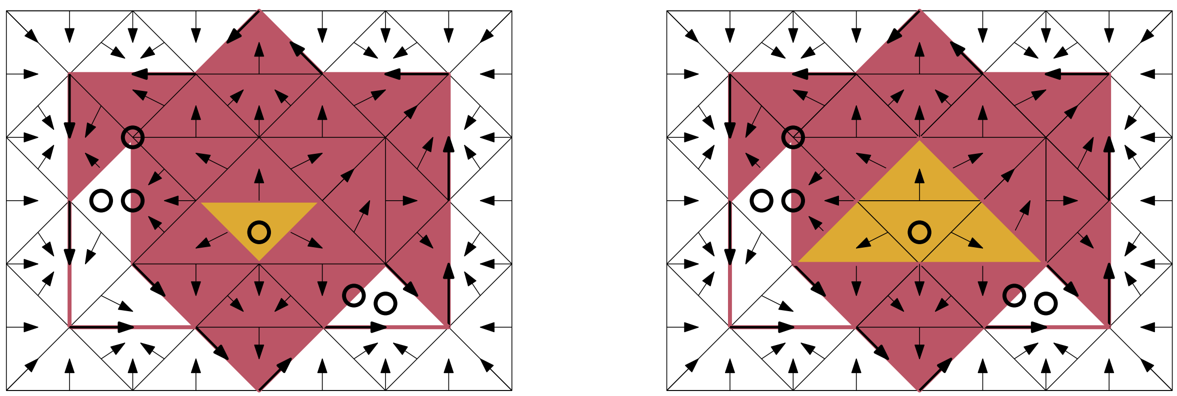
Solution: Make E Smaller



Dimension: 2

Conley Index Persistence

<u>Proposition [DMS20]:</u> Let (P, E) denote an index pair for A in N. If $V \subseteq E$ is a regular multivector such that $E' := E \setminus V$ is closed, then $\overline{(P, E')}$ is an index pair in N for A .



Conley Index Persistence

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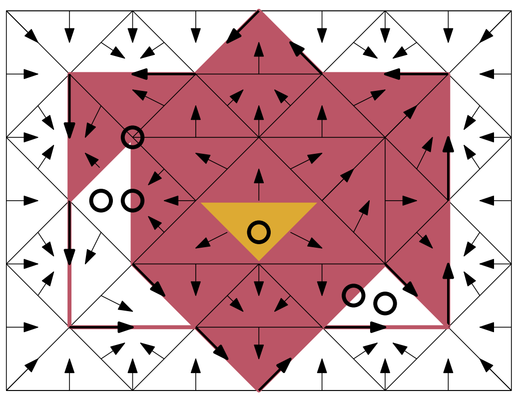
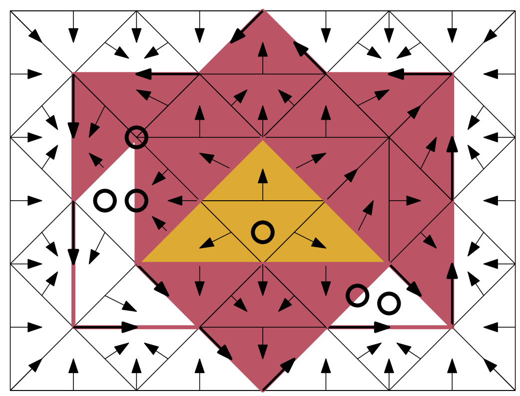
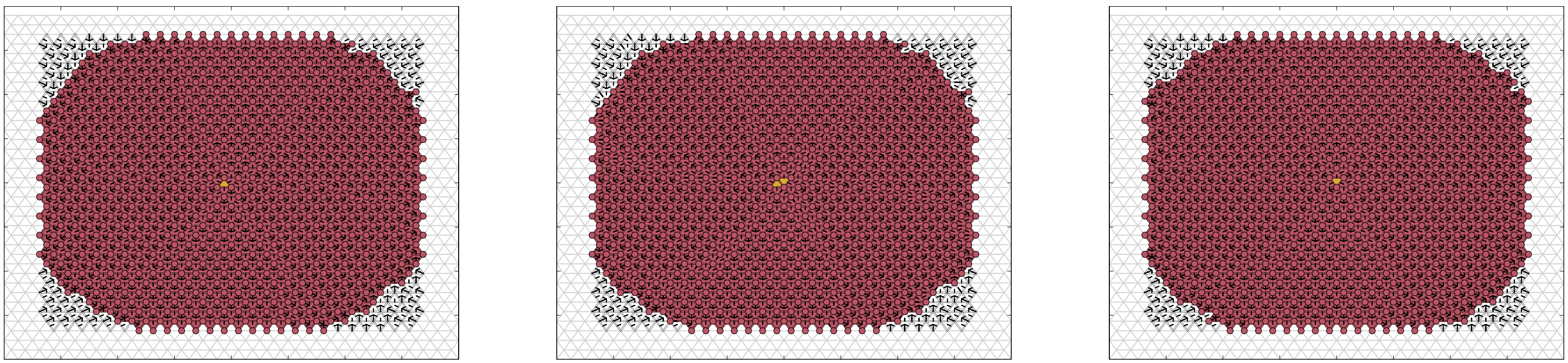




Multivector Removal Strategy

Remove as many multivectors as possible, up to a fixed distance away from the isolated invariant set.



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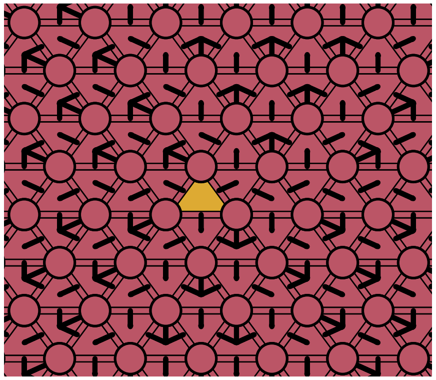

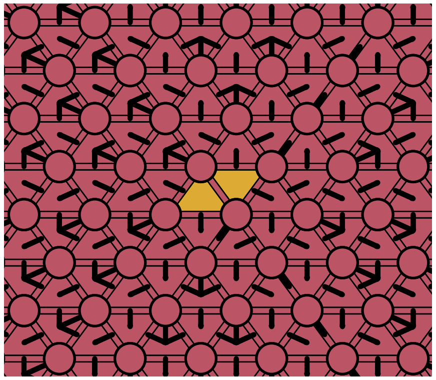

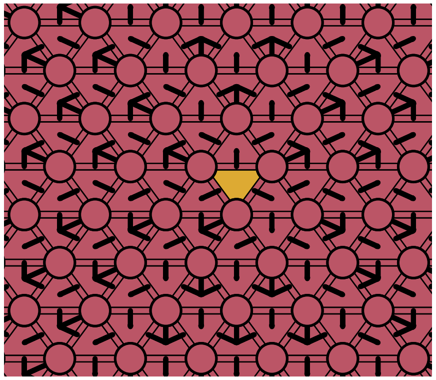

Multivector Removal Strategy

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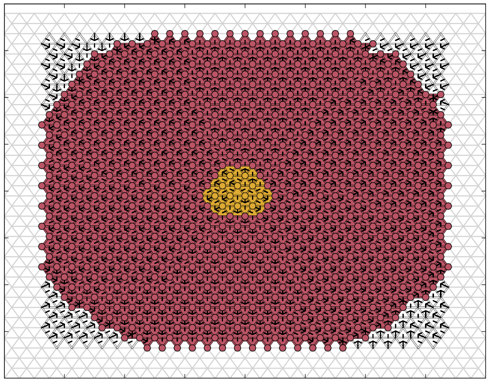
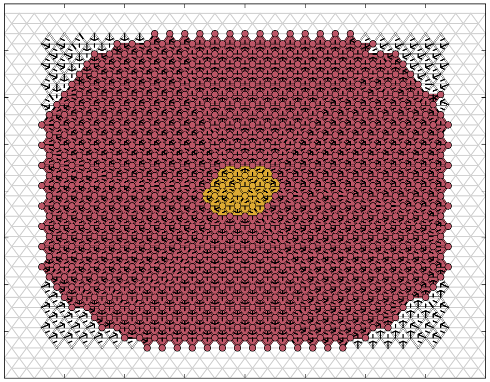
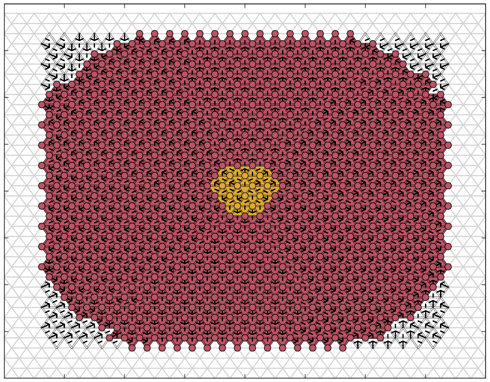
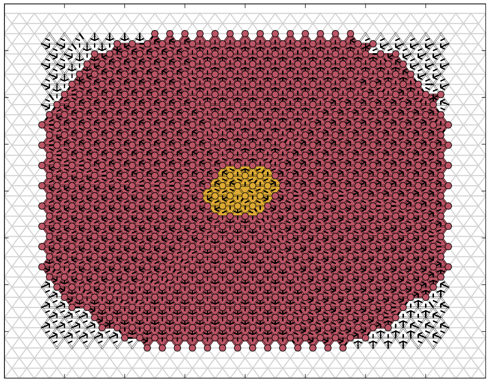




Algorithm

MakeNoiseResilient(P, E, A, δ):

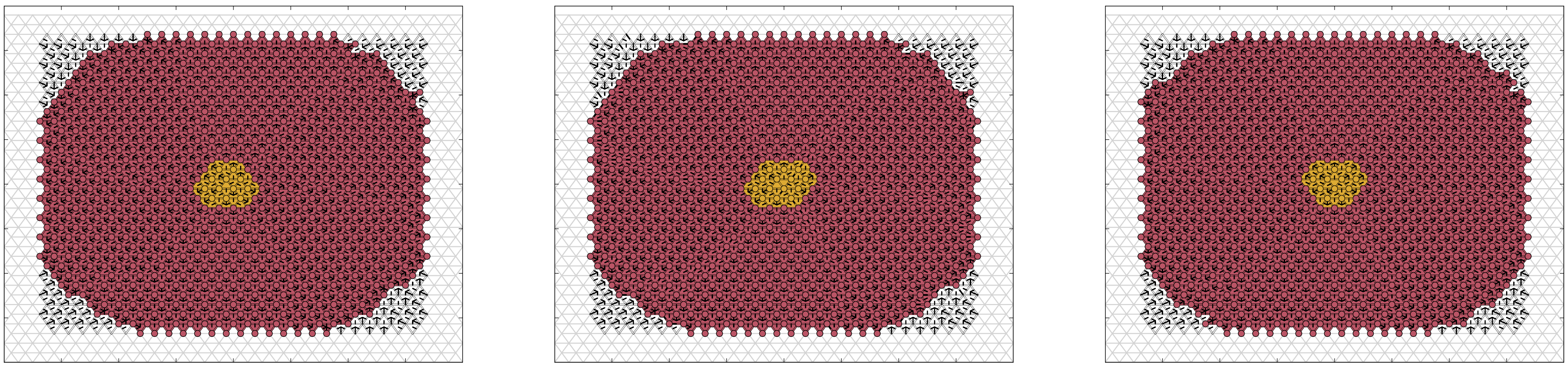
while there exists a regular multivector $V \subset E$ such that $E \setminus V$ is closed and $d(V, A) \leq \delta$:

$$E \leftarrow E \setminus V$$



Multivector Removal Strategy

Theorem [DMS'20]: This algorithm outputs index pairs



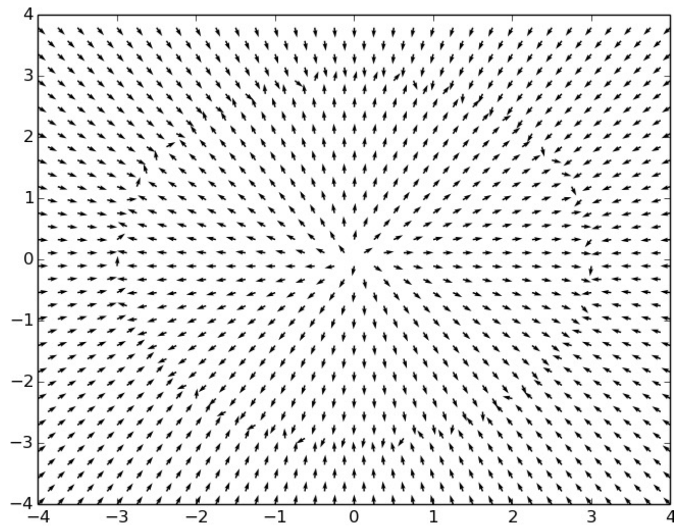
Dimension: 2

Overview & Outline

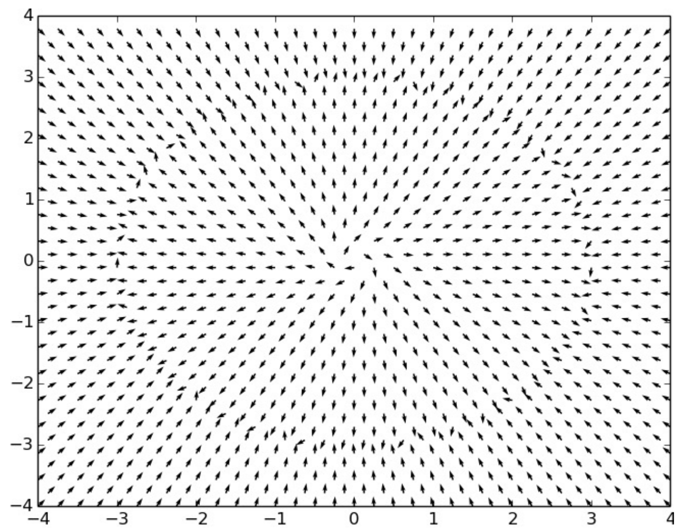
- Motivating Example and Persistence
- Combinatorial Dynamical Systems & Conley Index
- Conley Index Persistence
- Conley-Morse Graph Persistence

[DMS22] T. K. Dey, M. Mrozek, R. Slechta. "Persistence of Conley-Morse Graphs in Combinatorial Dynamical Systems." SIADS 2022.

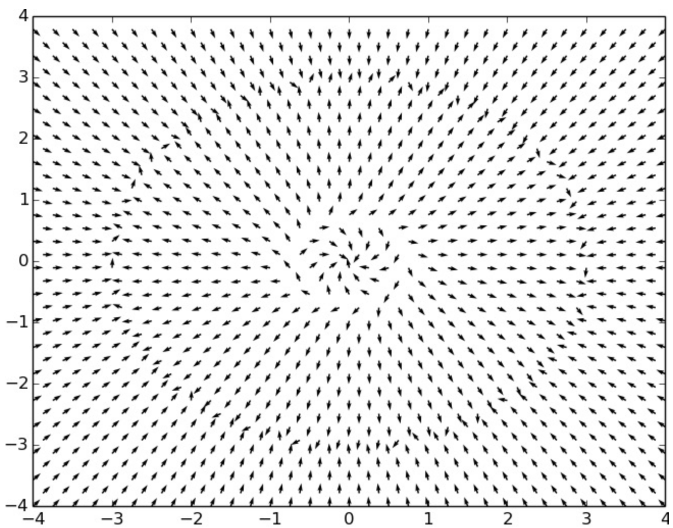
Motivating Example



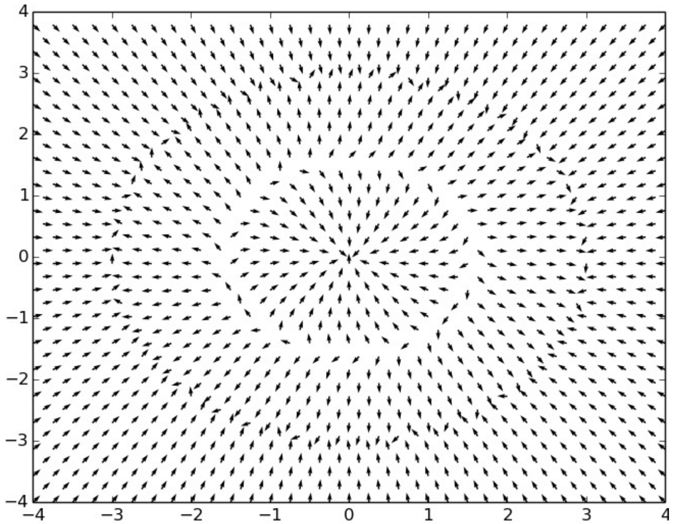
Motivating Example



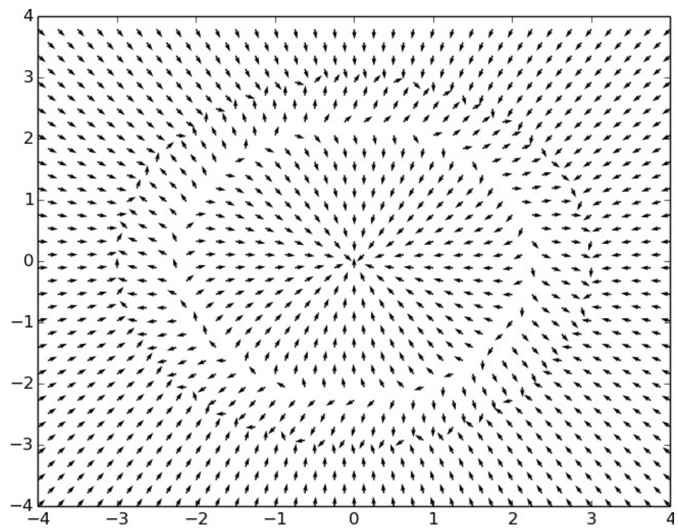
Motivating Example



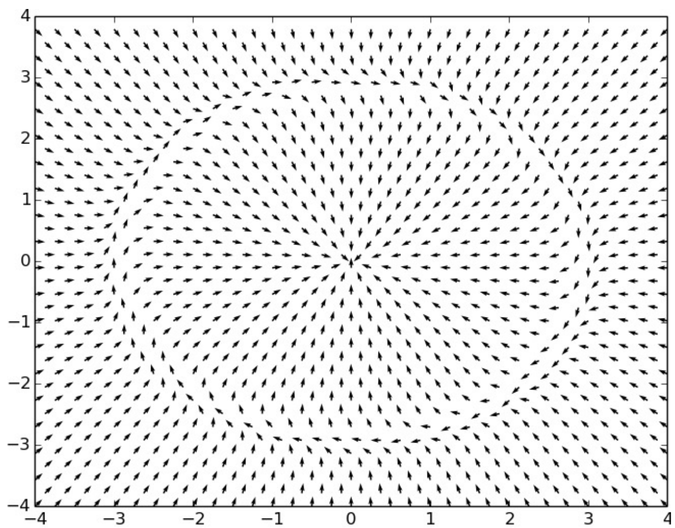
Motivating Example



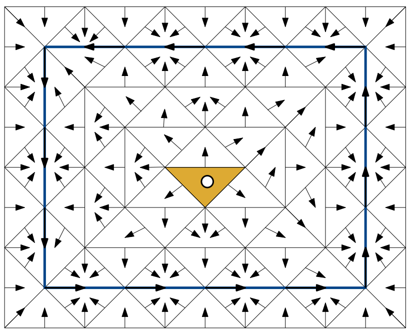
Motivating Example

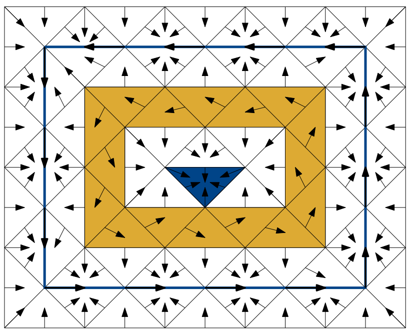


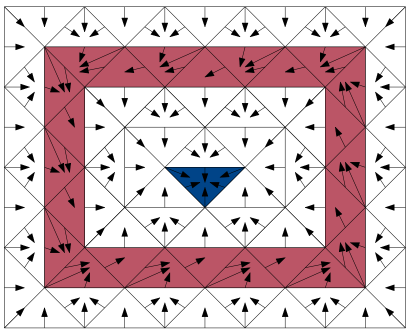
Motivating Example



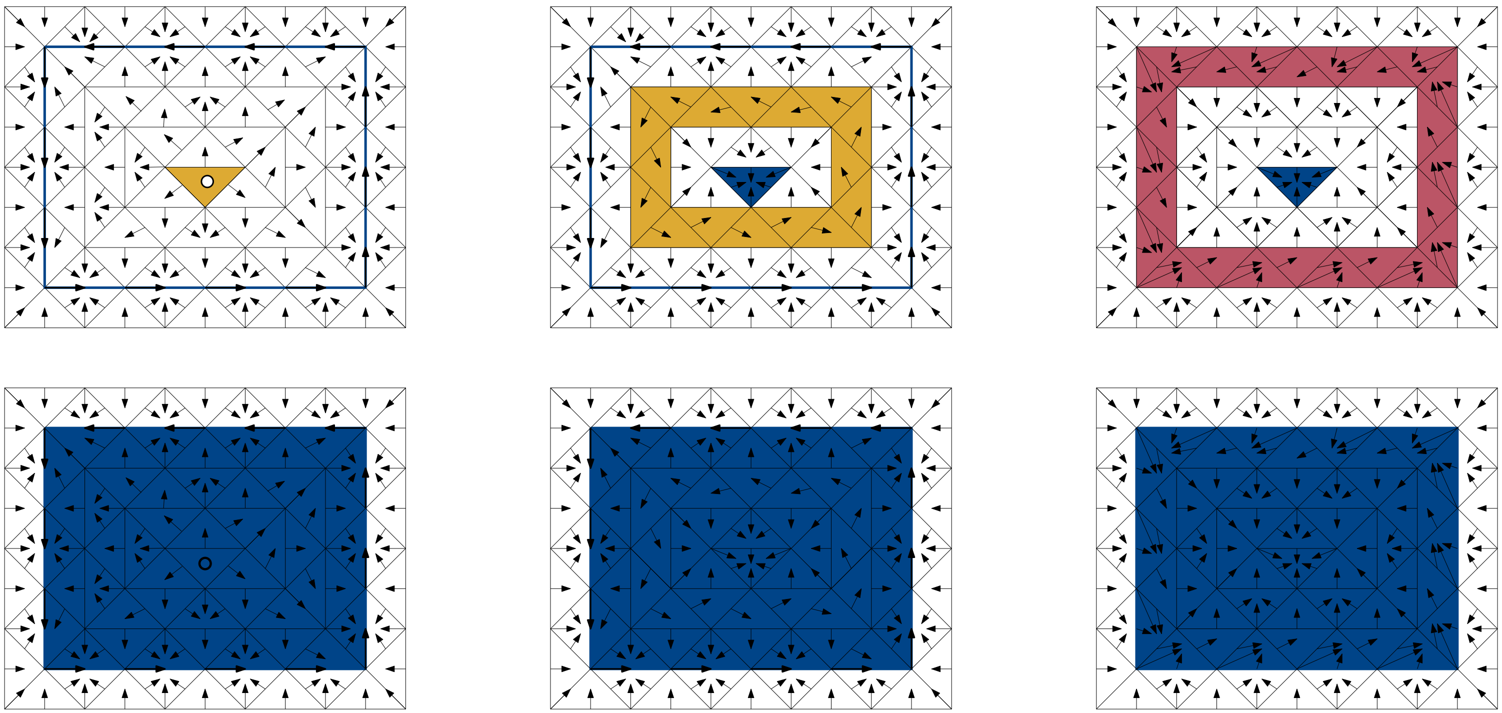
Motivating Example





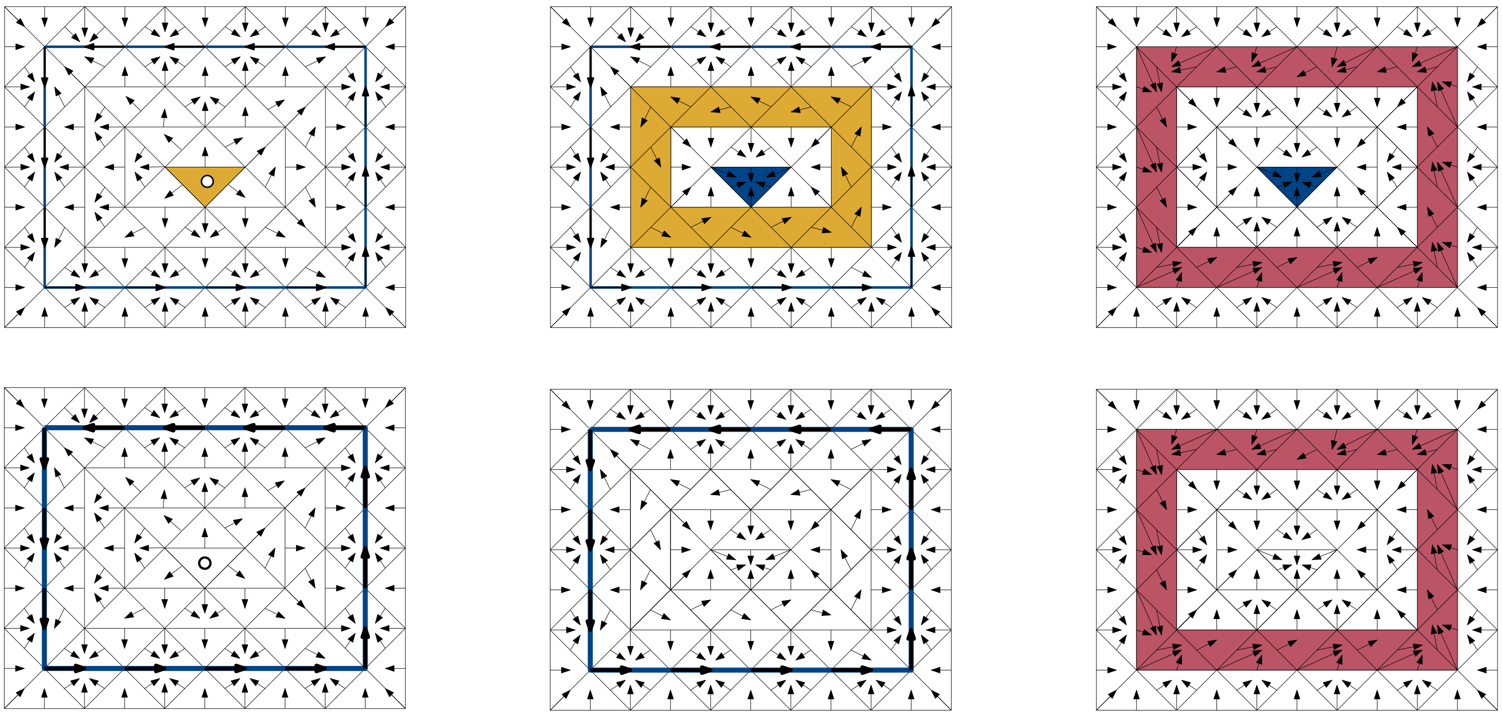


Motivating Example



Dimension: 0

Original Example

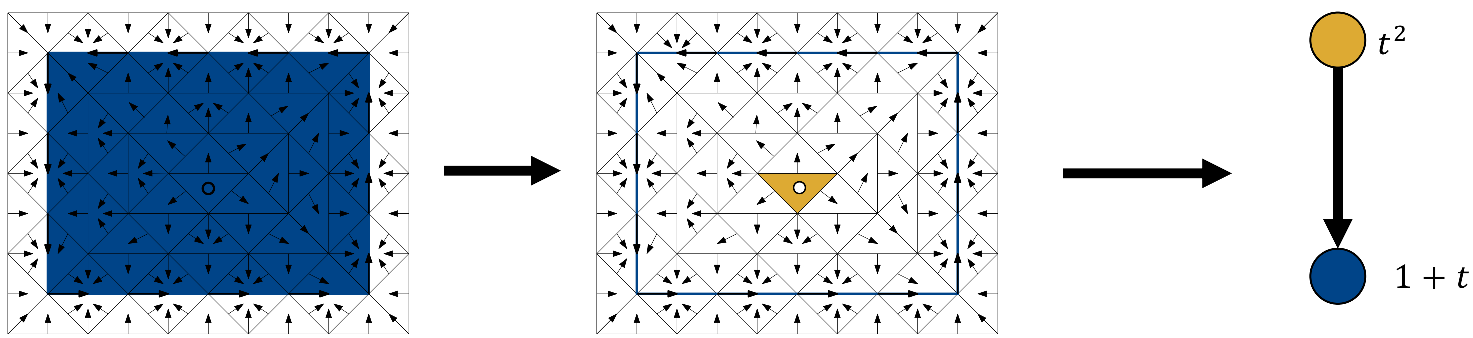


Dimension: 0

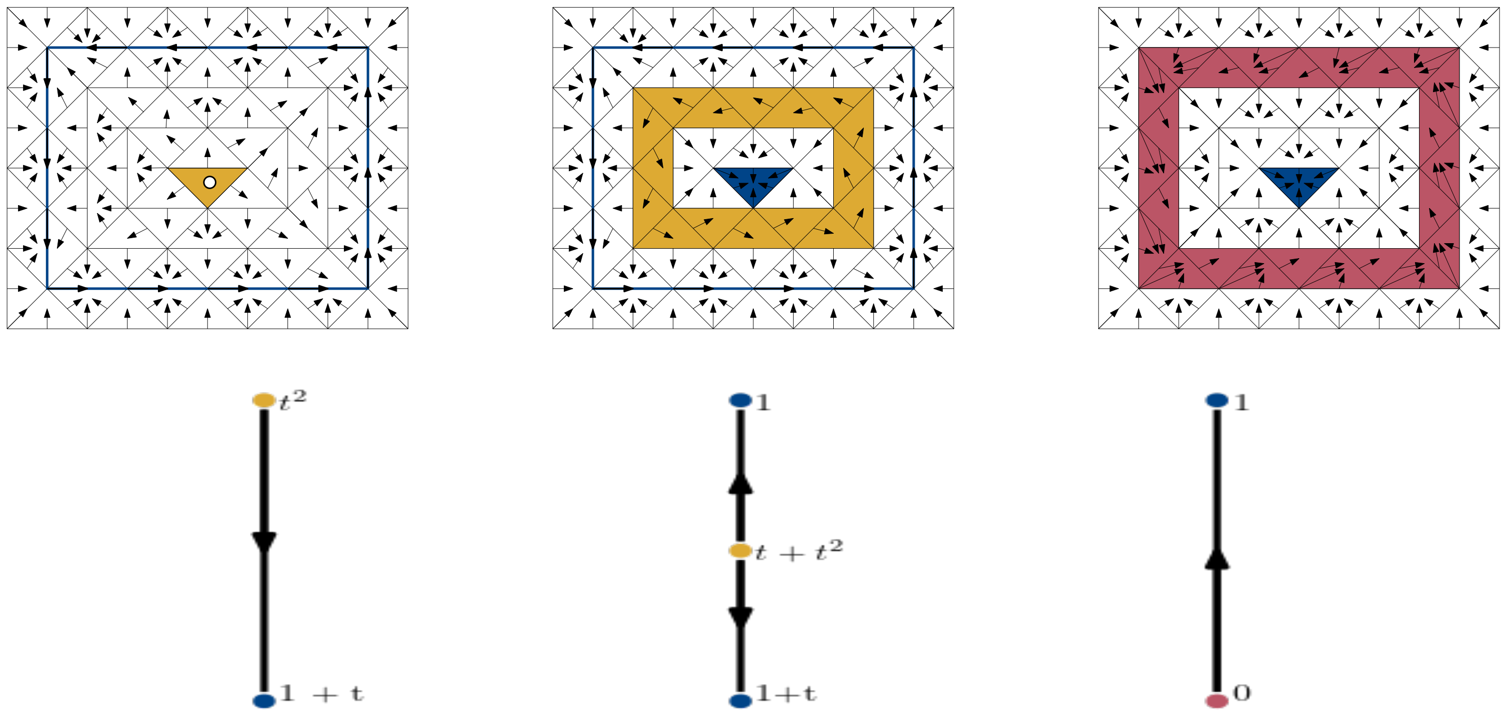
Dimension: 1

Conley-Morse Graph

A Morse decomposition graph equipped with information about the Conley Index



Conley-Morse Graph



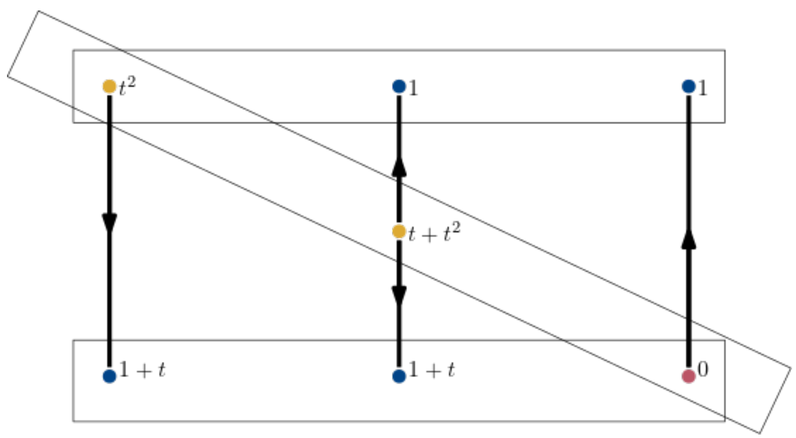
Conley-Morse Graph Persistence

Two types of filtrations:

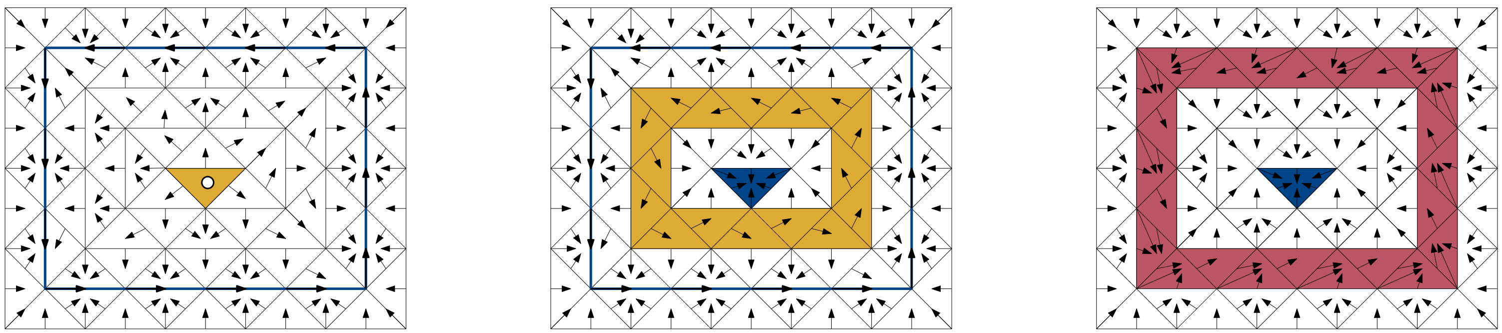
1. Graph Filtrations
2. Conley-Morse Filtrations

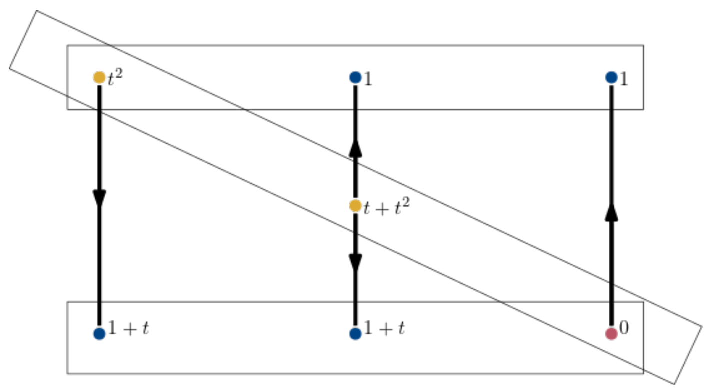
Conley-Morse Filtrations

1. Assume every isolated invariant set is isolated by the same N.
2. Fix index pair for each Morse set.
3. Find all “maximal” sequences of index pairs across Conley-Morse graphs with nontrivial intersection.

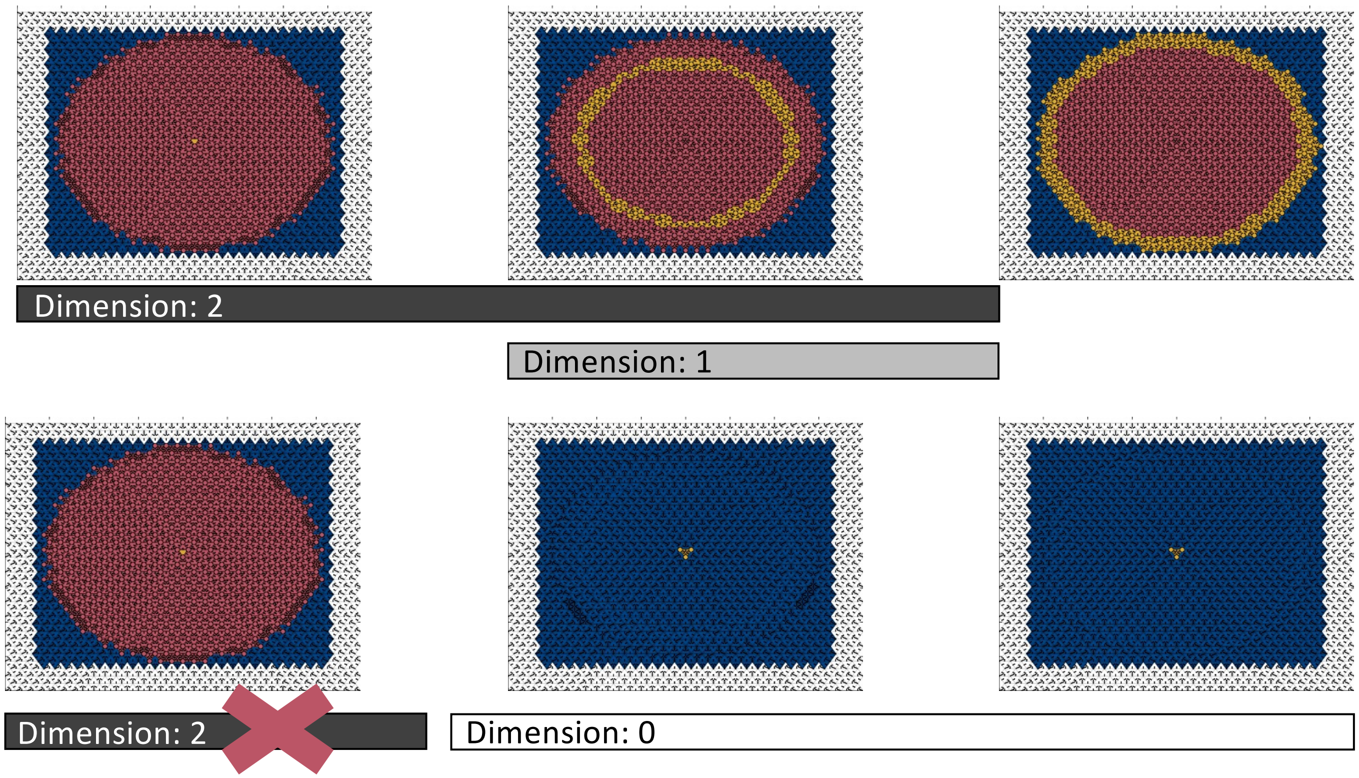


Conley-Morse Filtrations

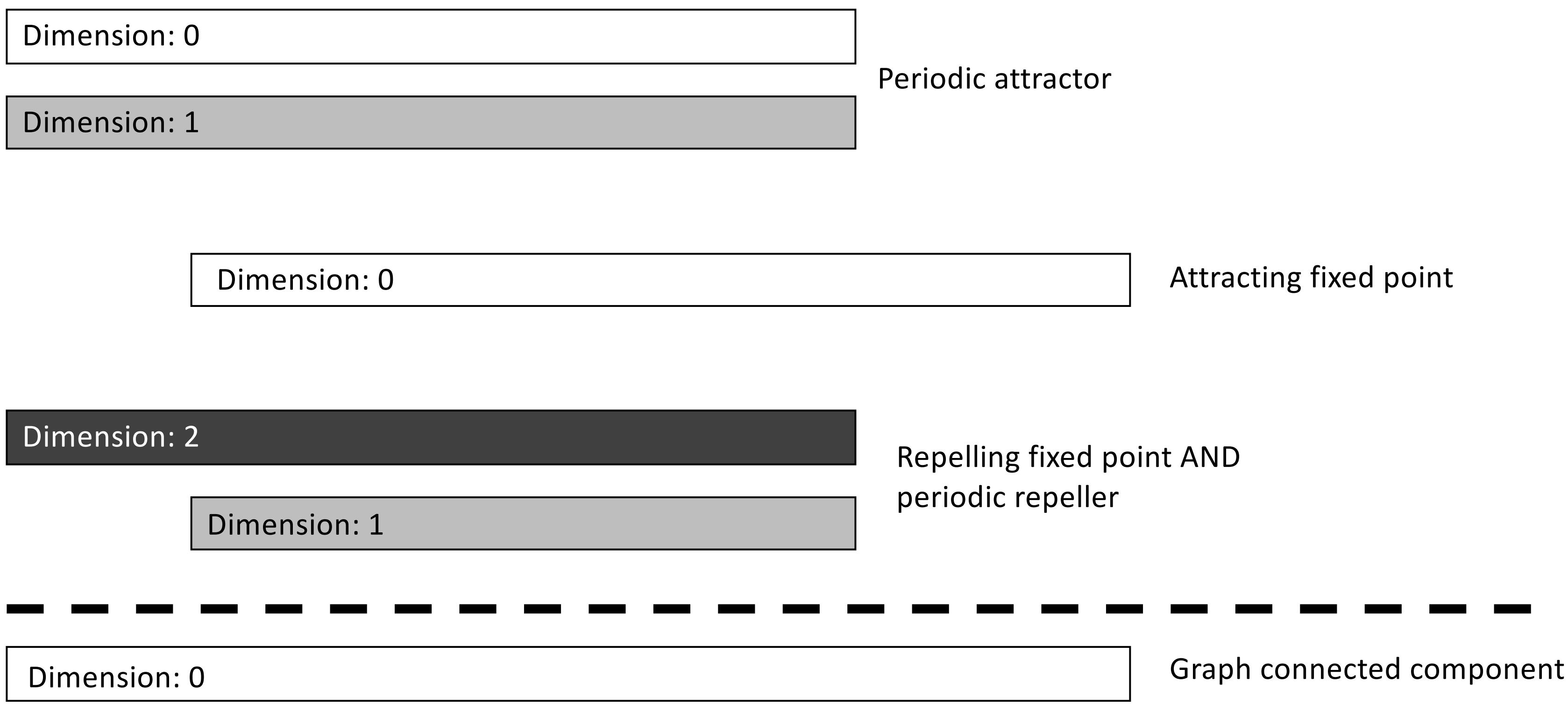




Conley-Morse Barcodes



Conley-Morse Graph Barcodes



Conclusion & Future Work

- In this presentation: devised method to capture changes in combinatorial dynamical systems. But…

- Stability?

- Inference?

References

[CDM09] G. Carlsson, V. de Silva, D. Morozov. “Zigzag Persistent Homology and Real Valued Functions.” SoCG ‘09

[DW07] T. Dey, R. Wenger. “Stability of Critical Points with Interval Persistence.” Discret. Comput. Geom. Volume 33, Issue 3.

[ELZ00] H. Edelsbrunner, D. Letscher, A. Zomordian “Toplogical Persistence and Simplification.” FOCS ‘00.

[LKMW19] M. Lipinski, J. Kubica, M. Mrozek, T. Wanner. “Conley-Morse-Forman theory for generalized combinatorial multivector fields on finite topological spaces.” Preprint.

[Mr17] M. Mrozek. “Conley-Morse-Forman Theory for Combinatorial Multivector Fields.” FOCM Volume 17, Issue 6.