Directed Homotopy Type Theory

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Outline

1. Homotopy theory via type theory
2. Desiderata for directed homotopy type theory
3. Directed homotopy type theory

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MORAL: Not only can types represent spaces, but homotopy type theory is the right setting in which to do homotopy theory.

This is because everything we can say or do respects equality / identity / homotopy (terms of $\mathrm{Id}_A(a,b)$).

Types $\to$ ∞-groupoids

$$\boxed{\dfrac{A : \mathrm{Type} \qquad a, b : A}{\mathrm{Id}_A(a,b) : \mathrm{Type}}} \qquad (\mathrm{Id}\text{-form})$$

$$\boxed{\dfrac{A : \mathrm{Type} \qquad a : A}{r_a : \mathrm{Id}_A(a,a)}} \qquad (\mathrm{Id}\text{-intro})$$

$\to$ We get a tower

$$\mathrm{Id}_A(a,b), \quad \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p,q), \quad \mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A(a,b)}(p,q)}(\alpha,\beta) \ \ldots$$

with canonical terms

$$r_a : \mathrm{Id}_A(a,a), \quad r_p : \mathrm{Id}_{\mathrm{Id}_A(a,b)}(p,p), \quad r_\alpha : \mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A(a,b)}(p,q)}(\alpha,\alpha) \ \ldots$$

Pitstop in $\mathbb{B}$

$$\frac{}{\mathbb{B} : \mathrm{Type}} \quad (\mathbb{B}\text{-form})$$

$$\frac{}{0 : \mathbb{B}} \qquad \frac{}{1 : \mathbb{B}} \quad (\mathbb{B}\text{-intro})$$

→ Get one type $\mathbb{B}$ with canonical terms $0, 1$

$$\frac{\begin{array}{l} x : \mathbb{B} \vdash D(x) : \mathrm{Type} \\ \vdash z : D(0) \\ \vdash u : D(1) \end{array}}{\begin{array}{l} x : \mathbb{B} \vdash \mathrm{ind}_{z,u}(x) : D(x) \\ \vdash z \equiv \mathrm{ind}_{z,u}(0) : D(0) \\ \vdash u \equiv \mathrm{ind}_{z,u}(1) : D(1) \end{array}}$$

($\mathbb{B}$-elim & $\mathbb{B}$-comp)

→ Behavior determined @ canonical terms

Pitstop in $\mathbb{N}$

$$\frac{}{\mathbb{N} : \mathrm{Type}} \quad (\mathbb{N}\text{-form})$$

$$\frac{}{0 : \mathbb{N}} \qquad \frac{n : \mathbb{N}}{sn : \mathbb{N}} \quad (\mathbb{N}\text{-intro})$$

→ Get one type $\mathbb{N}$ with canonical terms $0, s0, ss0, \ldots$

$$\frac{\begin{array}{l} x : \mathbb{N} \vdash D(x) : \mathrm{Type} \\ \vdash z : D(0) \\ x : \mathbb{N}, y : D(x) \vdash \sigma(y) : D(sx) \end{array}}{\begin{array}{l} x : \mathbb{N} \vdash \mathrm{ind}_{z,\sigma}(x) : D(x) \\ \vdash z \equiv \mathrm{ind}_{z,\sigma}(0) : D(0) \\ x : \mathbb{N} \vdash \sigma(\mathrm{ind}_{z,\sigma}(x)) \equiv \mathrm{ind}_{z,\sigma}(sx) : D(sx) \end{array}}$$

($\mathbb{N}$-elim & $\mathbb{N}$-comp)

→ Behavior determined @ canonical terms

Types $\to$ ∞-groupoids

$$\frac{x,y : A,\ z : \mathrm{Id}_A(x,y) \vdash D(z) : \mathrm{Type} \qquad x : A \vdash p(x) : D(r_x)}{x,y : A,\ z : \mathrm{Id}_A(x,y) \vdash \mathrm{ind}_p(z) : D(z)} \quad (\mathrm{Id}\text{-elim})$$

$$x : A \vdash p(x) \equiv \mathrm{ind}_p(p(x)) : D(r_x) \quad (\mathrm{Id}\text{-comp})$$

Prop. Every identity has an inverse.

Pf. Every reflexivity identity has an inverse (itself).

Prop. There is a composition of any $p : \mathrm{Id}_A(x,y)$ and $q : \mathrm{Id}_A(y,z)$.

Pf. There is a composition of any $p : \mathrm{Id}_A(x,y)$ and r_y (namely p).

Cor. Every type has the structure of an ∞-groupoid.

Fibrations & transport

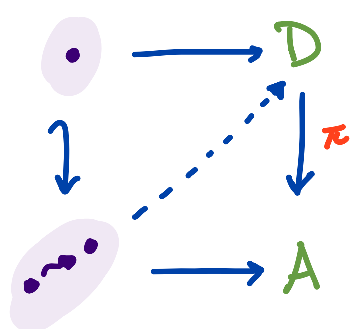
Any property or construction on terms of A.

Prop (transport). Given any $x:A \vdash D(x):\mathrm{Type}$ and $p:\mathrm{Id}_A(x,y)$, there is a function $p_*:D(x)\to D(y)$.

In fact, p_* is an equivalence and there is an identity $\mathrm{Id}_{\sum_{x:A} D(x)}(d, p_* d)$ for any $d:D(x)$.

Pf. If p is r_x, let p_* be the identity.

Cf.



$$\begin{tikzcd} D \arrow[r, equal] \arrow[d] & D \arrow[d, "\pi"] \\ D \times_A \mathrm{Path}(A) \arrow[r] \arrow[ur, dashed] & A \end{tikzcd}$$

Looks like a Hurewicz fibration in Top.

Fibrations & transport

(Let $\mathcal{C}$ be a finitely complete category.)

Thm (N) Identity types can be interpreted in any weak factorization system in $\mathcal{C}$ that

(1) is generated by a path object and

(2) is symmetric.

(1): There is a path object

$$X \xrightarrow{r} \mathrm{Path}(X) \xrightarrow{\varepsilon_0 \times \varepsilon_1} X \times X$$

functorial in $X \in \mathcal{C}$, and taking the mapping path factorization produces the wfs.

→ $\mathrm{Path}(X)$ plays the role of $\sum_{x,y:X} \mathrm{Id}_X(x,y)$ and r plays the role of reflexivity.

(2): There is an involution satisfying some properties...

$$\begin{tikzcd}
 & \mathrm{Path}(X) \arrow[dd, "2"] \arrow[dr, "\varepsilon_0 \times \varepsilon_1"] & \\
X \arrow[ur, "r"] \arrow[dr, "r"'] & & X \times X \\
 & \mathrm{Path}(X) \arrow[ur, "\varepsilon_1 \times \varepsilon_0"'] &
\end{tikzcd}$$

Fibrations & transport : Examples

- In any category, take Path(X) to be X or $X \times X$.
(Dependent types correspond to all morphisms or isomorphisms.)

- In $\mathbf{Cat}$, take Path(X) to be $X^{\bullet \cong \bullet}$.
(Dependent types correspond to isofibrations.)

- In $\mathbf{Top}$, take Path(X) to be TX (roughly X^I).
(Dependent types correspond to Hurewicz fibrations.)

- In Kan complexes, take Path(X) to be $X^{\Delta[1]}$
(Dependent types correspond to Kan fibrations.)

- which can be generalized to any Cisinski model category.
(Dependent types correspond to fibrations.)

Fibrations & transport : Examples

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- In Kan complexes, take Path(X) to be $X^{\Delta[1]}$
  (Dependent types correspond to Kan fibrations.)
- which can be generalized to any Cisinski model category.
  (Dependent types correspond to fibrations.)

Π-types : need LCCC

Type : need classifying fibration

+ others
(∞-toposes)

Univalence

- The univalence axiom characterizes identities in Type:

$$\mathrm{Id}_{\mathrm{Type}}(A,B) \simeq (A \simeq B)$$

- We can use it to characterize identities in other types:

Prop. $\mathrm{Id}_{A \to B}(f,g) \simeq \prod_{x:A} \mathrm{Id}_A(fx, gx)$

Univalence Principles

Prop. $\mathrm{Id}_{\mathrm{Set}}(S,T) \simeq (S \cong T)$

Prop. $\mathrm{Id}_{\mathrm{Group}}(G,H) \simeq (G \cong H)$ (Coquand-Danielsson)

Prop. $\mathrm{Id}_{\mathrm{Cat}}(\mathcal{C},\mathcal{D}) \simeq (\mathcal{C} \simeq \mathcal{D})$ (Ahrens-Kapulkin-Shulman)

Thm. (Ahrens-N-Shulman-Tsementzis) This pattern generalizes to encompass 'any' algebraic structure.

HoTT

- Transport + univalence $\longrightarrow$ Everything we can say or do respects these notions of sameness.

Advantages of HoTT

1. Proofs can be verified by a computer.

2. It is the 'theory' of homotopy theory (in the sense of model theory), and so results are not just valid in sSet, but in all models.

3. We can study algebraic structures with homotopical tools. In particular, everything is invariant under the appropriate notion of equivalence.

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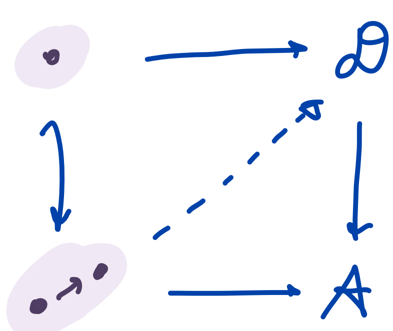
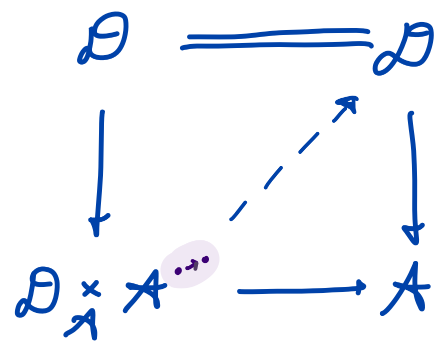
Directed transport

- Everything we can say or do should respect directed identities, in a directed way.
- Def: Given any $x : A \vdash D(x) : \mathrm{Type}$ and $p : \hom_A(x,y)$, there is a (noninvertible) $p_* : D(x) \to D(y)$.
- Cf. in $\underset{\sim}{\mathcal{C}at}$:





These are (the retract closure of) the Grothendieck opfibrations.

Example from rewriting:

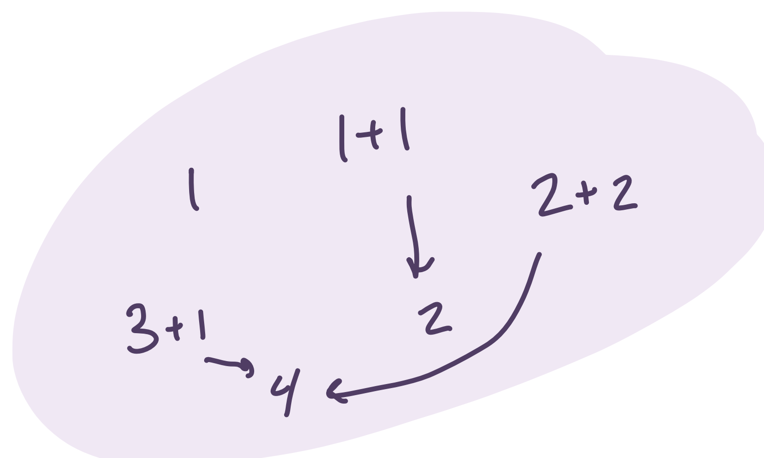
Consider $n:\mathbb{N} \vdash \mathrm{Vect}(n) : \mathrm{Type}$

where $\mathbb{N}$ is a directed homotopy type with terms like $3+1$, 4, ...

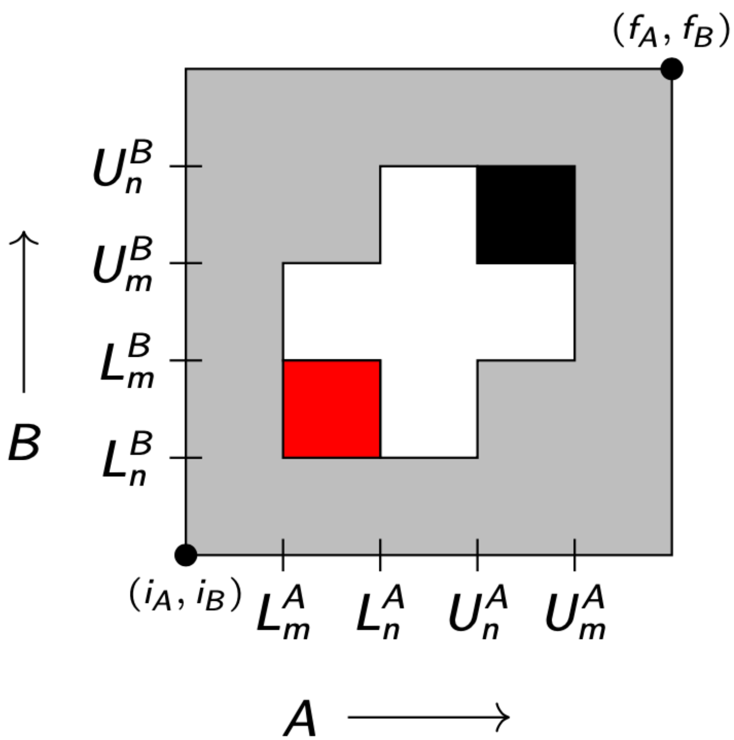
and directed paths like $\mathrm{plus\_one}_3 : \hom_{\mathbb{N}}(3+1, 4)$,

we need to be able to transport $\mathrm{Vect}(3+1) \to \mathrm{Vect}(4)$

along $\mathrm{plus\_one}_3$.



Example



Reachability:

$$x : F \vdash R(x) := \hom((i_A, i_B), x) : \mathrm{Type}$$

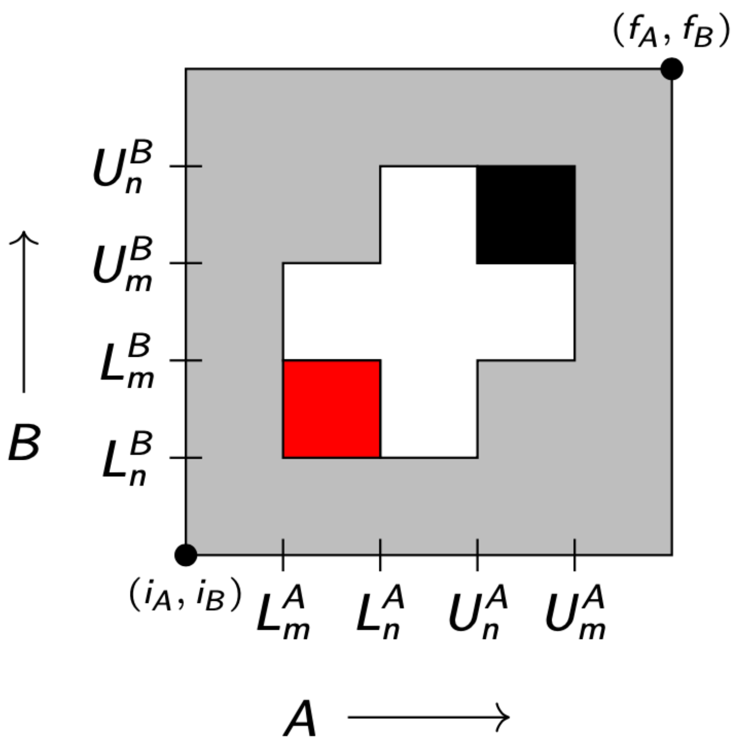
→ R can be transported along paths

Safety:

$$x : F \vdash S(x) := \hom(x, (f_A, f_B)) : \mathrm{Type}$$

→ S should be transported backwards along paths

Example



Reachability:

$$x :^{+} F \vdash R(x) := \hom((i_A, i_B), x) : \mathrm{Type}$$

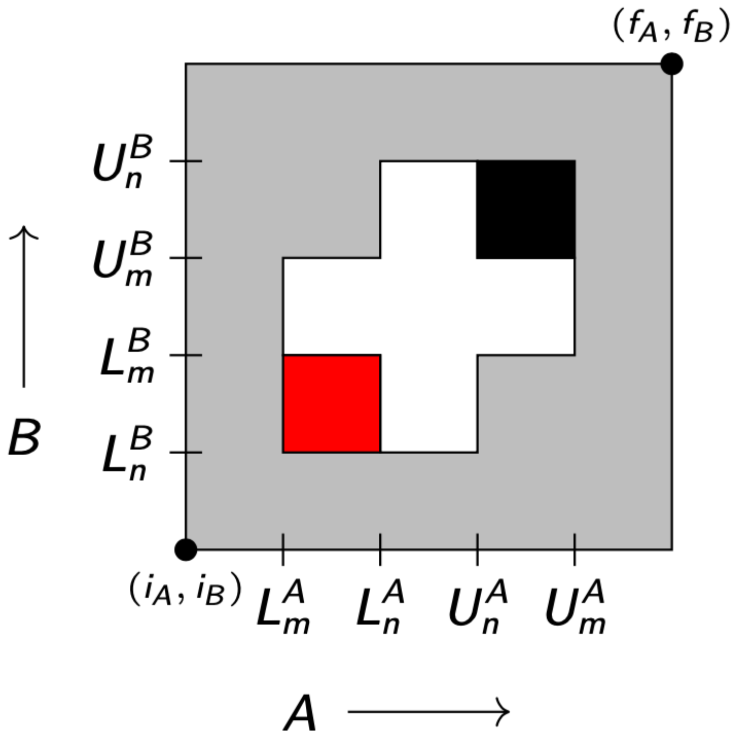
→ R can be transported along paths

Safety:

$$x :^{-} F \vdash S(x) := \hom(x, (f_A, f_B)) : \mathrm{Type}$$

→ S should be transported backwards along paths

Example



Then

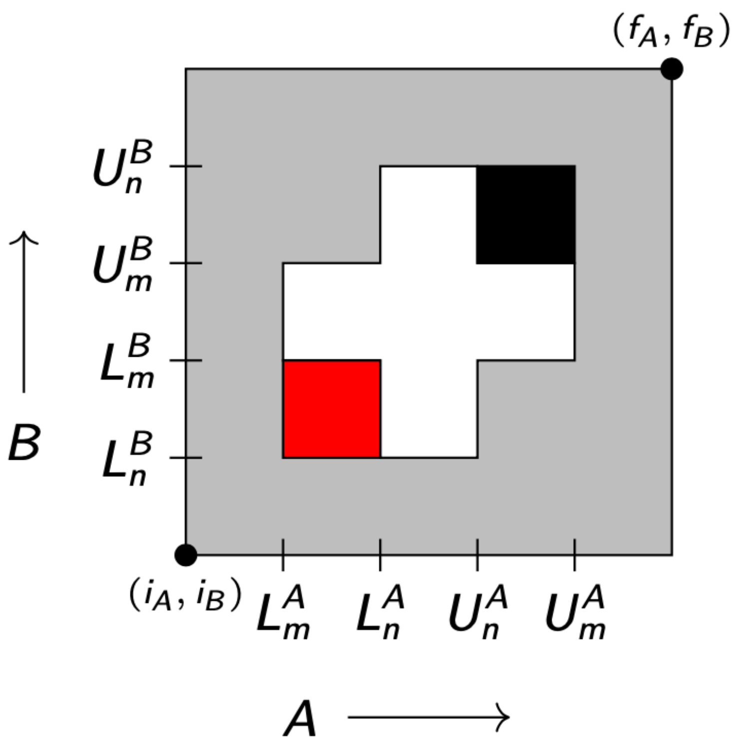
$$x : F \vdash \hom(x,x) : \mathrm{Type}$$

can only be transported along invertible directed paths.

And undirected homotopy should be expressible, as in

$$x \overset{-}{:} F,\ y \overset{+}{:} F,\ f : \hom_F(x,y),\ g : \hom_F(x,y) \vdash \mathrm{Id}_{\hom_F(x,y)}(f,g)$$

Example



Then

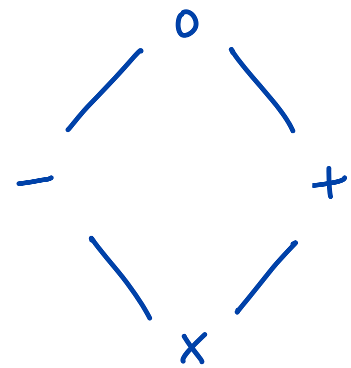
$$x \overset{\times}{:} F \vdash \mathrm{hom}(x,x) : \mathrm{Type}$$

can only be transported along invertible directed paths.

And undirected homotopy should be expressible, as in

$$x \overset{-}{:} F,\ y \overset{+}{:} F,\ f \overset{\circ}{:} \mathrm{hom}_F(x,y),\ g \overset{\circ}{:} \mathrm{hom}_F(x,y)$$
$$\vdash \mathrm{Id}_{\mathrm{hom}_F(x,y)}(f,g)$$

More notions of transport



Directed spaces:

invertible directed paths ⊆ directed paths ⊆ undirected paths

Categories:

isomorphisms ⊆ morphisms ⊆ localization

Even more notions of transport

Consider $x \overset{-}{:} A, y \overset{+}{:} A \vdash \hom_A(x,y)$.

We should be able to transport $\hom_A(x,y)$ along paths in the x or y variable without disturbing the other.

In $\underset{\sim}{\mathcal{C}at}$, we have a two-sided fibration.

$$\begin{tikzcd}
? \arrow[r] \arrow[d] \arrow[dr, phantom, "?"] & \sum_{x,y} \hom_A(x,y) \arrow[d, shift right, "\mathrm{DOM}"'] \arrow[d, shift left, "\mathrm{COD}"] \\
? \arrow[r] & A \quad A
\end{tikzcd}$$

For longer contexts, we have more complicated diagrams...

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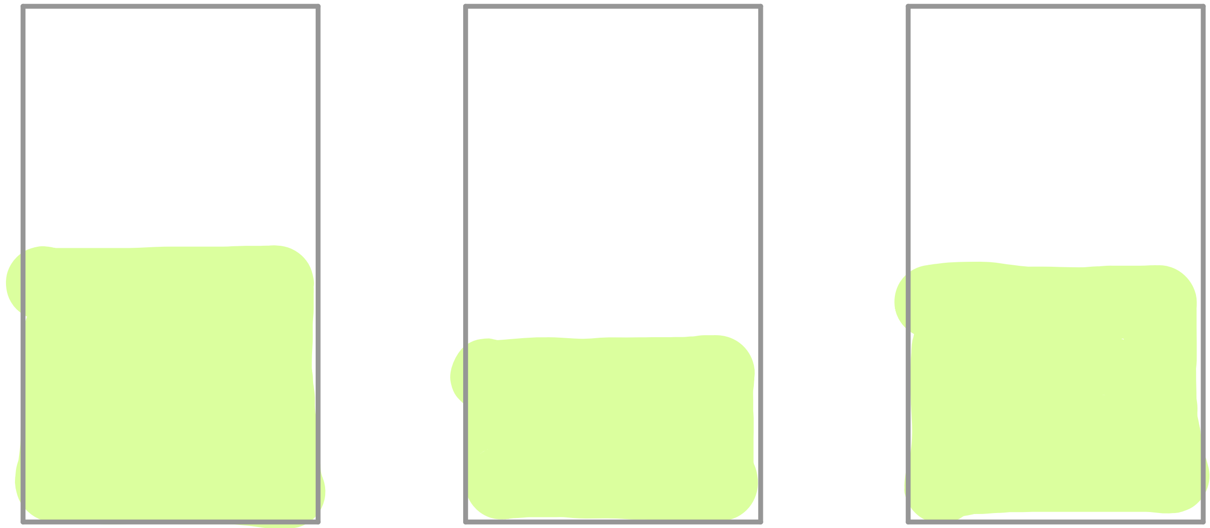
Results



Syntax

(type theory)

(N)

Connection

(comprehension category)

(Ahrens-N-vdWeide)

Semantics

(generalized weak factorization systems)

(vdBerg-McCloskey-N)

Syntax : first approximation

(Towards a directed homotopy type theory, North 2019)

- Only models

$$\begin{tikzcd} - \arrow[dr, dash] & & + \arrow[dl, dash] \\ & \times & \end{tikzcd}$$

- Uses operators op and core on types with $\iota : T^{core} \to T$, $\iota^{op} : T^{core} \to T^{op}$

$$\boxed{\dfrac{A : \mathrm{Type} \quad a : A^{op} \quad b : A}{\hom_A(a,b) : \mathrm{Type}}} \ (\hom\text{-form}) \qquad \boxed{\dfrac{A : \mathrm{Type} \quad a : A^{core}}{1_a : \hom_A(a,a)}} \ (\hom\text{-intro})$$

- There are left and right versions of the elimination and computation rules that allow for
  - forward transport along homomorphisms in A
  - backward transport along homomorphisms in A^{op}
  - both along homomorphisms in A^{core}.
- Model in $\underset{\sim}{\mathcal{C}at}$.

Syntax : first approximation

(Towards a directed homotopy type theory, North 2019)

- Only modes

$$\begin{tikzcd}[column sep=small] - \arrow[dr, dash] & & + \arrow[dl, dash] \\ & \times & \end{tikzcd}$$

Problems

- Uses operators op and core on types with $\iota : T^{core} \to T$, $\iota^{op} : T^{core} \to T^{op}$

$$\boxed{\dfrac{A:\mathrm{Type} \quad a:A^{op} \quad b:A}{\hom_A(a,b):\mathrm{Type}}} \ \text{(hom-form)} \qquad \boxed{\dfrac{A:\mathrm{Type} \quad a:A^{core}}{1_a:\hom_A(a,a)}} \ \text{(hom-intro)}$$

- There are left and right versions of the elimination and computation rules that allow for
    - forward transport along homomorphisms in A
    - backward transport along homomorphisms in A^{op}
    - both along homomorphisms in A^{core}.
- Model in $\underset{\sim}{\mathcal{C}at}$.

Syntax: second approximation (to appear)

- Use ~~hom~~ rules from above.
- Change the notion of dependency so that

$$x \overset{\times}{:} A \vdash D(x) : \mathrm{Type} \qquad x \overset{=}{:} A \vdash D(x) : \mathrm{Type} \qquad x \overset{+}{:} A \vdash D(x) : \mathrm{Type} \qquad x \overset{\circ}{:} A : D(x) : \mathrm{Type}$$

produce the four kinds of transport.

- This walls Id off from hom to prevent them from collapsing into each other.
- Models in any category $\mathscr{C}$ with the following kind of weak factorization system.

Semantics : generalized weak factorization systems

(jww vdBerg, McClosky, to appear)

- Recall : Models of the identity type are generated by a symmetric functorial path object

$$\begin{tikzcd} X \arrow[r, "r"] & \sum_{x,y} \mathrm{Id}(x,y) \arrow[r, "\varepsilon_0 \times \varepsilon_1"] & X \times X \end{tikzcd}$$

left class of wfs + right class of wfs $\longrightarrow$ transport

- In intended models of the hom-type we also have a functorial path object :

$$\begin{tikzcd} & & X \\ X \arrow[r, "r"] & \sum_{x,y} \hom(x,y) \arrow[ur, "\varepsilon_0"] \arrow[dr, "\varepsilon_1"'] & \\ & & X \end{tikzcd}$$

- We generalize the notion of weak factorization system to encompass various shapes.
- Two-sided fibrations in $\underset{\sim}{\mathrm{Cat}}$ are captured by the theory.

Thank

you!