Tracking Dynamical Features via Continuation and Persistence

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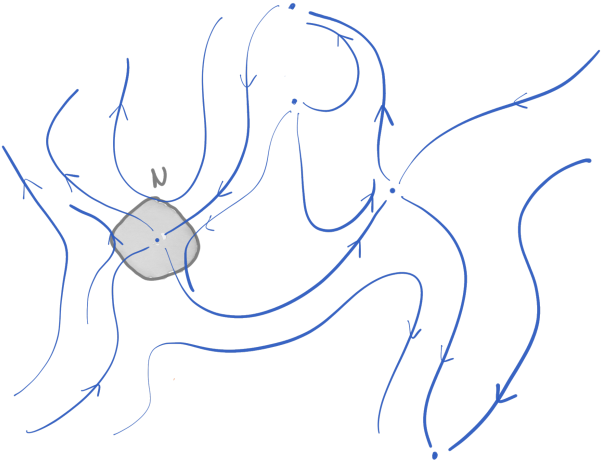
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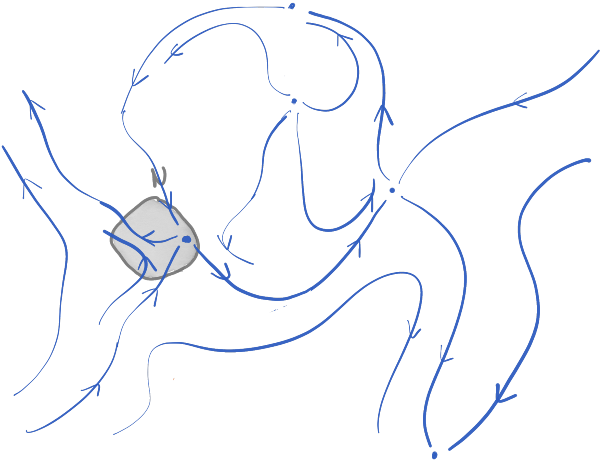
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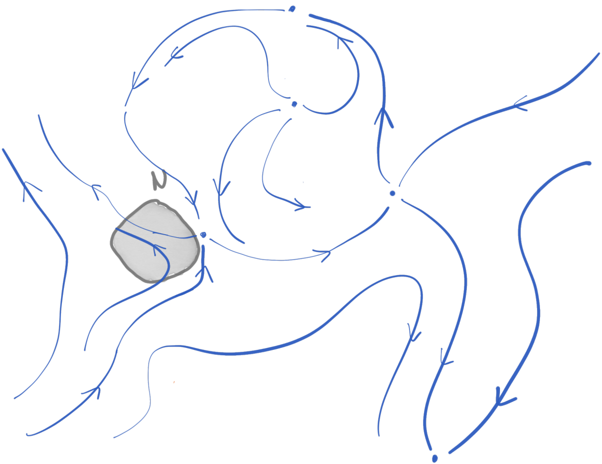
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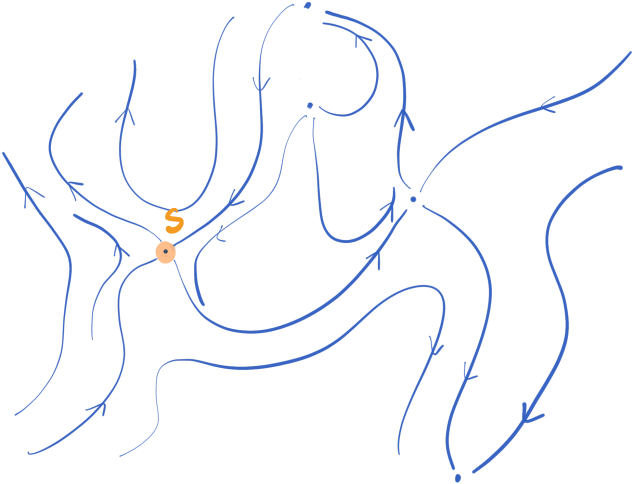
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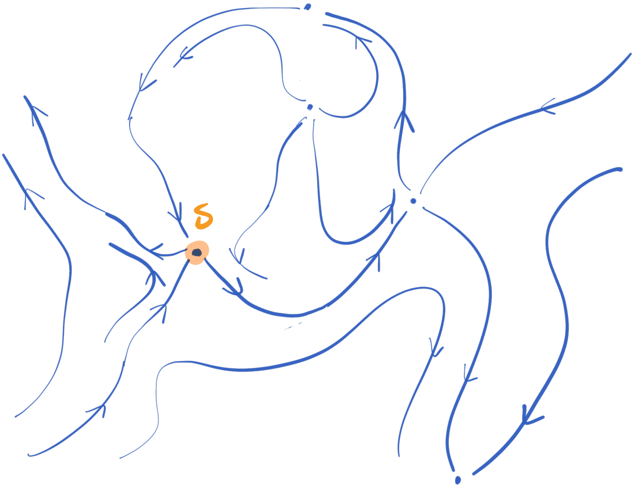
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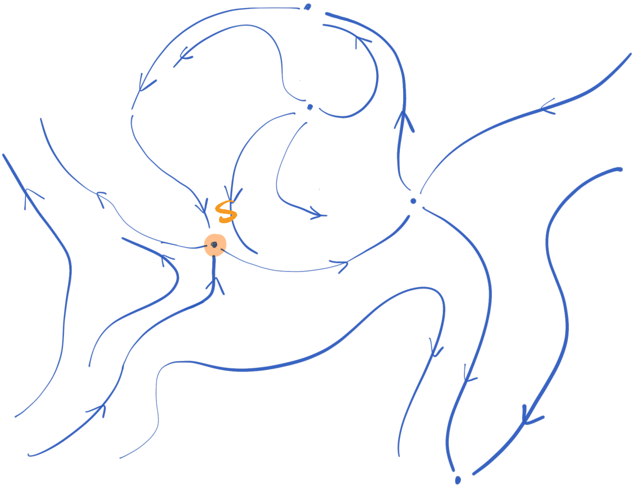
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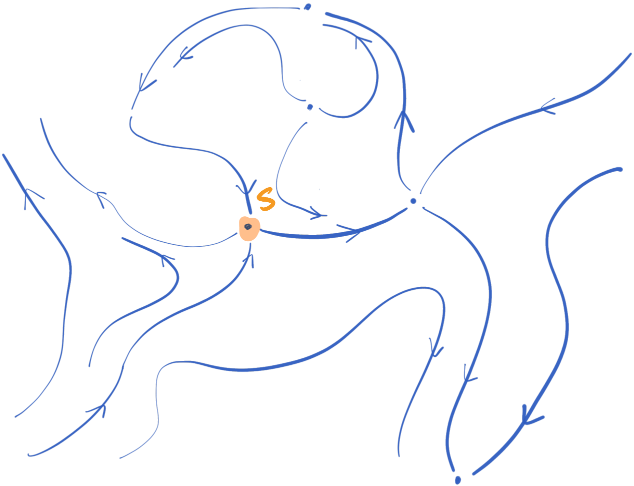
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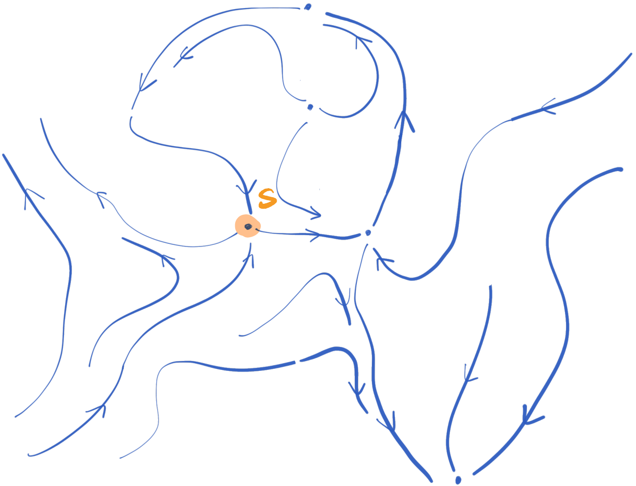
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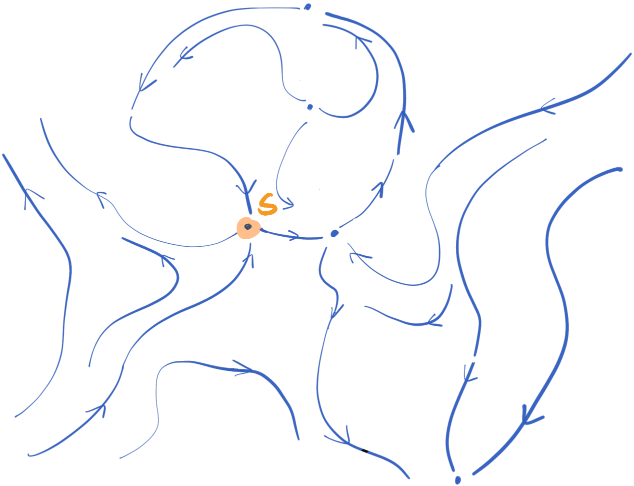
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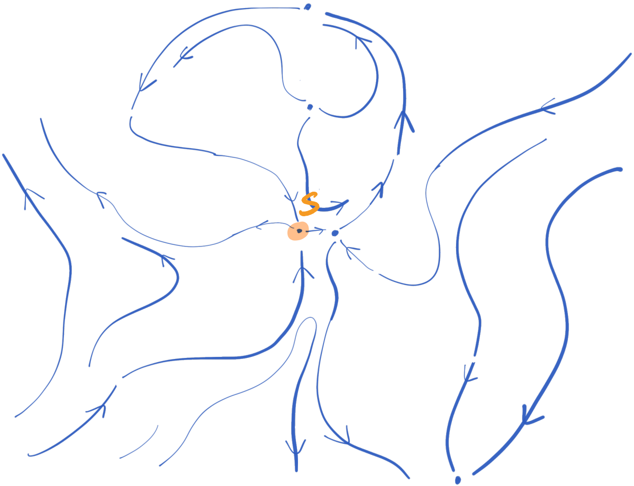
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Let $\varphi(x,t) : X \times \mathbb{R} \to X$ be a continuous flow on a compact metric space.

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A compact set N is an **isolating neighborhood** if $\operatorname{inv} N \subseteq \operatorname{int} N$.

An invariant set S which admits an isolating neighborhood such that $\operatorname{inv} N = S$ is called an **isolated invariant set**.

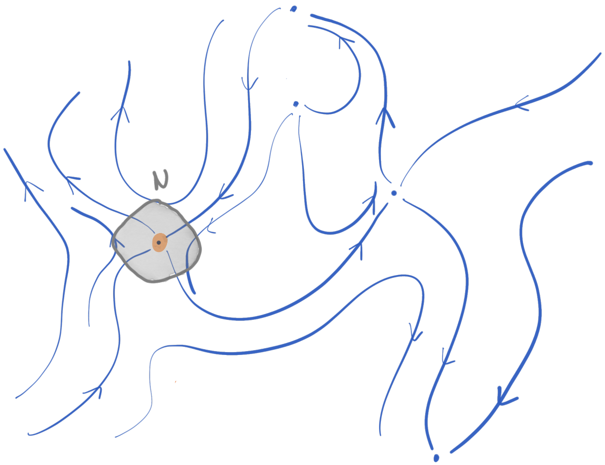
Continuation

Let $\varphi_p(x,t) : X \times \mathbb{R} \to X$ be a flow parametrized by $p \in [a,b] \subset \mathbb{R}$. An isolated invariant set S_a in φ_a **continues** to another isolated invariant set S_b in φ_b if there exist a sequence of compact sets $N_0, N_1, \dots, N_k$ and a sequence of intervals $\{[a_i, b_i] \subset [a,b] \mid i \in 0, 1, \dots, k\}$ such that

- $a_0 = a$ and $b_k = b$,
- $[a_i, b_i] \cap [a_{i+1}, b_{i+1}] \neq \emptyset$ for all $i \in \{0, 1, \dots, k-1\}$,
- N_i is an isolating neighbourhood in $\varphi_p(x,t)$ with $p \in [a_i, b_i]$,
- $\mathrm{inv}_{\varphi_a}(N_0) = S_a$ and $\mathrm{inv}_{\varphi_b}(N_k) = S_b$.

Theoerm 1.7, Conley & Easton, 1971

Denote $\Phi(X)$ a space of flows $\varphi : X \times \mathbb{R} \to X$ on the compact metric space X endowed with the compact open topology.
Let N be an isolating neighborhood for a flow $\varphi \in \Phi(X)$. Then there exists an open neighborhood $U_\varphi \subset \Phi(X)$ such that N is an isolating neighborhood for every $\psi \in U_\varphi$.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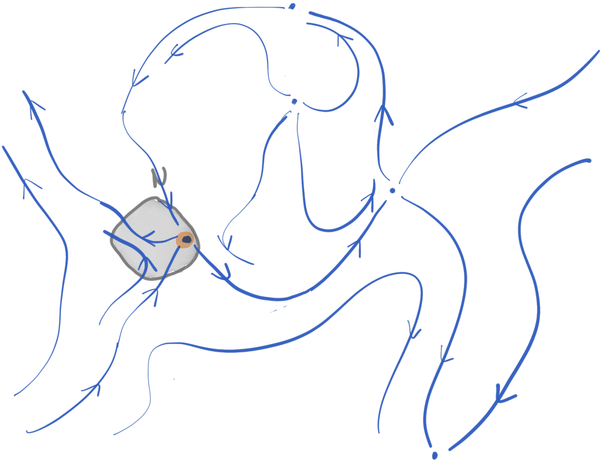
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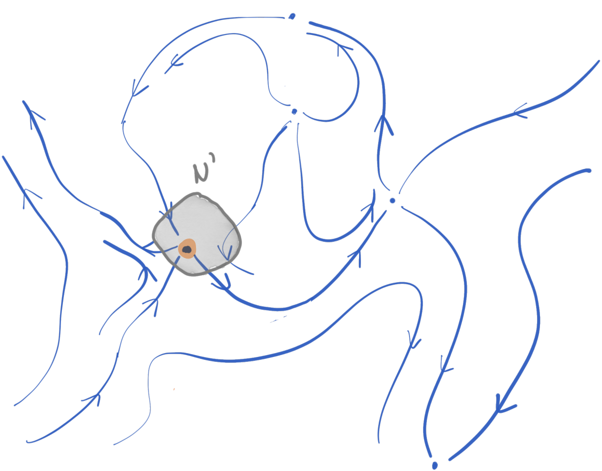
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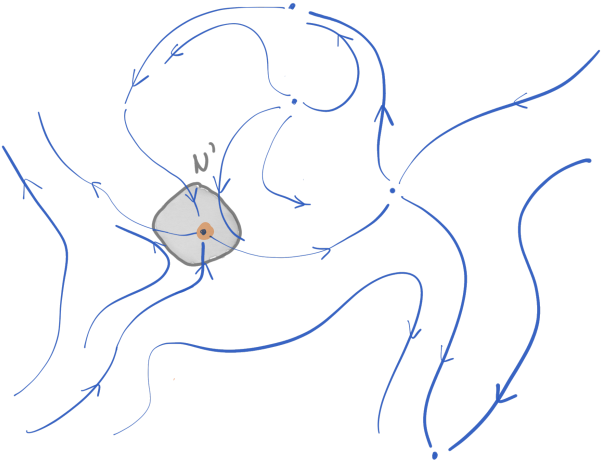
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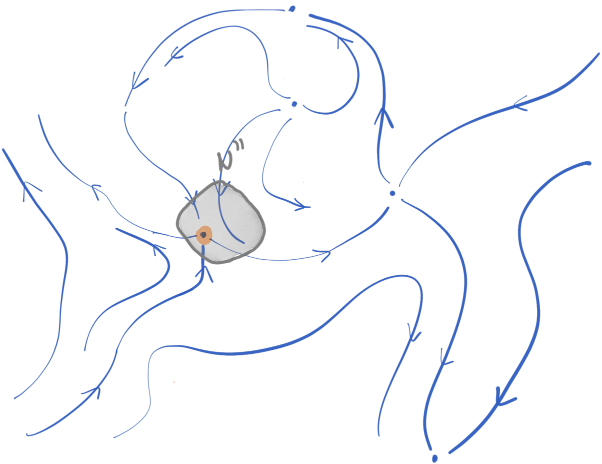
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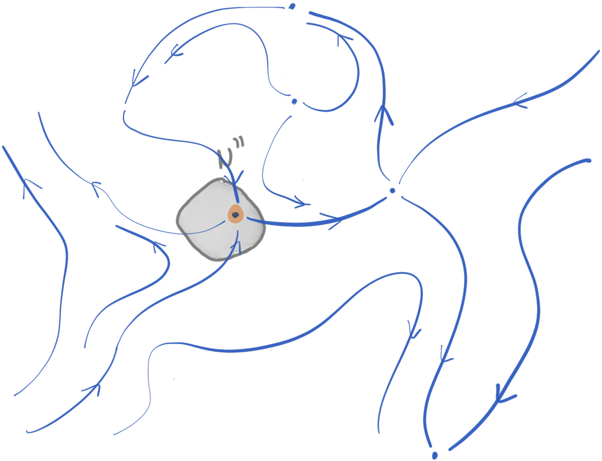
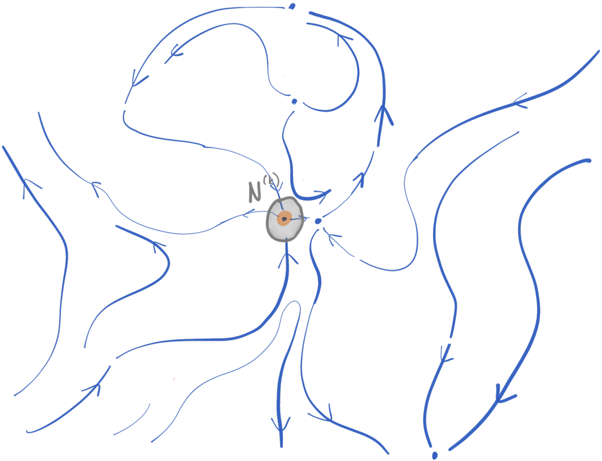


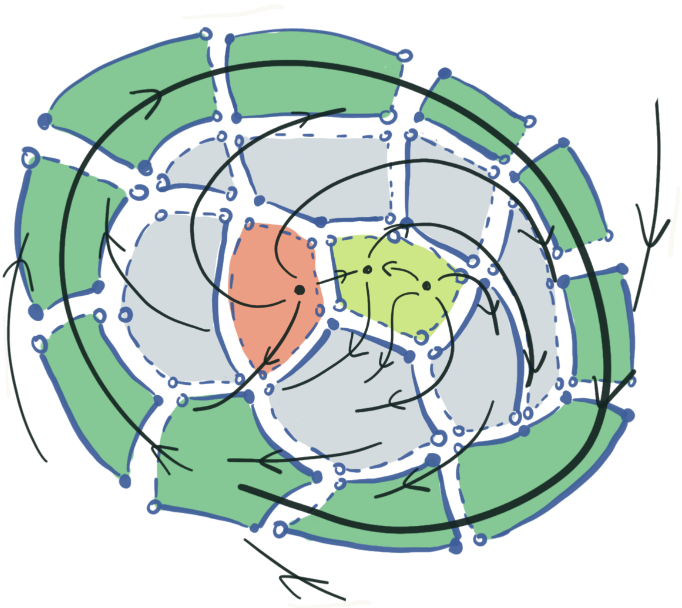

$N^{(k)}$

Multivector fields theory

Multivector as a dynamical black box

Multivector as a dynamical black box

Multivector as a dynamical black box



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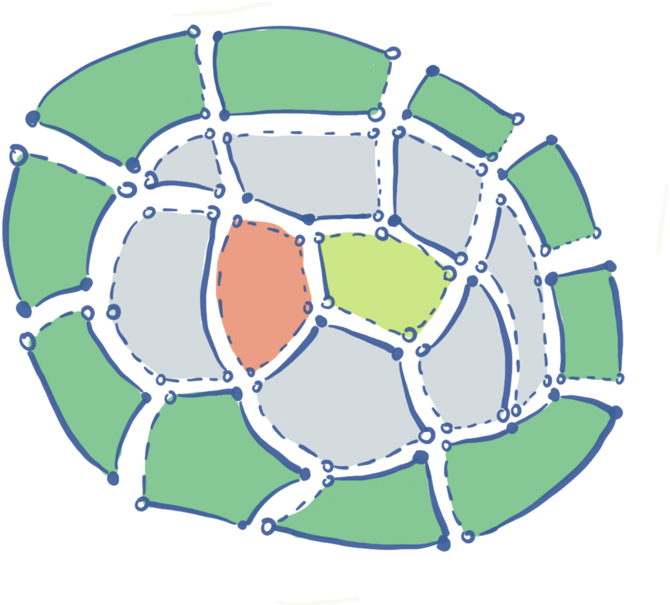

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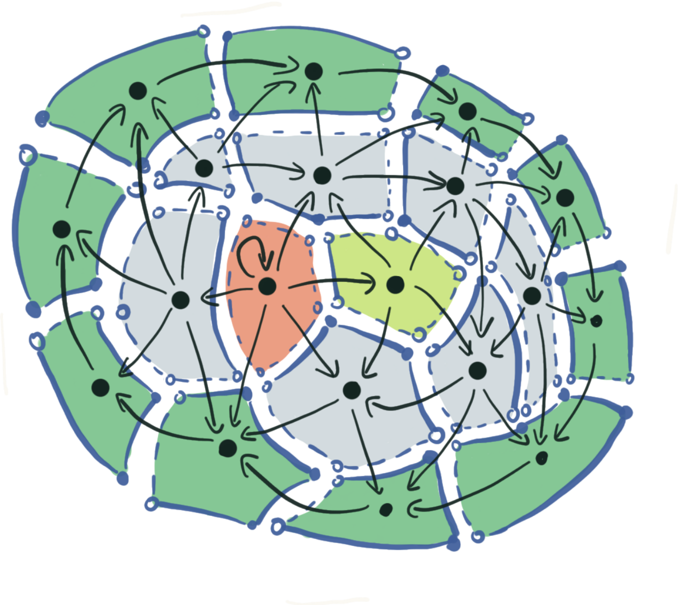
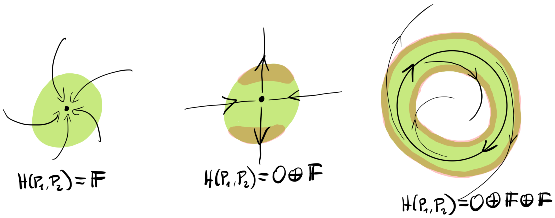


A compact set N is a **Ważewski set** if $N^- := \{x \in N \mid \forall_{\epsilon>0}\varphi(x,[0,\epsilon]) \not\subset N\}$ is closed.

Ważewski principle

If N is a Ważewski set and $H_*(N, N^-) \neq 0$ then $\operatorname{inv} N \neq \emptyset$.



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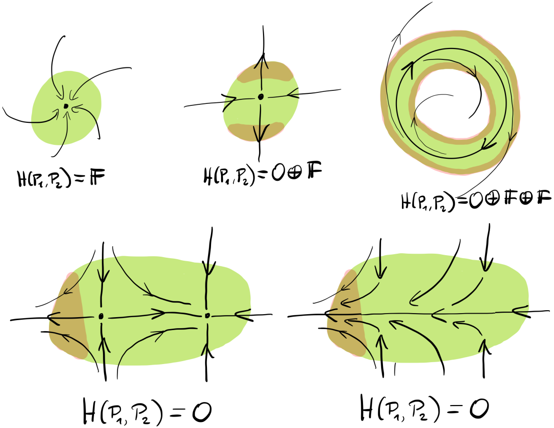


Alexandrov Theorem

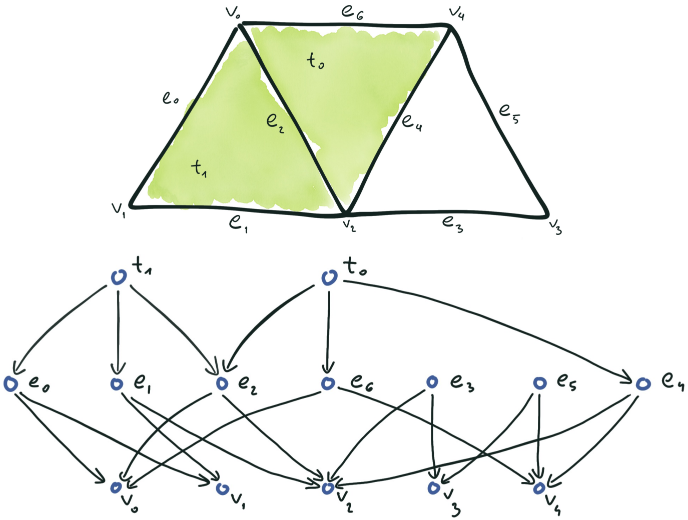
Alexandrov Theorem (1937)

For a preorder $\leq$ on a finite set X, there is a topology $\mathcal{T}_{\leq}$ on X whose open sets are the upper sets with respect to $\leq$. For a topology $\mathcal{T}$ on a finite set X, there is a preorder $\leq_{\mathcal{T}}$ where $x \leq_{\mathcal{T}} y$ if and only if $x \in \mathrm{cl}_{\mathcal{T}}\, y$. The correspondences $\mathcal{T} \mapsto \leq_{\mathcal{T}}$ and $\leq \mapsto \mathcal{T}_{\leq}$ are mutually inverse. Under these correspondences continuous maps are transformed into order-preserving maps and vice versa. Moreover, the topology $\mathcal{T}$ is T_0 (Kolmogorov) if and only if the preorder $\leq_{\mathcal{T}}$ is a partial order.

Finite topological spaces <=> Partially ordered sets

Continuous maps <=> Order preserving maps

Simplicial complex as a finite topological space



Homology of finite topological spaces

McCord Theorem, (McCord, 1966)

There exists a map

$$\mu_{(X,\mathcal{T})} : |\mathcal{K}(X,\mathcal{T})| \to (X,\mathcal{T})$$

such that it is continuous and a weak homotopy equivalence. Moreover, if $f : (X,\mathcal{T}_X) \to (Y,\mathcal{T}_Y)$ is a continuous map of two finite T_0 topological spaces, then the following diagrams commute:

$$\begin{array}{ccc} |\mathcal{K}(X,\mathcal{T}_X)| & \xrightarrow{|\mathcal{K}(f)|} & |\mathcal{K}(Y,\mathcal{T}_Y)| \\ \downarrow{\scriptstyle \mu_{(X,\mathcal{T}_X)}} & & \downarrow{\scriptstyle \mu_{(Y,\mathcal{T}_Y)}} \\ (X,\mathcal{T}_X) & \xrightarrow{f} & (Y,\mathcal{T}_Y) \end{array} \qquad \begin{array}{ccc} H(|\mathcal{K}(X,\mathcal{T}_X)|) & \xrightarrow{|\mathcal{K}(f)|_*} & H(|\mathcal{K}(Y,\mathcal{T}_Y)|) \\ \downarrow{\scriptstyle \mu_{(X,\mathcal{T}_X)_*}} & & \downarrow{\scriptstyle \mu_{(Y,\mathcal{T}_Y)_*}} \\ H(X,\mathcal{T}_X) & \xrightarrow{f_*} & H(Y,\mathcal{T}_Y) \end{array}$$

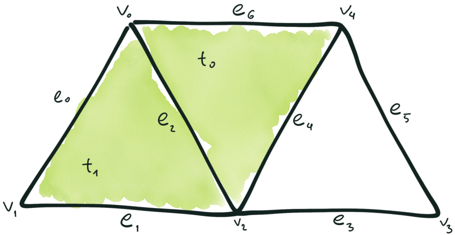
Let X be a finite topological space and $A \subset X$. Then

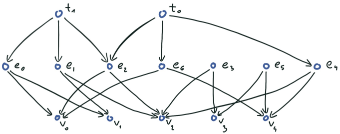
$$H(X) \cong H(|\mathcal{K}(X)|) \cong H^{\Delta}(\mathcal{K}(X)).$$

$$H(X,A) \cong H(|\mathcal{K}(X)|, |\mathcal{K}(A)|) \cong H^{\Delta}(\mathcal{K}(X),\mathcal{K}(A)).$$

Let $(\mathcal{P}, \leq)$ be a partial order.

$A \subset \mathcal{P}$ is an **upper set** (**open**) iff $x \in A$ and $y \geq x$ implies $y \in A$.
$A \subset \mathcal{P}$ is a **down set** (**closed**) iff $x \in A$ and $y \leq x$ implies $y \in A$.
$A \subset \mathcal{P}$ is **convex** (**locally closed**) iff $x \leq y \leq z$ with $x, z \in A,\ y \in \mathcal{P}$ implies $y \in A$.





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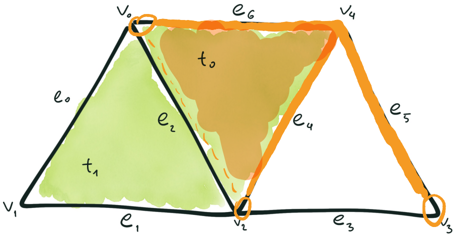

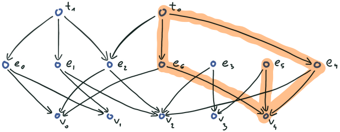

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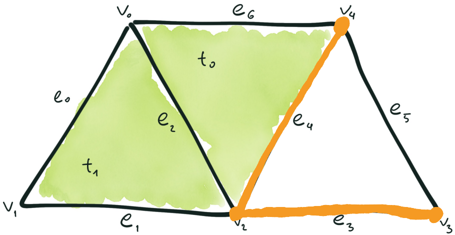

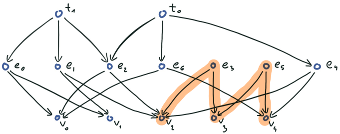

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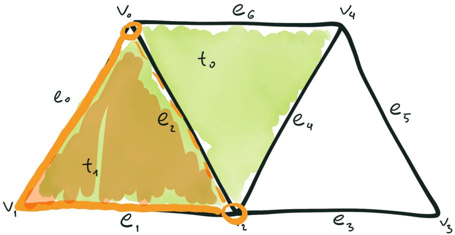
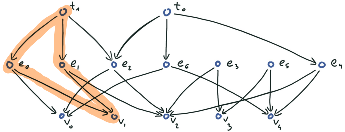
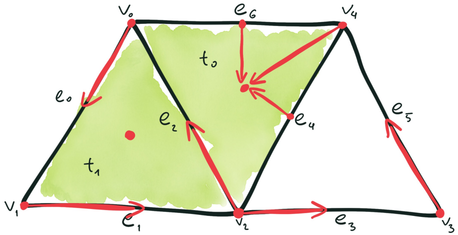




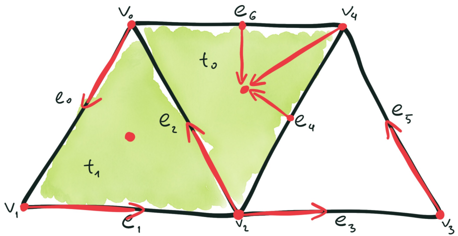
Combinatorial Multivector Fields for FTop

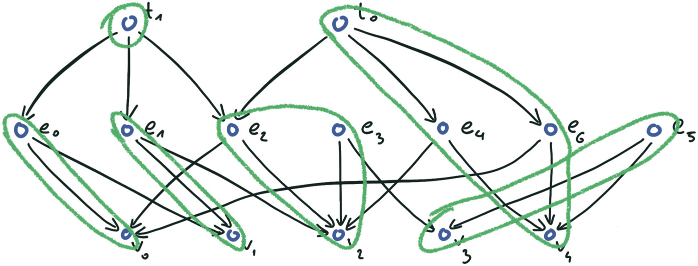
Let X be a finite topological space.

A **multivector** is a locally closed subset of X.
Combinatorial multivector field (MVF) $\mathcal{V}$ on X is a collection of multivectors, such that $\mathcal{V}$ is a partition of X.



Dynamics of MVF for FTop

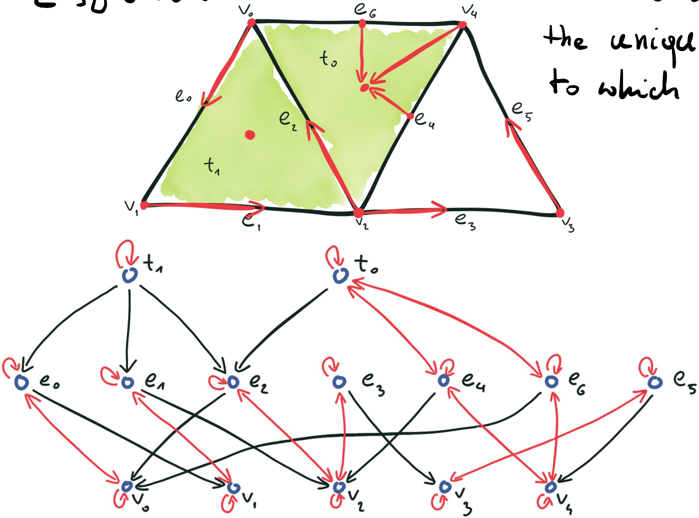




Dynamics of MVF for FTop

$$F_{\mathcal{V}}(x) := [x]_{\mathcal{V}} \cup \operatorname{cl} x$$

where $[x]_{\mathcal{V}}$ is the unique mv to which x belongs



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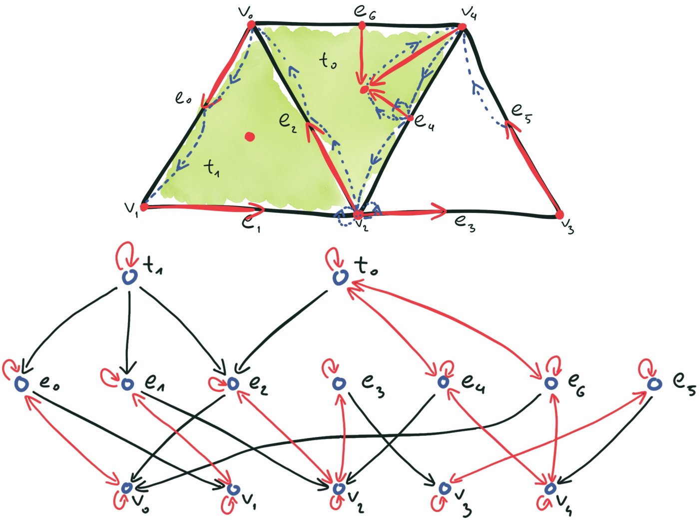
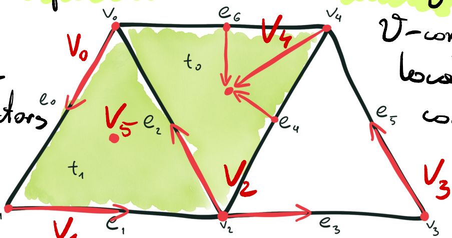
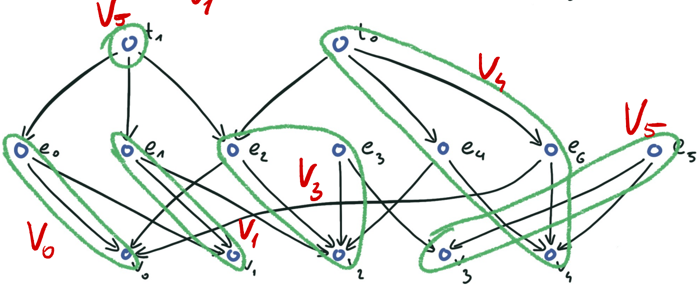


Dynamics of MVF for FTop

set A is $\mathcal{V}$-compatible if A can be expressed as a union of multivectors from $\mathcal{V}$

$\langle A\rangle_{\mathcal{V}}$ - the minimal $\mathcal{V}$-compatible and locally closed set containing A





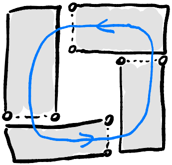
$$\langle\{v_0,e_2\}\rangle_{\mathcal{V}} = V_0 \cup V_1 \cup V_2 \cup V_5$$

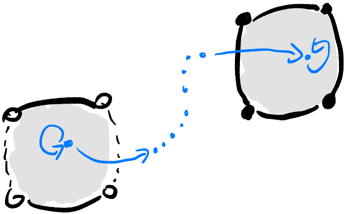
Essential solutions and invariant sets

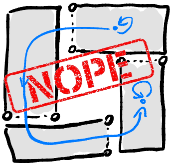
A map $\varphi : \mathbb{Z} \to X$ is a **full solution for** $\mathcal{V}$ iff $\forall_{i \in \mathbb{Z}}\ \varphi(i+1) \in F_{\mathcal{V}}(\varphi(i))$. We denote a set of full solutions in X by $\operatorname{Sol}(X)$.

A multivector $V \in \mathcal{V}$ is **critical** if $H(\operatorname{cl} V, \operatorname{mo} V) \neq 0$, otherwise V is **regular**.

A full solution $\varphi : \mathbb{Z} \to X$ is **essential** if for every regular $x \in \operatorname{im} \varphi$ the set $\{t \in \mathbb{Z} \mid \varphi(t) \notin [x]_{\mathcal{V}}\}$ is either left- and right-unbounded. A set of all essential solutions in a set $A \subseteq X$ with $\varphi(0) = x$ is denoted $\operatorname{eSol}(x, A)$.







Isolated invariant sets

Invariant part of $A \subseteq X$ is

$$\mathrm{Inv}(A) := \{x \in A \mid \mathrm{eSol}(x, A) \neq \emptyset\}$$

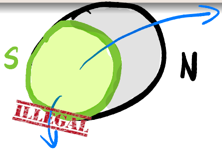
We say that A is **invariant** iff $\mathrm{Inv}(A) = A$.

A closed set N **isolates** an invariant set $S \subseteq N$ if the following two conditions holds

a) every path in N with endpoints in S is a path in S,

b) $\Pi_{\mathcal{V}}(S) \subseteq N$.

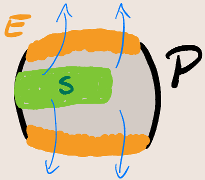
In this case, we also say that N is an isolating set for S. An invariant set S is **isolated** if there exists a closed set N meeting the above conditions.



Conley index

Let S be isolated invariant set under $\mathcal{V}$, and let P and E denote closed sets where $E \subseteq P$. If the following conditions hold, then (P, E) is an **index pair** for S:

1) $F_{\mathcal{V}}(P \setminus E) \subseteq P$,
2) $F_{\mathcal{V}}(E) \cap P \subseteq E$,
3) $S = \operatorname{inv}_{\mathcal{V}}(P \setminus E)$.



The **combinatorial homology Conley index** of an isolated invariant set S is defined as $\mathrm{Con}(S) := H(P, E)$, where (P, E) is an index pair for S.

Theorem 5.16 (LKMW, 2020)

Let (P, E) and (P', E') be index pairs for an isolated invariant set S then $H(P, E) \cong H(P', E')$.

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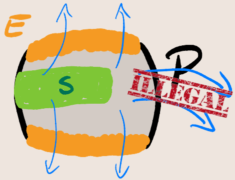

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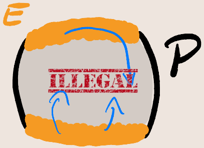


The **combinatorial homology Conley index** of an isolated invariant set S is defined as $\text{Con}(S) := H(P, E)$, where (P, E) is an index pair for S.

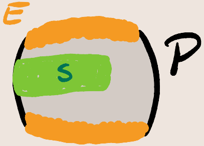
Theorem 5.16 (LKMW, 2020)

Let (P, E) and (P', E') be index pairs for an isolated invariant set S then $H(P, E) \cong H(P', E')$.

Conley index

Let S be isolated invariant set under $\mathcal{V}$, and let P and E denote closed sets where $E \subseteq P$. If the following conditions hold, then (P, E) is an **index pair** for S:

1) $F_{\mathcal{V}}(P \setminus E) \subseteq P$,
2) $F_{\mathcal{V}}(E) \cap P \subseteq E$,
3) $S = \mathrm{inv}_{\mathcal{V}}(P \setminus E)$.



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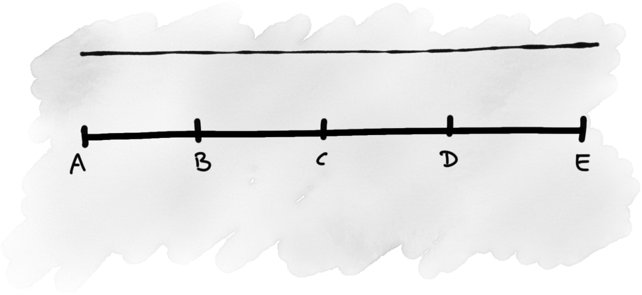
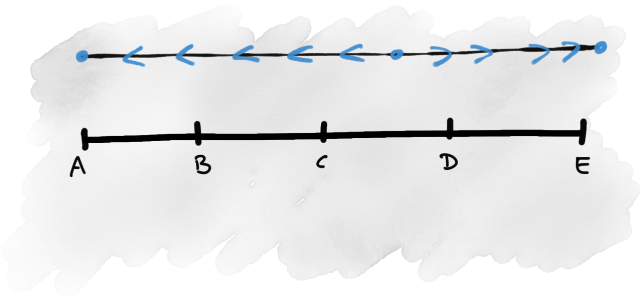
Combinatorial perturbation

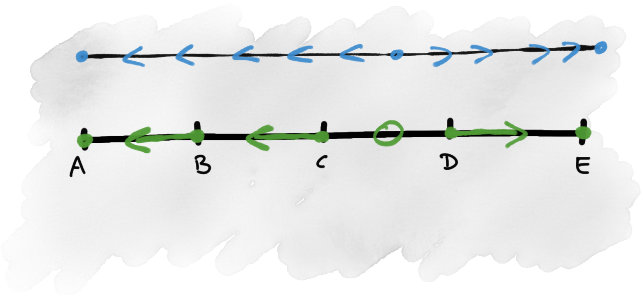
Combinatorial perturbation



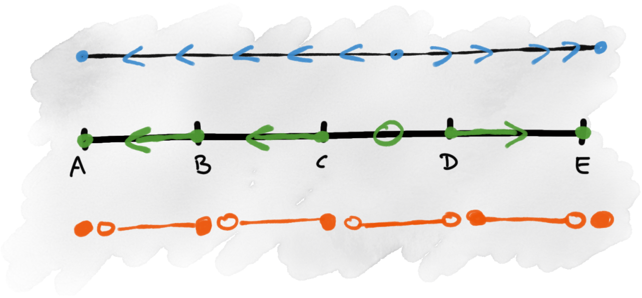
Combinatorial perturbation



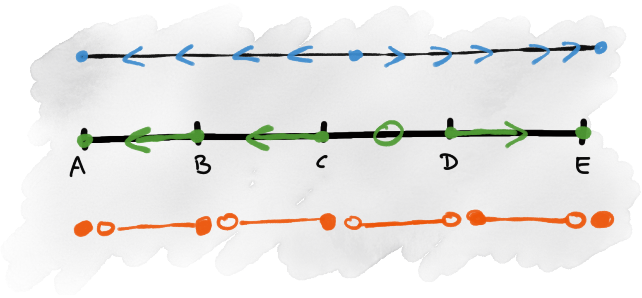
Combinatorial perturbation



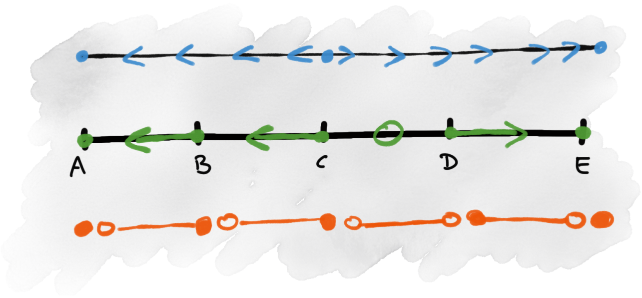
Combinatorial perturbation



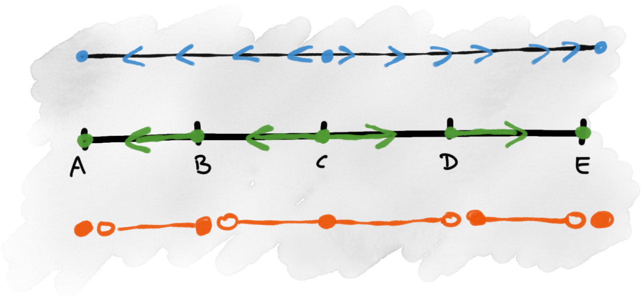
Combinatorial perturbation



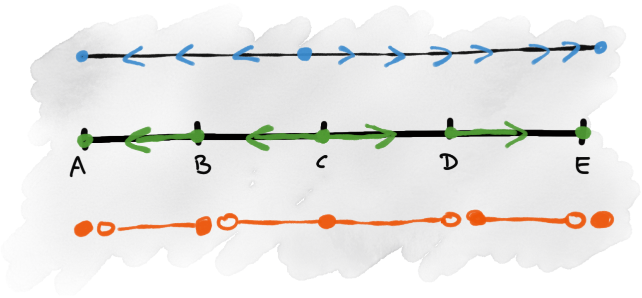
Combinatorial perturbation



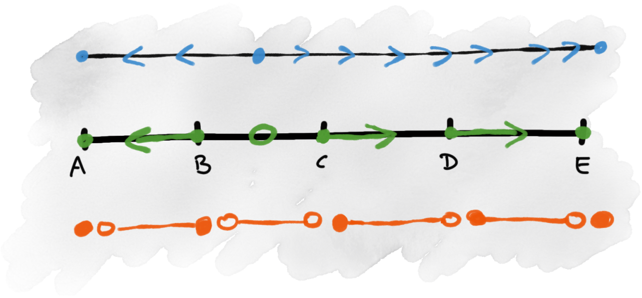
Combinatorial perturbation



Combinatorial perturbation



Combinatorial perturbation

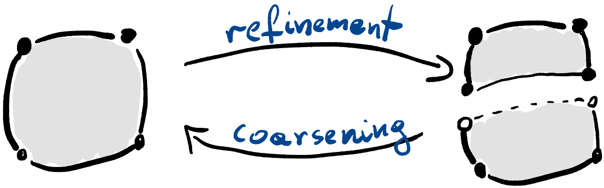


Multivector fields space

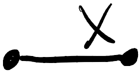
For two families of sets $\mathcal{A}$ and $\mathcal{B}$ we write $\mathcal{A} \sqsubseteq \mathcal{B}$ if for every $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$ such that $A \subseteq B$.

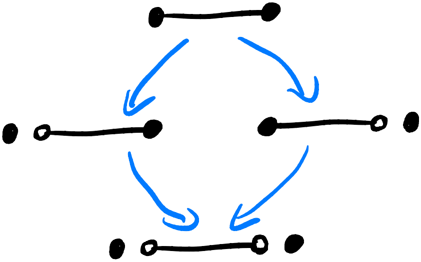
An atomic rearrangements of multivector fields:

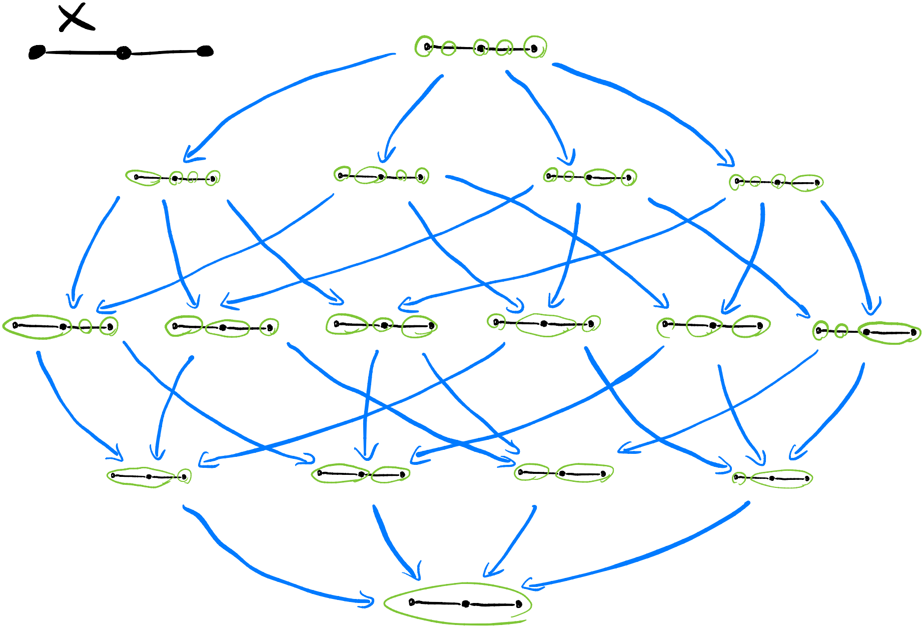
- $\mathcal{V}$ is an **atomic refinement** of $\mathcal{W}$ if $\mathcal{V} \sqsubseteq \mathcal{W}$ and $|\mathcal{V} \setminus \mathcal{W}| = 1$
- $\mathcal{V}$ is an **atomic coarsening** of $\mathcal{W}$ if $\mathcal{V} \sqsupseteq \mathcal{W}$ and $|\mathcal{V} \setminus \mathcal{W}| = 2$



MVF(X) - a family of all multivector fields on X with a topology induced by $\sqsubseteq$.






X

Combinatorial continuation of an isolated invariant set

Combinatorial continuation of an isolated invariant set

Let S_1, S_2, ..., S_n denote a sequence of isolated invariant sets under the multivector fields $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_n$, where each $\mathcal{V}_i$ is defined on a fixed simplicial complex K. We say that isolated invariant set S_1 **continues** to isolated invariant set S_n whenever there exists a sequence of index pairs (P_1, E_1), (P_2, E_2), ..., (P_{n-1}, E_{n-1}) where (P_i, E_i) is an index pair for both S_i and S_{i+1}. Such a sequence is a **sequence of connecting index pairs**.

The tracking

$$v_0 \qquad v_1 \qquad v_2 \qquad v_3 \quad \ldots \quad v_n$$

The tracking

$v_0 \quad v_1 \quad v_2 \quad v_3 \quad \ldots \quad v_n$

s_0

The tracking

$$v_0 \quad v_1 \quad v_2 \quad v_3 \quad \dots \quad v_n$$

$$s_0 \longrightarrow ?$$

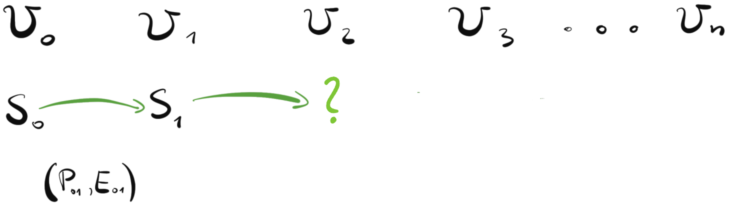
The tracking

$$U_0 \quad U_1 \quad U_2 \quad U_3 \quad \dots \quad U_n$$

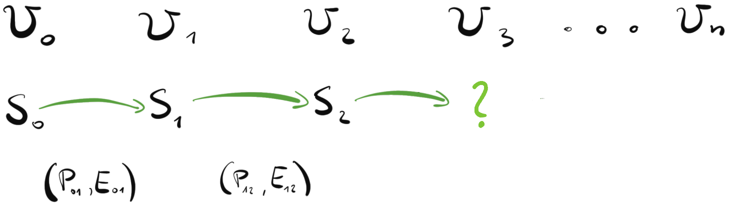
$$S_0 \longrightarrow S_1$$

$$(P_{01}, E_{01})$$

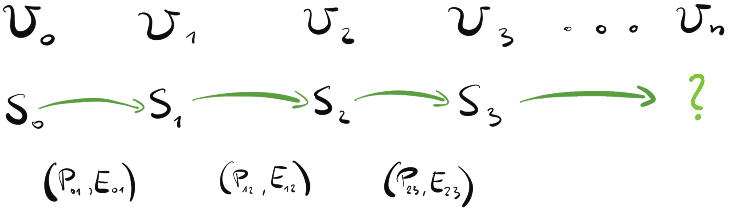
The tracking



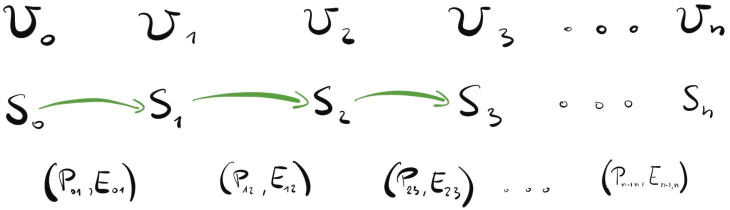
The tracking



The tracking



The tracking



Tracking Protocol

$\langle A \rangle_{\mathcal{V}}$ is the minimal locally closed set

Attempt to track via continuation:

1. If $\mathcal{V}'$ is an atomic refinement of $\mathcal{V}$, then take $S' := \mathrm{inv}_{\mathcal{V}'}(S)$.
2. If $\mathcal{V}'$ is an atomic coarsening of $\mathcal{V}$, and the unique merged multivector V has the property that $V \subseteq S$, then take $S' := \mathrm{inv}_{\mathcal{V}'}(S)$.
3. If $\mathcal{V}'$ is an atomic coarsening of $\mathcal{V}$, and the unique merged multivector V has the property that $V \cap S = \emptyset$, then take $S' := \mathrm{inv}_{\mathcal{V}'}(S) = S$.
4. If $\mathcal{V}'$ is an atomic coarsening of $\mathcal{V}$ and the unique merged multivector V satisfies the formulae $V \cap S \neq \emptyset$ and $V \not\subseteq S$, then consider $A = \langle S \cup V \rangle_{\mathcal{V}'}$. If $\mathrm{inv}_{\mathcal{V}}(A) = S$, then take $S' := \mathrm{inv}_{\mathcal{V}'}(A)$.
5. Else, it is impossible to track via continuation.

Theorem 11 (Dey, L., Mrozek, Slechta; 2022)

Let $\mathcal{V}$ and $\mathcal{V}'$ denote multivector fields where $\mathcal{V}'$ is an atomic refinement of $\mathcal{V}$. Let A be a $\mathcal{V}$-compatible and convex set. The pair $(\mathrm{cl}(A), \mathrm{mo}(A))$ is an index pair for both $\mathrm{inv}_{\mathcal{V}}(A)$ under $\mathcal{V}$ and $\mathrm{inv}_{\mathcal{V}'}(A)$ under $\mathcal{V}'$.

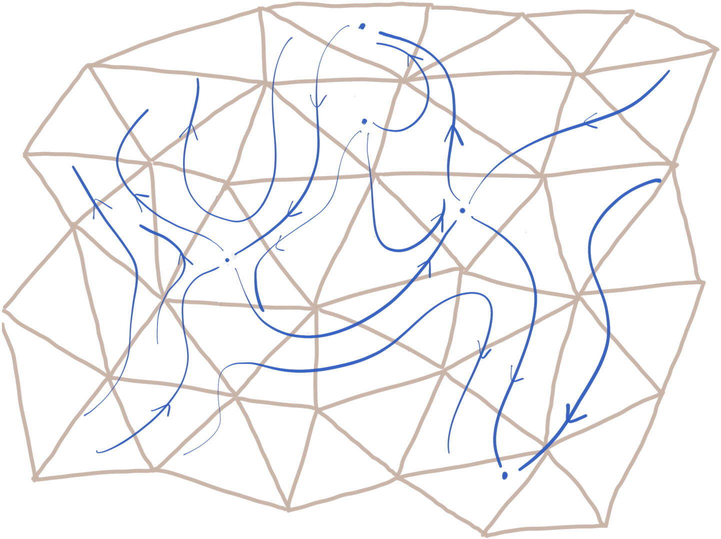
Theorem 12 (Dey, L., Mrozek, Slechta; 2022)

Let $\mathcal{V}$ and $\mathcal{V}'$ denote multivector fields where $\mathcal{V}'$ is an atomic coarsening of $\mathcal{V}$. Let A be a convex and $\mathcal{V}$-compatible set, and let $V \in \mathcal{V}'$ be the unique merged multivector. If $V \subseteq A$ or $V \cap A = \emptyset$, then $(\mathrm{cl}(A), \mathrm{mo}(A))$ is an index pair for both $\mathrm{inv}_{\mathcal{V}}(A)$ and $\mathrm{inv}_{\mathcal{V}'}(A)$.

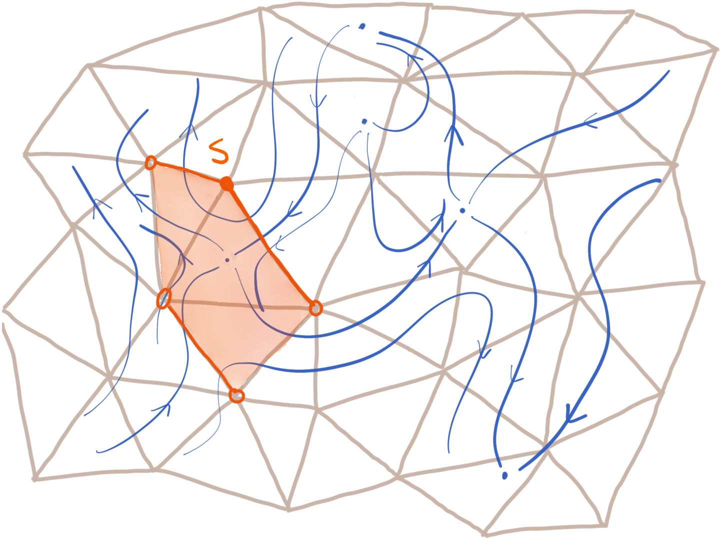
Theorem 13 (Dey, L., Mrozek, Slechta; 2022)

Let S denote an isolated invariant set under $\mathcal{V}$ and let $\mathcal{V}'$ denote an atomic coarsening of $\mathcal{V}$ where the unique merged multivector $V \in \mathcal{V}' \setminus \mathcal{V}$ satisfies the formulae $V \cap S \neq \emptyset$ and $V \not\subseteq S$. Furthermore, let $A := \langle S \cup V \rangle_{\mathcal{V}'}$. If $S \neq \mathrm{inv}_{\mathcal{V}}(A)$, then there does not exist an isolated invariant set S' under $\mathcal{V}'$ for which there is an index pair (P, E) satisfying $\mathrm{inv}_{\mathcal{V}}(P \setminus E) = S$ and $\mathrm{inv}_{\mathcal{V}'}(P \setminus E) = S'$.

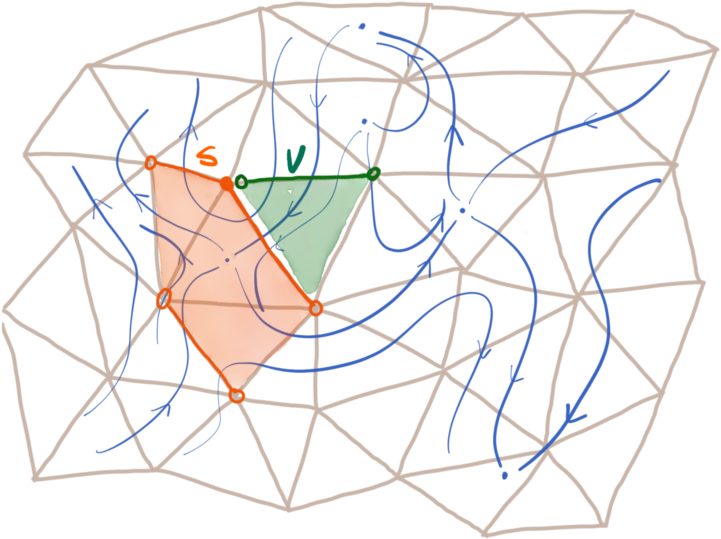
Continuation of an isolated invariant set



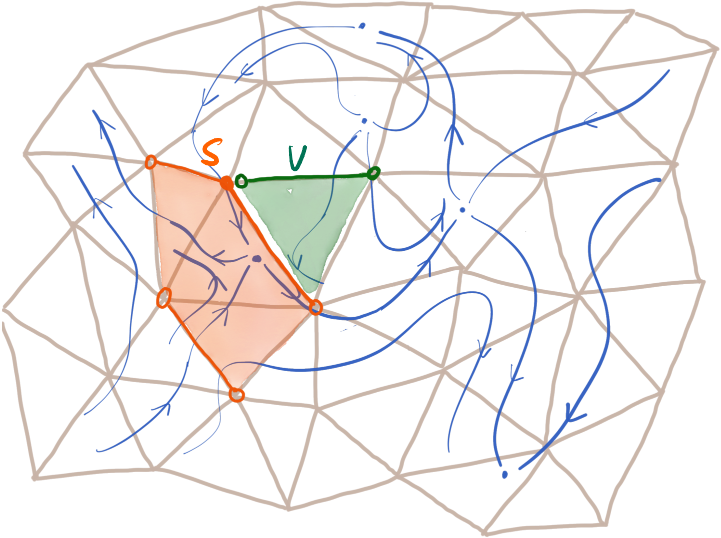
Continuation of an isolated invariant set



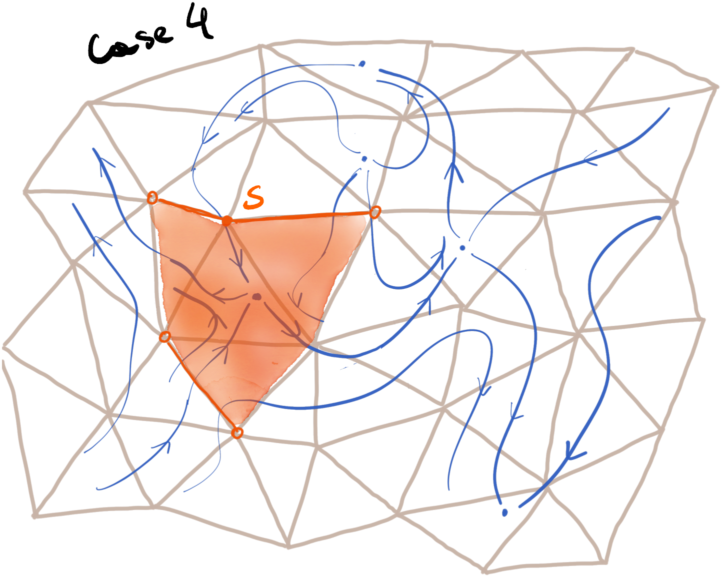
Continuation of an isolated invariant set



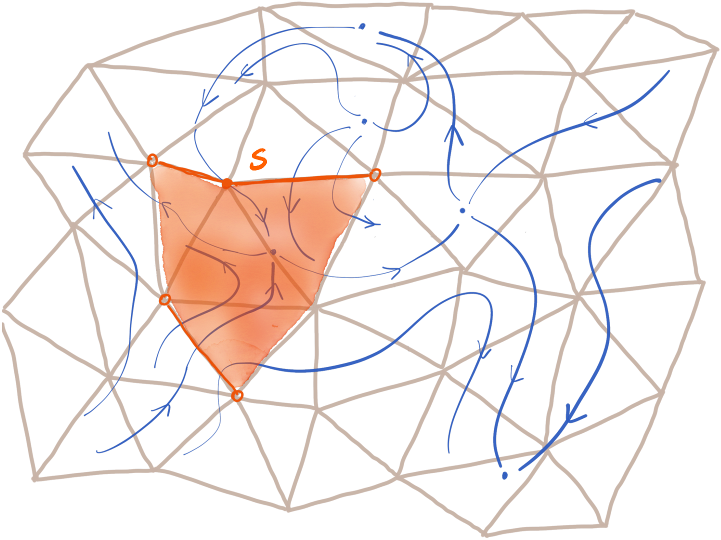
Continuation of an isolated invariant set



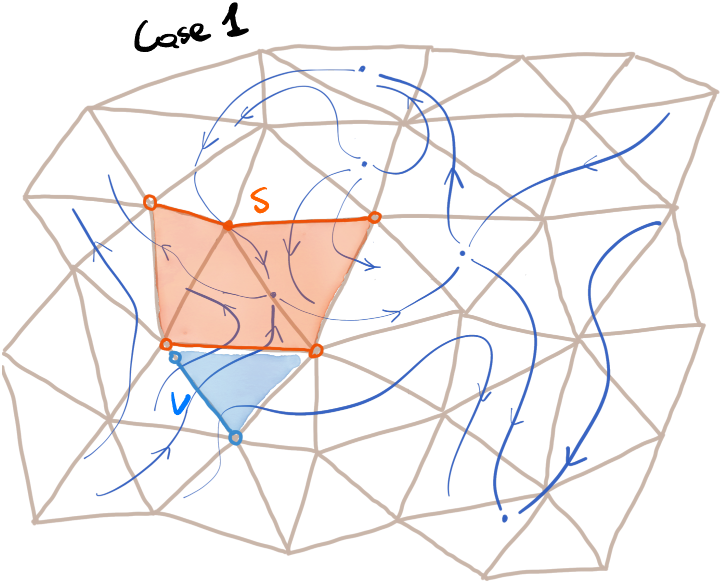
Continuation of an isolated invariant set



Continuation of an isolated invariant set



Continuation of an isolated invariant set



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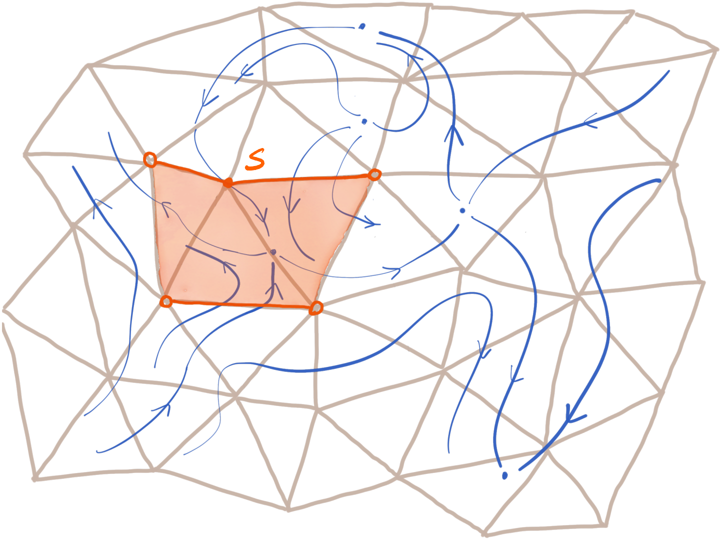
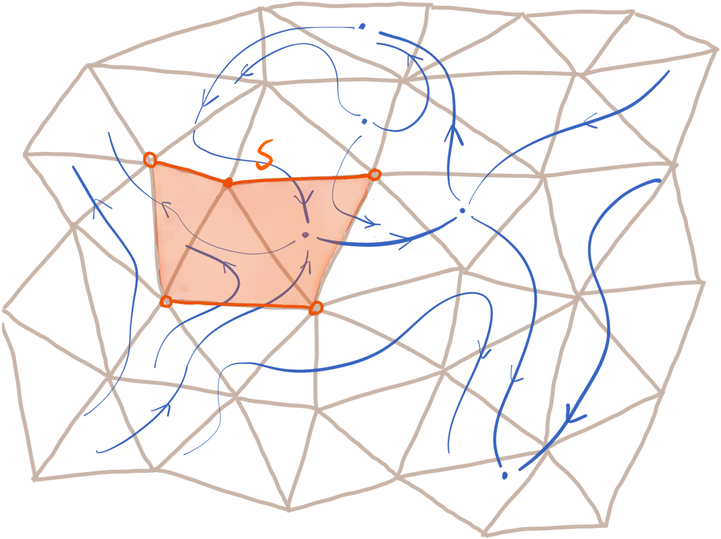
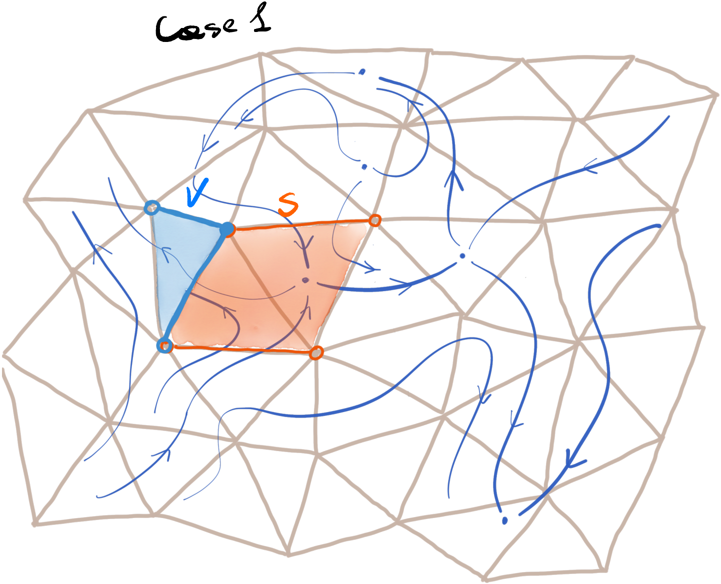


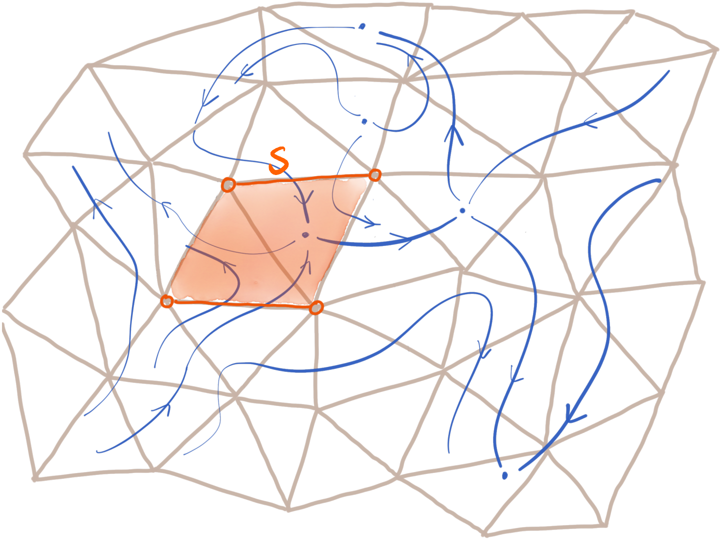
Continuation of an isolated invariant set



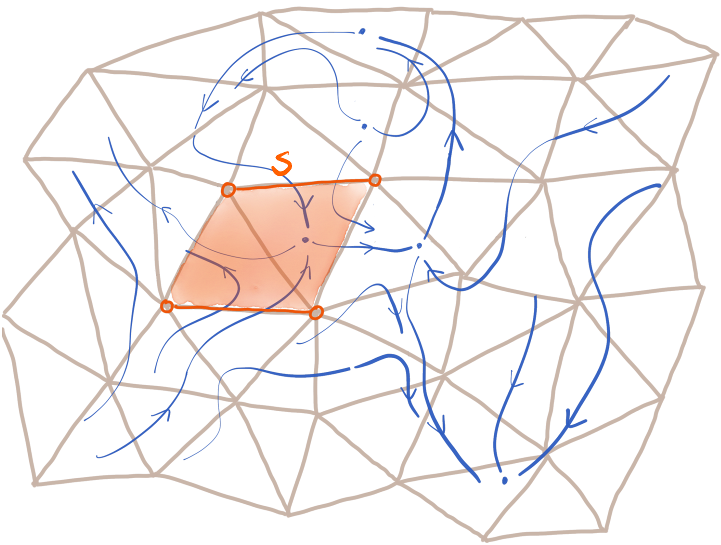
Continuation of an isolated invariant set



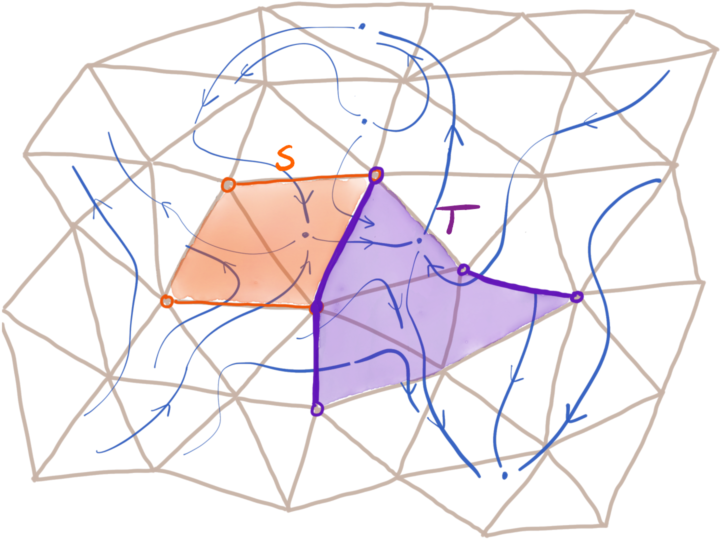
Continuation of an isolated invariant set



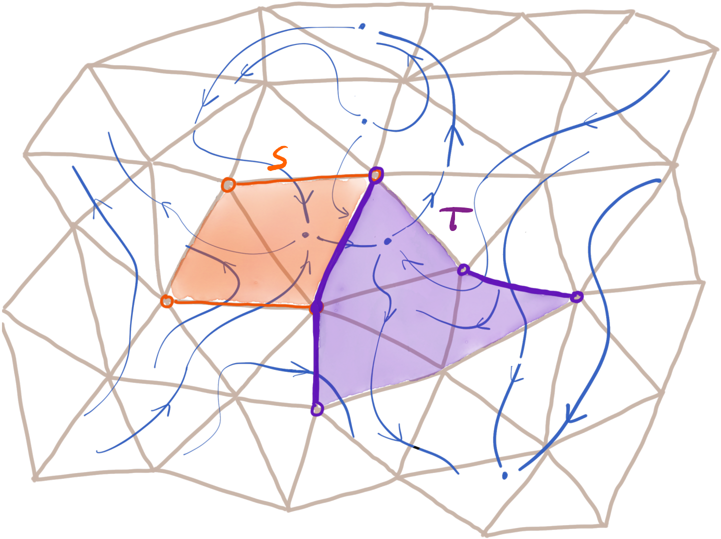
Continuation of an isolated invariant set



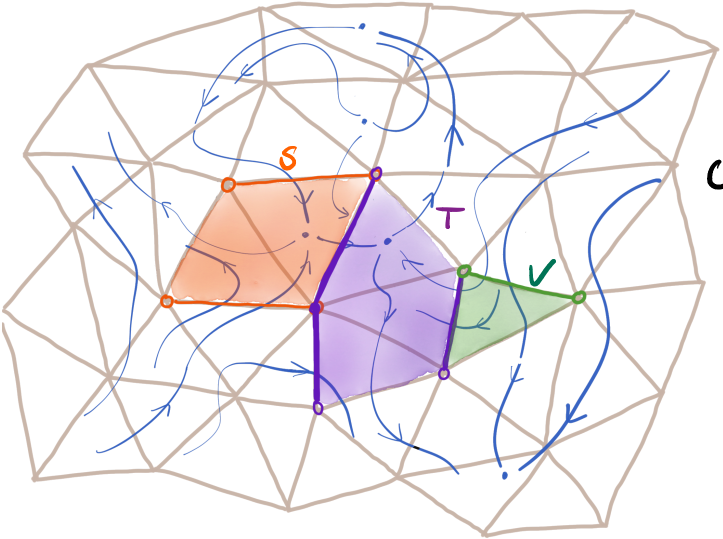
Continuation of an isolated invariant set



Continuation of an isolated invariant set

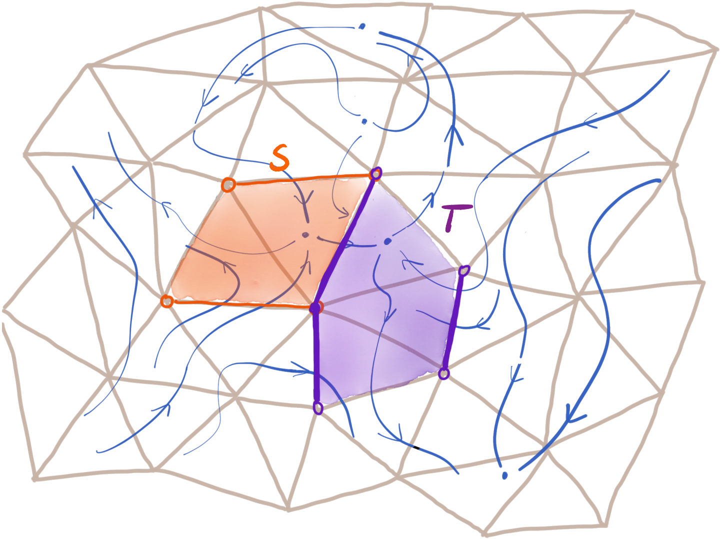


Continuation of an isolated invariant set



Case 3

Continuation of an isolated invariant set



The canonicity of the choice

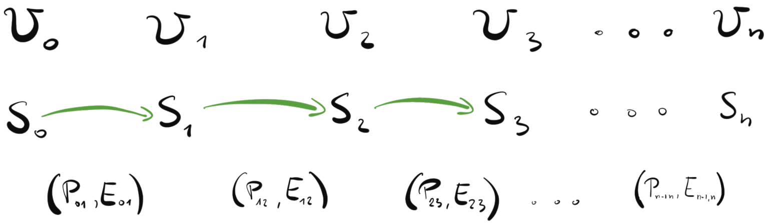
Theorem 25; Dey, L., Mrozek, Slechta (2022)

Let S be an isolated invariant set under $\mathcal{V}$, and let S' denote an isolated invariant set under $\mathcal{V}'$ that is obtained by applying the Tracking Protocol. If S' is obtained via Steps 1, 2, or 3, then $S' \subseteq S$.

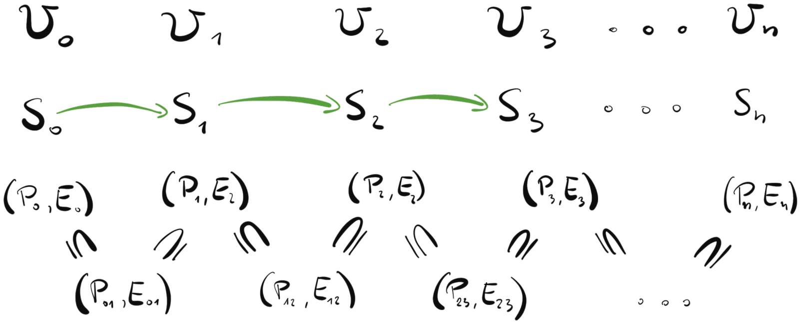
Theorem 26; Dey, L., Mrozek, Slechta (2022)

Let S be an isolated invariant set under $\mathcal{V}$, and let S' denote an isolated invariant set under $\mathcal{V}'$ that is obtained by applying the Tracking Protocol. If S' is obtained via Step 4 then $S \subseteq S'$ or $S' \subseteq S$.

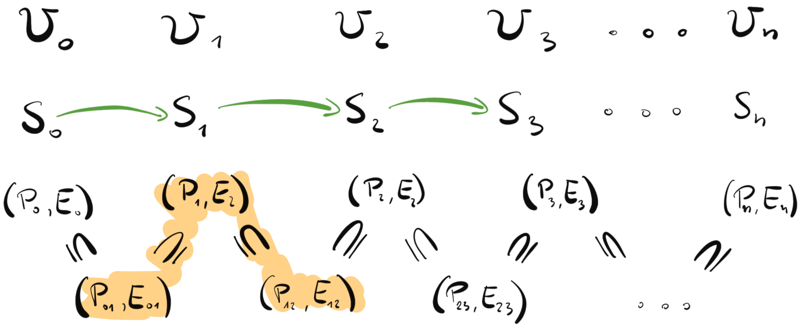
Continuation in terms of persistence



Continuation in terms of persistence



Continuation in terms of persistence



$$(P, E) \supseteq (\mathrm{cl}\, S, \mathrm{mo}\, S) \subseteq (P', E')$$

$$(P, E) \supseteq (\mathrm{cl}\, S, \mathrm{mo}\, S) \subseteq (P', E')$$

$$(\mathrm{cl}(S), \mathrm{mo}(S)) \subseteq$$
$$(\mathrm{pf}_{\mathcal{V}'}(\mathrm{cl}(S), P'), \mathrm{pf}_{\mathcal{V}'}(\mathrm{mo}(S), P')) \supseteq$$
$$(P' \cap \mathrm{pf}_{\mathcal{V}'}(\mathrm{cl}(S), P'), E' \cap \mathrm{pf}_{\mathcal{V}'}(\mathrm{mo}(S), P')) \subseteq (P', E')$$

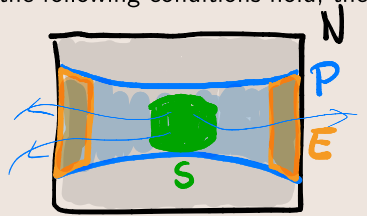
$$(P, E) \supseteq (\mathrm{cl}\, S, \mathrm{mo}\, S) \subseteq (P', E')$$

$$\begin{aligned}(P, E) &\supseteq (P \cap \mathrm{pf}_{\mathcal{V}_i}(\mathrm{cl}(S), P), E \cap \mathrm{pf}_{\mathcal{V}}(\mathrm{mo}(S), P)) \\ &\subseteq (\mathrm{pf}_{\mathcal{V}}(\mathrm{cl}(S), P), \mathrm{pf}_{\mathcal{V}}(\mathrm{mo}(S), P)) \\ &\supseteq (\mathrm{cl}(S), \mathrm{mo}(S)) \subseteq \\ &\quad (\mathrm{pf}_{\mathcal{V}'}(\mathrm{cl}(S), P'), \mathrm{pf}_{\mathcal{V}'}(\mathrm{mo}(S), P')) \supseteq \\ &\quad (P' \cap \mathrm{pf}_{\mathcal{V}'}(\mathrm{cl}(S), P'), E' \cap \mathrm{pf}_{\mathcal{V}'}(\mathrm{mo}(S), P')) \subseteq (P', E')\end{aligned}$$

Connecting index pairs

Let S be isolated invariant set under $\mathcal{V}$ isolated by N. Let P and E be closed sets such that $E \subseteq P$. If the following conditions hold, then (P, E) is an **index pair in** N for S:

1) $\Pi_{\mathcal{V}}(P \setminus E) \subseteq N$,
2) $\Pi_{\mathcal{V}}(E) \cap N \subseteq E$,
3) $\Pi_{\mathcal{V}}(P) \cap N \subseteq P$,
4) $S = \mathrm{inv}_{\mathcal{V}}(P \setminus E)$.



Theorem 21; Dey, L., Mrozek, Slechta (2022)

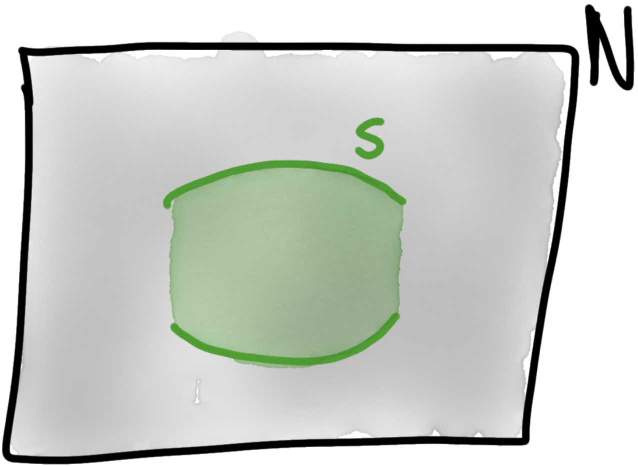
Let (P, E) and (P', E') denote index pairs for S in N under $\mathcal{V}$. The pair $(P \cap P', E \cap E')$ is an index pair for S in N under $\mathcal{V}$.

Theorem 28; Dey, L., Mrozek, Slechta (2022)

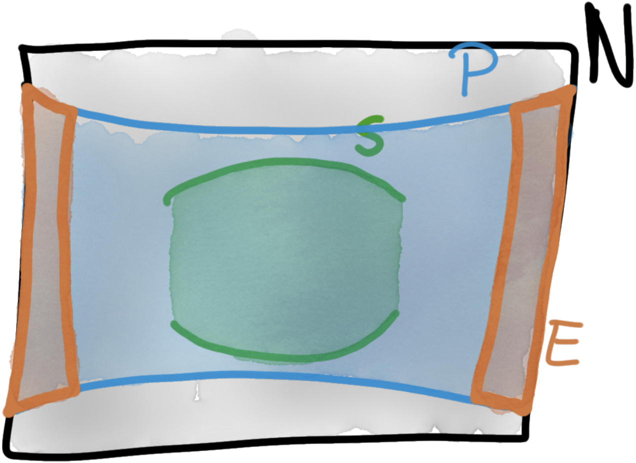
Let (P, E) and (P', E') denote index pairs for S under $\mathcal{V}$ such that $P \subseteq P'$ and $E \subseteq E'$. Then the inclusion $i : (P, E) \hookrightarrow (P', E')$ induces an isomorphism in homology.

The **push-forward of a set** A **in** N is defined as
$\mathrm{pf}_{\mathcal{V}}(A, N) := \{x \in N \mid \exists \rho \in \mathrm{Sol}(x, N), k \in \mathbb{N}\ \rho(0) \in A,\ \rho(k) = x\}$.

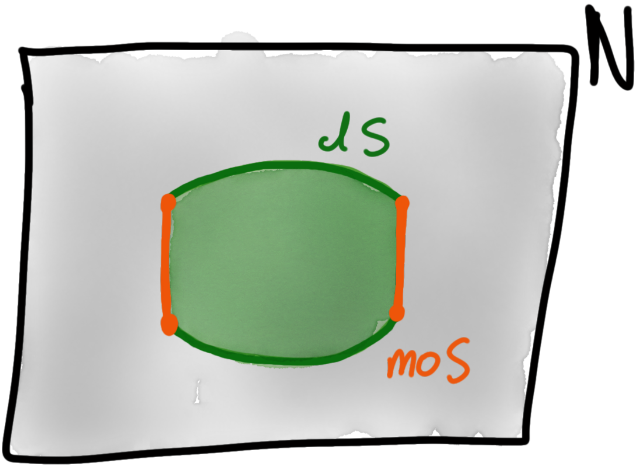
$$\begin{aligned}(\mathrm{cl}(S), \mathrm{mo}(S)) &\subseteq (\mathrm{pf}_{\mathcal{V}'}(\mathrm{cl}(S), P'), \mathrm{pf}_{\mathcal{V}'}(\mathrm{mo}(S), P')) \\ &\supseteq (P' \cap \mathrm{pf}_{\mathcal{V}'}(\mathrm{cl}(S), P'), E' \cap \mathrm{pf}_{\mathcal{V}'}(\mathrm{mo}(S), P')) \\ &\subseteq (P', E')\end{aligned}$$


N
S

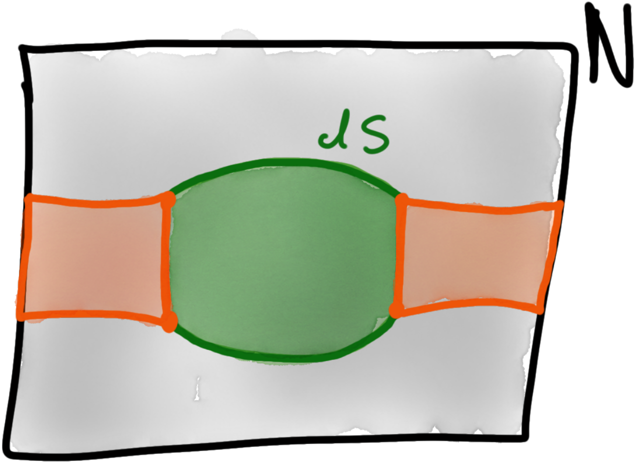
(P, E) in N



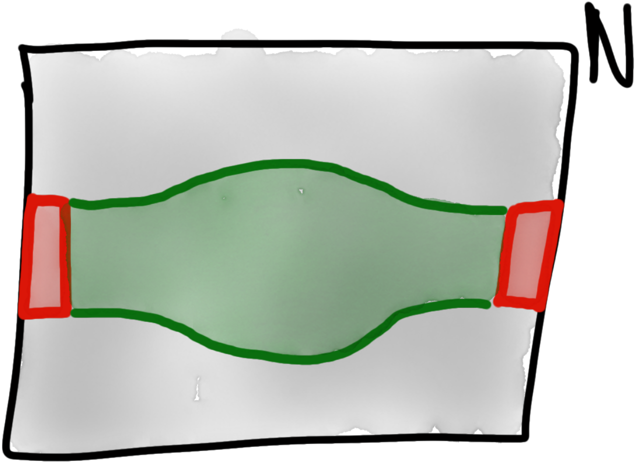
(dS, moS)

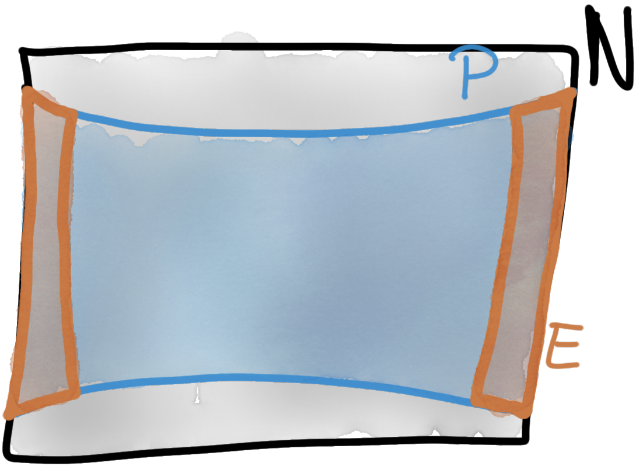


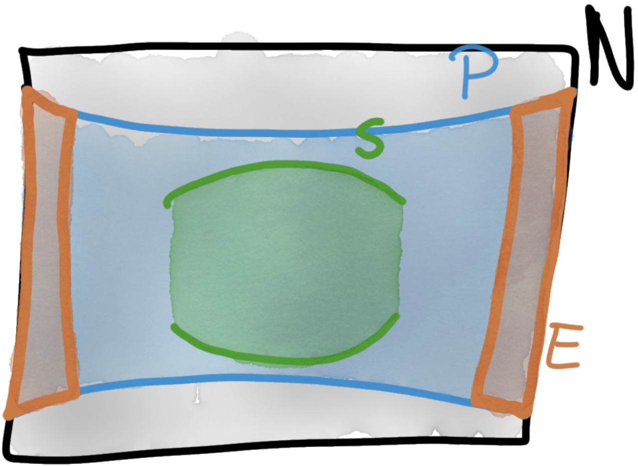
$$(dS, \mathrm{mo}\,S) \subseteq (\mathrm{pf}(dS, N), \mathrm{pf}(\mathrm{mo}S, N))$$



$(\mathrm{pf}(\partial S, N)\cap P, \mathrm{pf}(\mathrm{mo}\,S, N)\cap E)$




N
P
E


N
P
S
E

$$
\begin{aligned}
(P,E) &\supseteq (P \cap \mathrm{pf}_{\mathcal{V}_i}(\mathrm{cl}(S),P), E \cap \mathrm{pf}_{\mathcal{V}}(\mathrm{mo}(S),P)) \\
&\subseteq (\mathrm{pf}_{\mathcal{V}}(\mathrm{cl}(S),P), \mathrm{pf}_{\mathcal{V}}(\mathrm{mo}(S),P)) \\
&\supseteq (\mathrm{cl}(S), \mathrm{mo}(S)) \subseteq \\
&\quad (\mathrm{pf}_{\mathcal{V}'}(\mathrm{cl}(S),P'), \mathrm{pf}_{\mathcal{V}'}(\mathrm{mo}(S),P')) \supseteq \\
&\quad (P' \cap \mathrm{pf}_{\mathcal{V}'}(\mathrm{cl}(S),P'), E' \cap \mathrm{pf}_{\mathcal{V}'}(\mathrm{mo}(S),P')) \subseteq (P',E')
\end{aligned}
$$

Theorem 22; Dey, L., Mrozek, Slechta (2022)

For every $k \geq 0$, the k-dimensional barcode of a connecting sequence of index pairs $\{(P_i, E_i)\}_{i=1}^{n}$ has m bars $[1, n]$ if $\dim H_k(P_1, E_1) = m$.

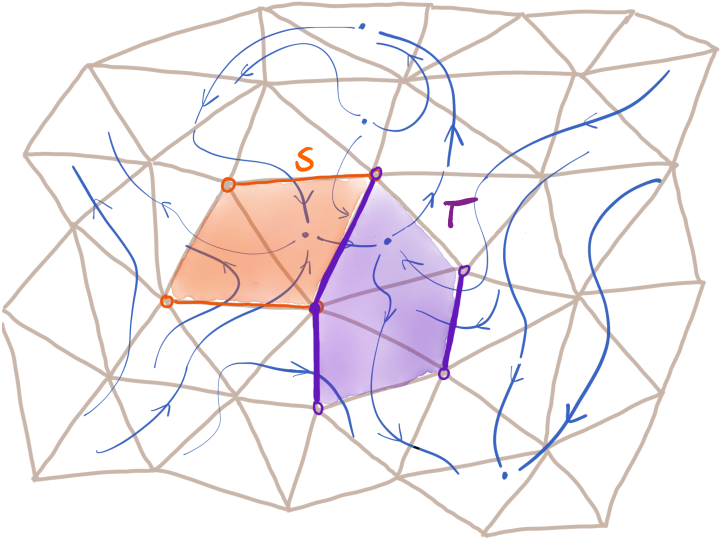
Beyond continuation

Tracking Protocol - continuation

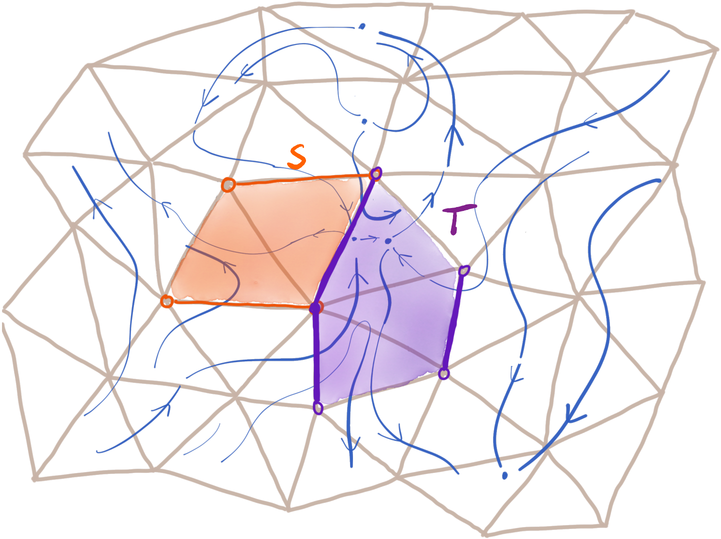
If it is impossible to track via continuation, then attempt to track via persistence:

6. If $A := \langle S \cup V \rangle_{\mathcal{V}}$, then take $S' := \mathrm{inv}_{\mathcal{V}'}(A)$. If S and S' have a common isolating set, then use the technique from the next slide to find a zigzag filtration connecting them.
7. Otherwise, there is no natural choice of S'.

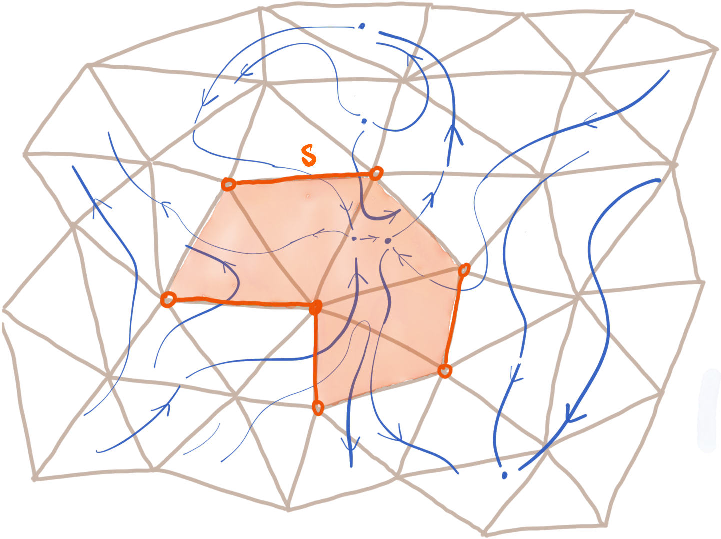
Persistence of an isolated invariant set



Persistence of an isolated invariant set



Persistence of an isolated invariant set



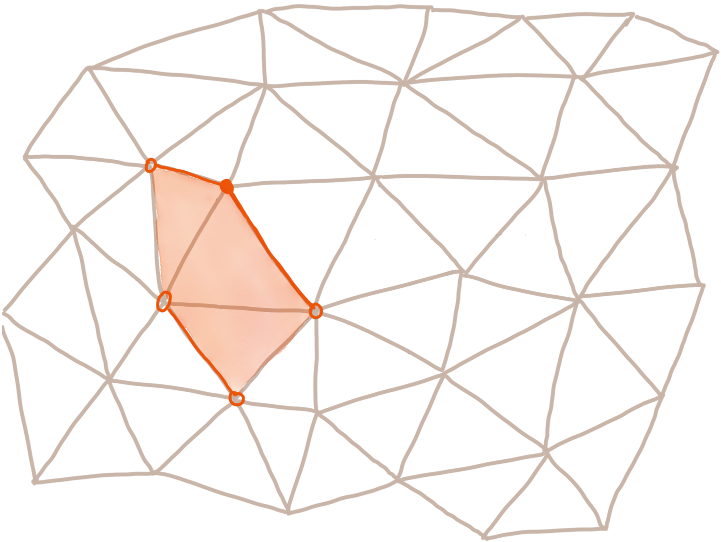
The canonicity of the choice

Theorem 23; Dey, L., Mrozek, Slechta (2022)

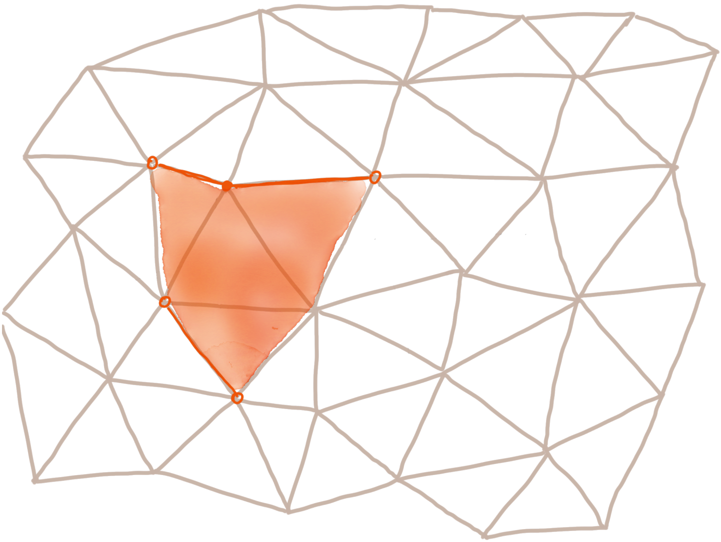
Let S' denote an isolated invariant set under $\mathcal{V}'$ that is obtained from applying Step 6 of the Tracking Protocol to the isolated invariant set S under $\mathcal{V}$. If S'' is an isolated invariant set under V' where $S \subseteq S''$, then $S' \subseteq S''$.

$$
\begin{aligned}
&(\mathsf{cl}(S), \mathsf{mo}(S)) \subseteq (\mathsf{pf}_{\mathcal{V}}(\mathsf{cl}(S), B), \mathsf{pf}_{\mathcal{V}}(\mathsf{mo}(S), B)) \supseteq \\
&(\mathsf{pf}_{\mathcal{V}}(\mathsf{cl}(S), B) \cap \mathsf{pf}_{\mathcal{V}'}(\mathsf{cl}(S'), B), \mathsf{pf}_{\mathcal{V}}(\mathsf{mo}(S), B) \cap \mathsf{pf}_{\mathcal{V}'}(\mathsf{mo}(S'), B)) \\
&\qquad \subseteq (\mathsf{pf}_{\mathcal{V}'}(\mathsf{cl}(S'), B), \mathsf{pf}_{\mathcal{V}'}(\mathsf{mo}(S'), B)) \supseteq (\mathsf{cl}(S'), \mathsf{mo}(S'))
\end{aligned}
$$

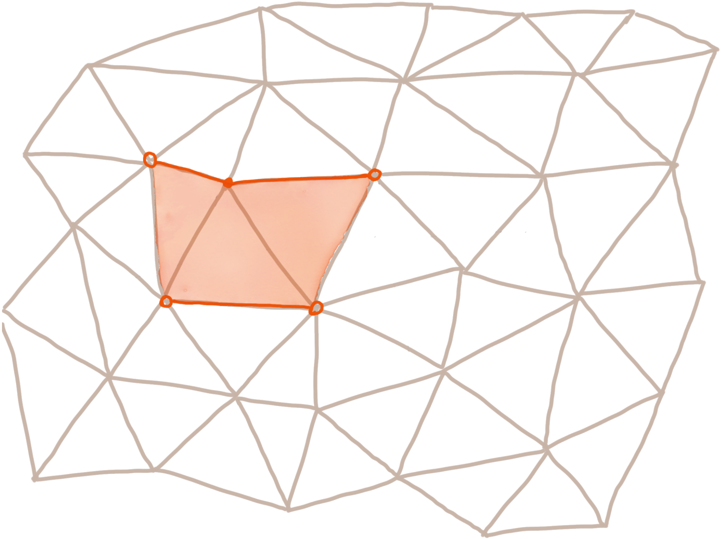
$$(P, E) \supseteq (P \cap \mathrm{pf}_{\mathcal{V}}(\mathrm{cl}(S), P), E \cap \mathrm{pf}_{\mathcal{V}}(\mathrm{mo}(S), P)) \subseteq$$
$$(\mathrm{pf}_{\mathcal{V}}(\mathrm{cl}(S), P), \mathrm{pf}_{\mathcal{V}}(\mathrm{mo}(S), P)) \supseteq (\mathrm{cl}(S), \mathrm{mo}(S))$$



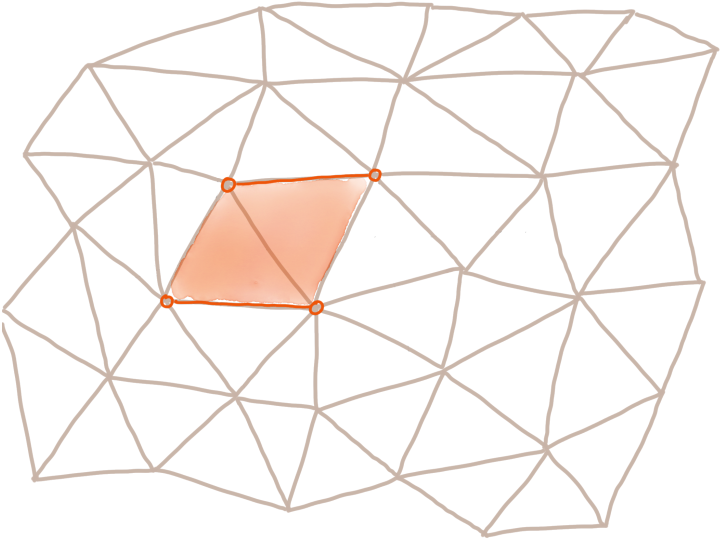
H_1 —



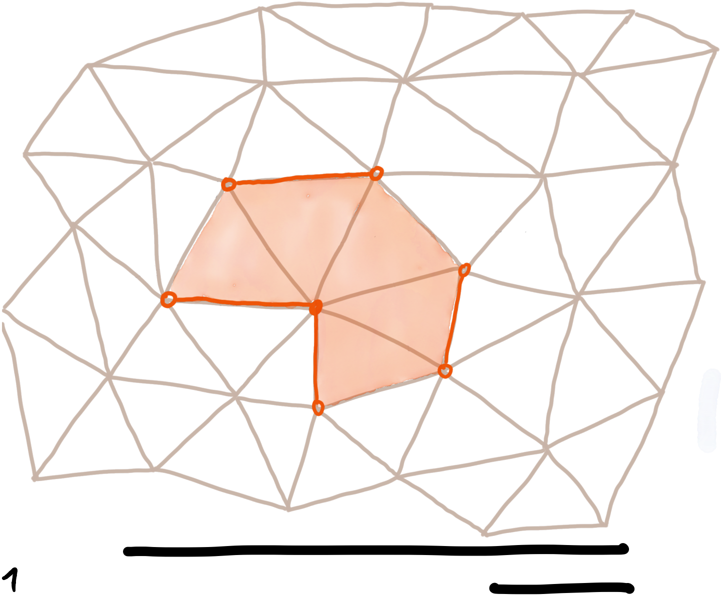
H_1



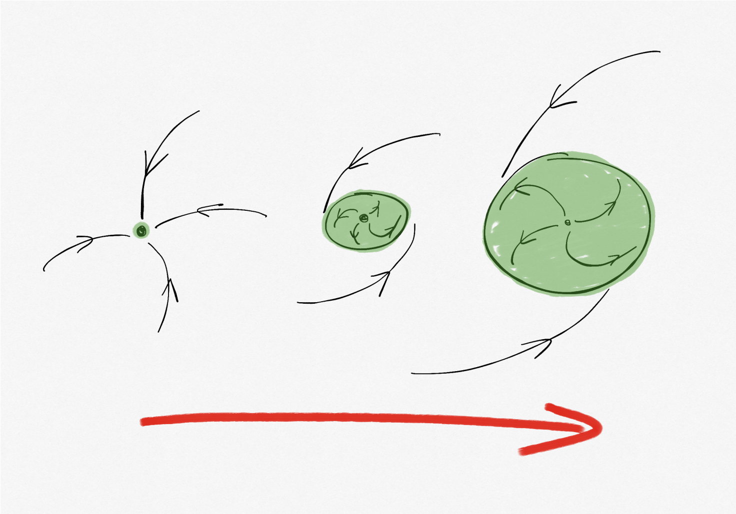
H_1

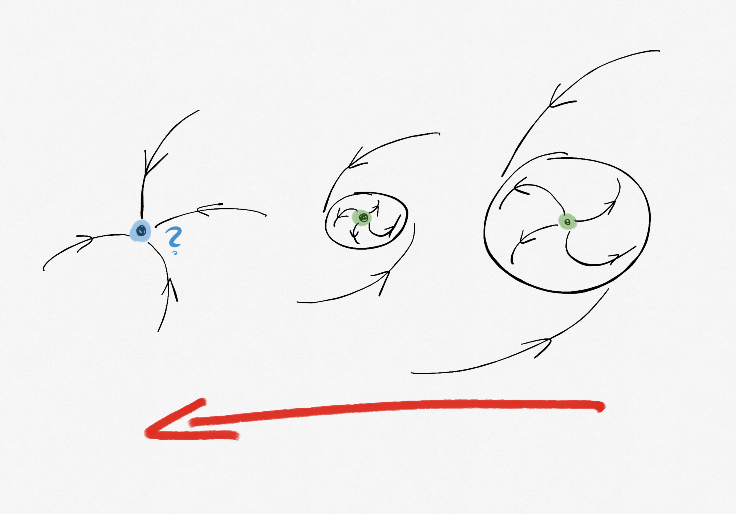


H_1



H_1





Thank you!

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