Simplicial Models for Epistemic Logic

GETCO 2022

Jérémy Ledent

Monday 30 May, 2022

Introduction

The distributed computing setting

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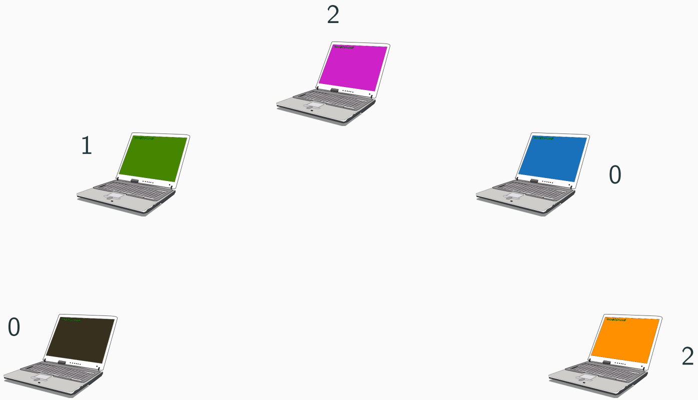

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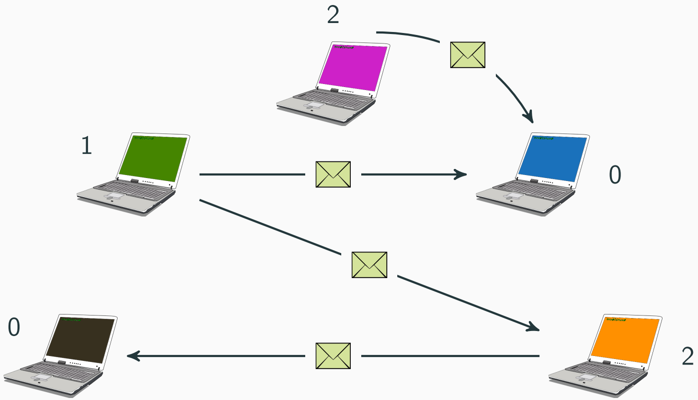

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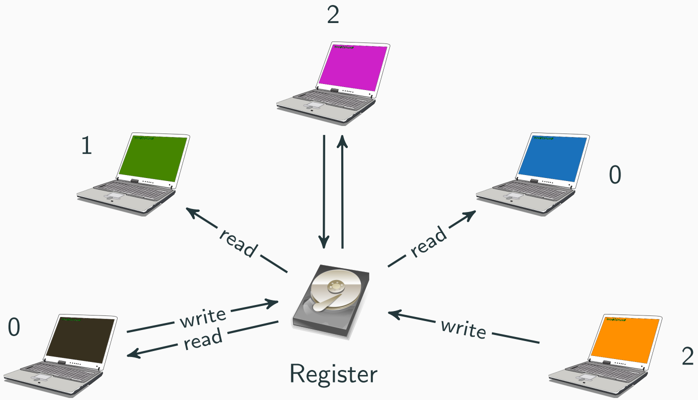

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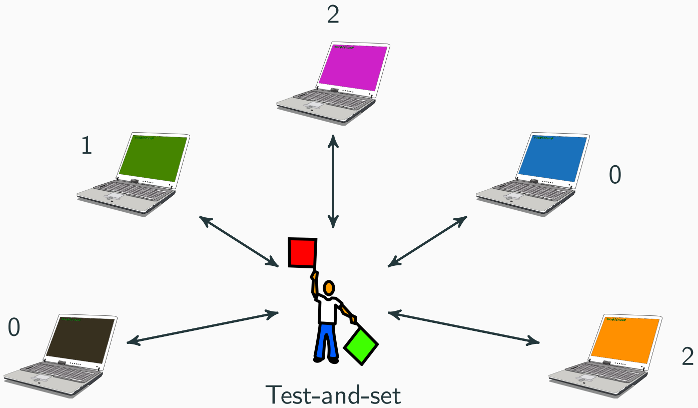

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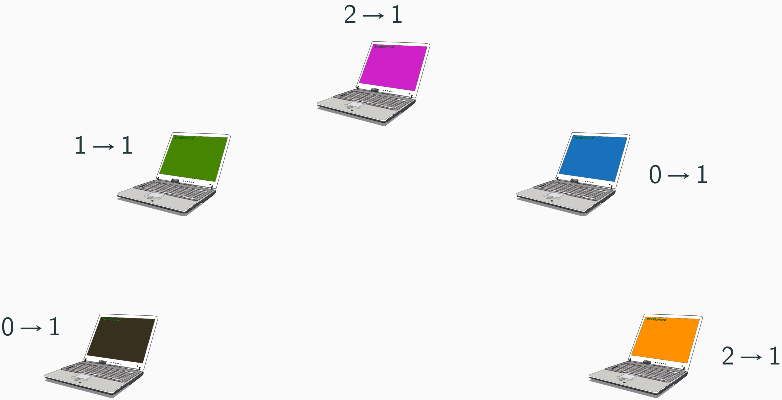

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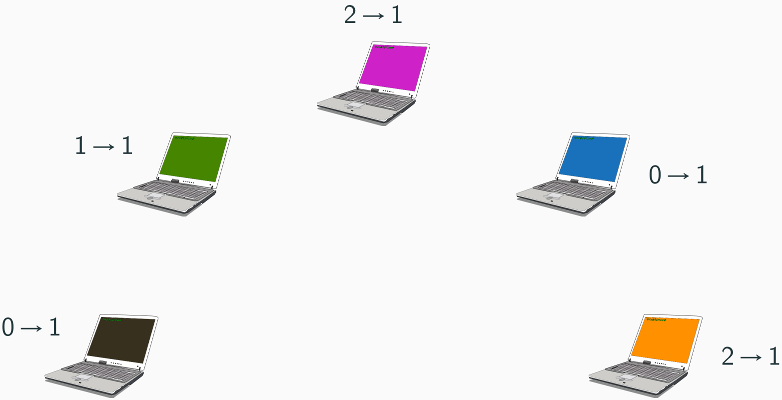

Task specification: $(0,1,2,0,2) \to (1,1,1,1,1)$ ✓ or ✗ ?

Distributed Computability

Goal: prove impossibility results in distributed computing.

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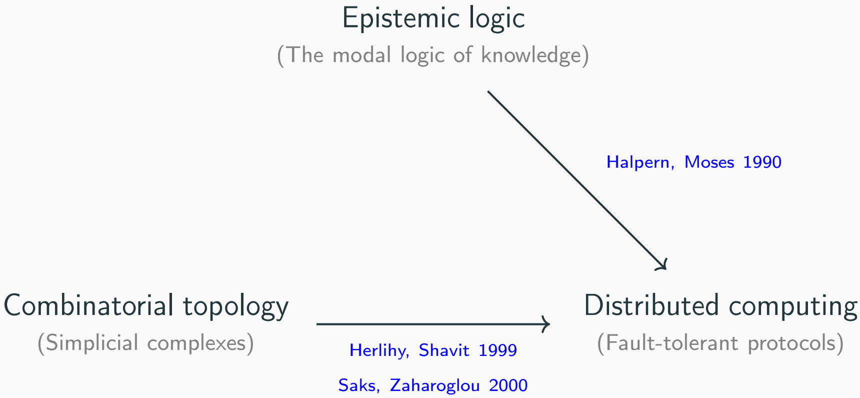
Various methods :

- Valency arguments (e.g. "FLP impossibility")
- Epistemic logic (Halpern and Moses 1990)
- Combinatorial topology (Herlihy and Shavit 1999)

Brief overview of this talk



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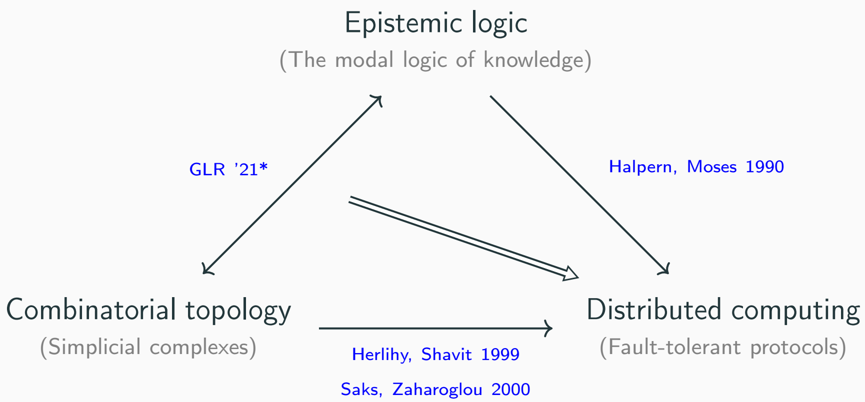

**A Simplicial Complex Model for Dynamic Epistemic Logic to study Distributed Task Computability.*

Goubault, Ledent, Rajsbaum (2021)

Epistemic Logic

Epistemic Logic: Syntax

Let Ag be a finite set of agents and At a set of atomic propositions.

Syntax:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi \qquad\qquad p \in \mathrm{At},\ a \in \mathrm{Ag}$$

Example formula: $K_a \neg K_b \varphi$ where $a, b \in \mathrm{Ag}$

"a knows that b doesn't know that the formula φ is true."

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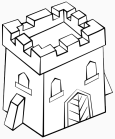
In distributed computing:

Agents	$\longleftrightarrow$	Processes
Atomic propositions	$\longleftrightarrow$	Facts about the system

Example: the two generals problem

Two divisions of the same army, commanded by general A and general B, are surrounding an enemy fortress.

A



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Example: the two generals problem

Two divisions of the same army, commanded by general A and general B, are surrounding an enemy fortress.

- They must attack simultaneously.

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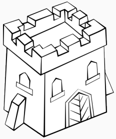

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- They communicate by sending messengers.

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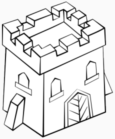

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Two divisions of the same army, commanded by general A and general B, are surrounding an enemy fortress.

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- Messengers might be captured by the enemy, in which case, the message is never received.

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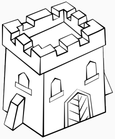

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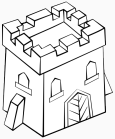

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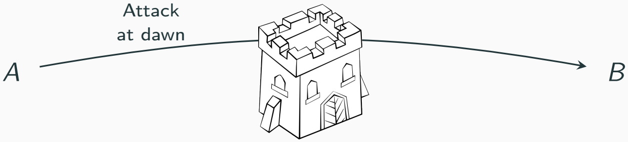

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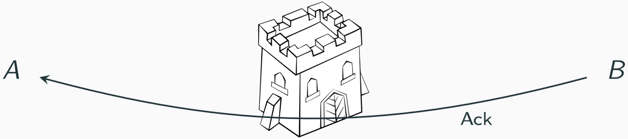

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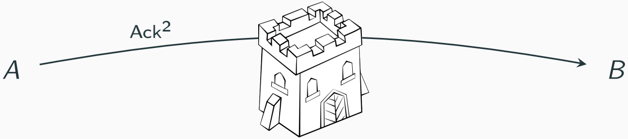


Simplicial Models for Epistemic Logic

Chromatic Simplicial Complexes

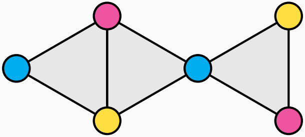
Definition

A **chromatic simplicial complex** is given by (V, S, χ) where:

- (V, S) is a simplicial complex,
- $\chi : V \to \mathrm{Ag}$ is a *coloring* map,

such that every simplex $X \in S$ has all vertices of distinct colors.

Example: a pure chromatic simplicial complex of dimension 2.



Pure Simplicial Models

Assume the number of agents is $|\text{Ag}| = n+1$.

Definition

A pure simplicial model is given by $\mathscr{C} = (V, S, \chi, \ell)$ where:

- (V, S, χ) is a pure chromatic simplicial complex of dimension n.
- $\ell : V \to \mathscr{P}(\text{At})$ is a valuation function.

Example: Consider four cards, $1, 2, 3, 4$, and three agents,
We deal one card to each agent, and keep the remaining card hidden.

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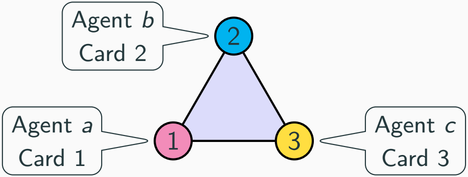

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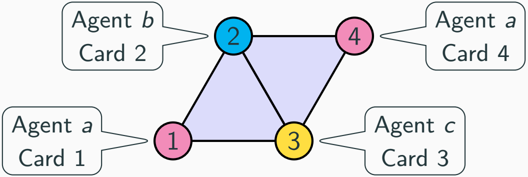

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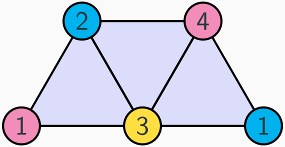

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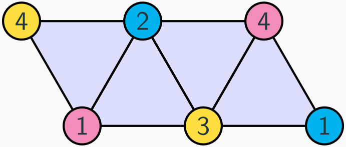

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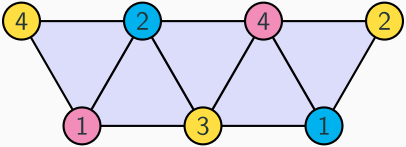

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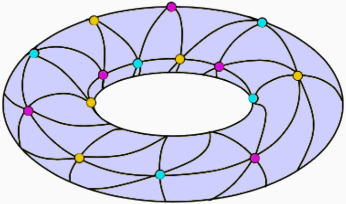

Epistemic Logic: Semantics

We define the validity relation $\mathscr{C}, X \models \varphi$, where:

- $\mathscr{C}$ is a simplicial model,
- $X \in \mathrm{Facet}(\mathscr{C})$ is a world of $\mathscr{C}$,
- φ is an epistemic logic formula.

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By induction on φ:

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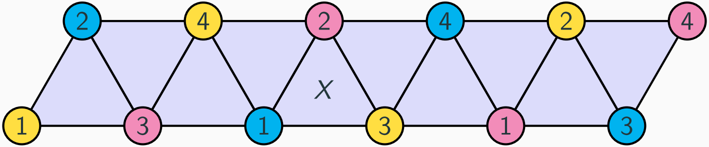
Example: $\mathscr{C}, X \models K_a K_b \,\mathrm{value(c)} \neq 1$ Agents: a, b, c



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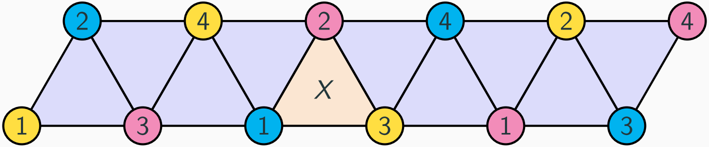

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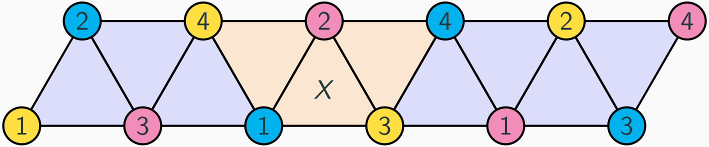

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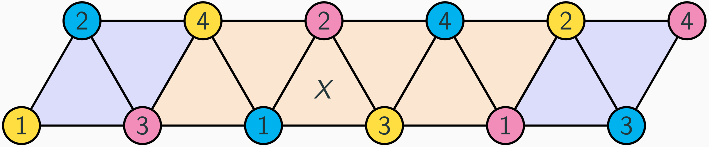


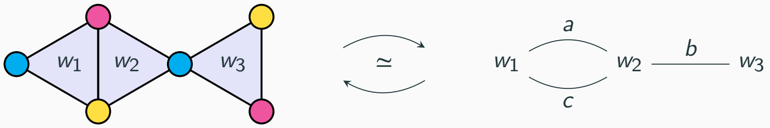
Equivalence with Kripke models

Suppose the number of agents is $|\mathrm{Ag}| = n + 1$.

Theorem (Goubault, Ledent, Rajsbaum (2018, 2021))

The category of pure simplicial models of dimension n is equivalent to the category of proper and local Kripke models.

Example: with three agents, $\mathrm{Ag} = \{$ a, b, c $\}$,



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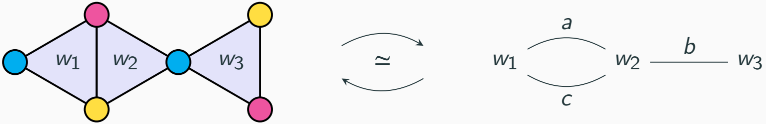
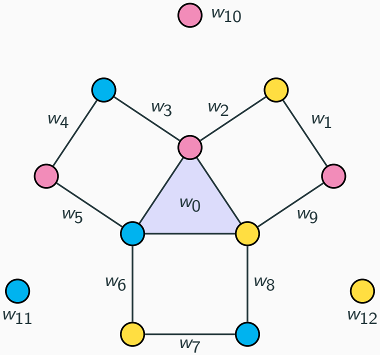


Generalizing Simplicial Models

What about impure simplicial models?

Impure simplicial complexes.

- Common in distributed computing.
- They model systems with detectable crashes.



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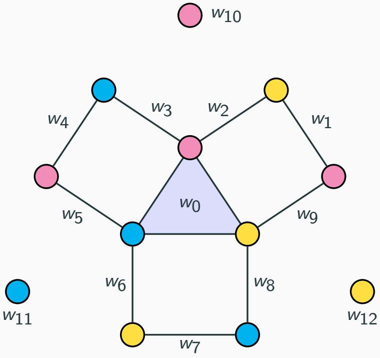
Contributions:

- Find an equivalent class of Kripke models.
- Axiomatise the logic.

A Simplicial Model for KB4: Epistemic Logic with Agents That May Die.

Goubault, Ledent, Rajsbaum (STACS 22)



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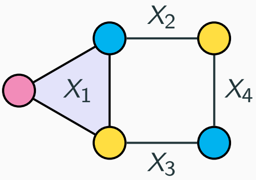
Example: with $\mathrm{Ag} = \{$ a, b, c $\}$ and $\mathrm{At} = \{p\}$.



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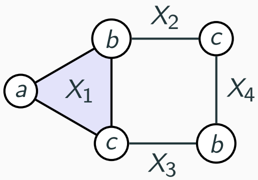

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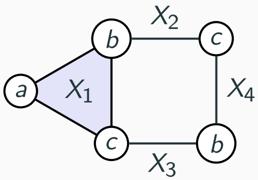

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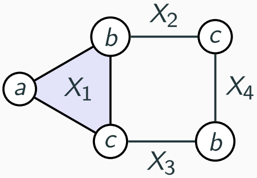

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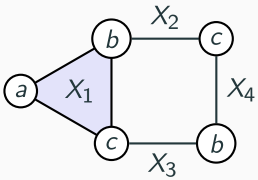

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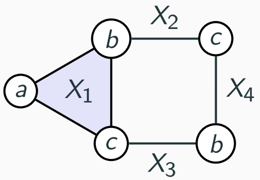

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KB4: Epistemic logic with agents that may die

Define the following formulas, for an agent $a \in \mathrm{Ag}$:

$$\mathrm{dead}(a) := K_a \mathrm{false} \qquad \mathrm{alive}(a) := \neg \mathrm{dead}(a)$$

One can check that:

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One can check that:

$$\mathscr{C}, w \models \mathrm{alive}(a) \quad \text{iff} \quad a \in \chi(w)$$

Example: Some valid formulas in KB4:

- Dead agents know everything: $\mathbf{KB4} \vdash \mathrm{dead}(a) \implies K_a \varphi$.
- Alive agents know they are alive: $\mathbf{KB4} \vdash \mathrm{alive}(a) \implies K_a \, \mathrm{alive}(a)$.
- Alive agents satisfy Axiom **T**: $\mathbf{KB4} \vdash \mathrm{alive}(a) \implies (K_a \varphi \Rightarrow \varphi)$.

Simplicial set models

Definition

A pre-simplicial set is given by a sequence of sets $(S_n)_{n \in \mathbb{N}}$, together with maps $d_i^n : S_n \to S_{n-1}$ for every $n \in \mathbb{N}$ and $0 \leq i \leq n$, satisfying the *simplicial identities*.

$$S_0 \;\underset{d_1}{\overset{d_0}{\leftleftarrows}}\; S_1 \;\overset{d_0,\, d_1,\, d_2}{\longleftarrow\!\!\longleftarrow\!\!\longleftarrow}\; S_2 \;\overset{d_0,\, d_1,\, d_2,\, d_3}{\longleftarrow\!\!\longleftarrow\!\!\longleftarrow\!\!\longleftarrow}\; S_3 \qquad \cdots$$

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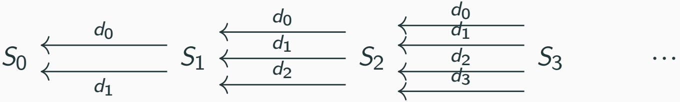
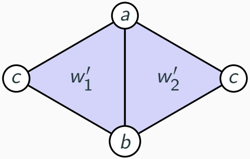
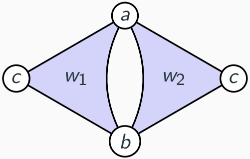




v.s.

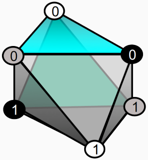


Idea:

- Define simplicial models based on (pre-)simplicial sets.
- What is the associated logic?
- What are some use cases?

Applications to Distributed Computing

Topological characterization of task solvability (Herlihy et al.)



Input complex

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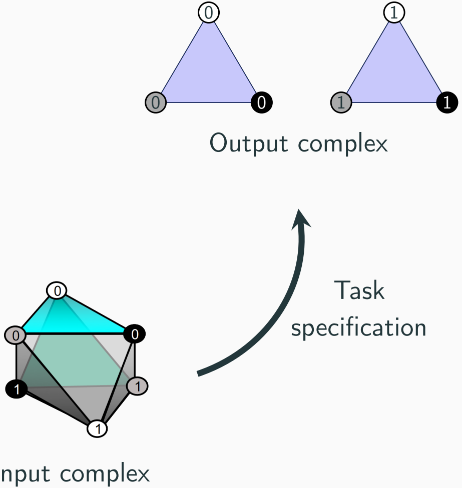

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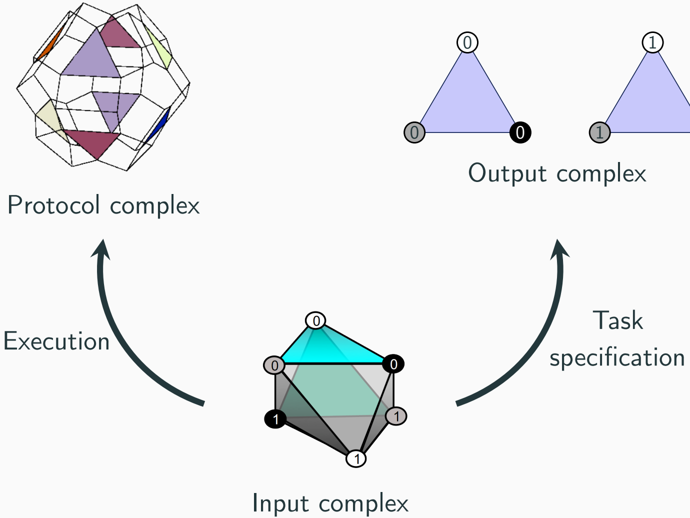

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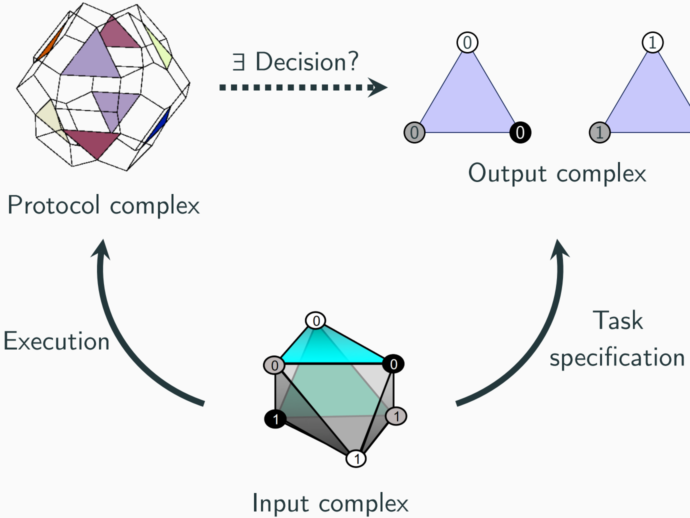

Epistemic proofs of impossibility

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Let $\delta : \mathscr{C} \longrightarrow \mathscr{C}'$ be a morphism of simplicial models, and let φ be a positive formula. Then:

$$\mathscr{C}', \delta(X) \models \varphi \qquad \textit{implies} \qquad \mathscr{C}, X \models \varphi$$

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Recipe for impossibility proofs:

- Assume by contradiction that $\delta : \mathscr{P} \longrightarrow \mathscr{O}$ exists.
- Choose a suitable formula φ such that:
- φ is true everywhere in the output model
- φ is false somewhere in the protocol model

Results

Goubault, Ledent, Rajsbaum (2018, 2021)

- ✓ **Consensus**: impossbility proof using common knowledge.
- ✓ **Approximate agreement**: impossibility proof using iterated knowledge.
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Goubault, Lazić, Ledent, Rajsbaum, *A Dynamic Epistemic Logic analysis of Equality Negation* (2019)

- ✗ **Equality negation**: no formula can prove impossibility.

Research directions

1 – Enrich the logic

Distributed knowledge. $D_B\,\varphi$, where $B \subseteq \mathrm{Ag}$.

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Other topological operators?

Distributed computing	Topology	Logic
consensus	connectedness	common knowledge
k-set agreement	k-connectedness	???

2 – Logical invariants of topological spaces

Topology vs logic: can we characterize topological properties via logical formulas?

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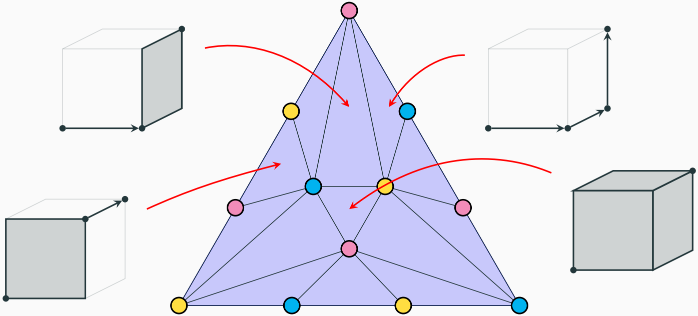
Topology vs logic: can we characterize topological properties via logical formulas?

Examples:

- Is there a φ such that $\mathscr{C} \models \varphi$ iff $\mathscr{C}$ is a (pseudo-)manifold?
- Is there a sound and complete axiomatization for the class of collapsible simplicial models?
- Which logical formulas are preserved under subdivision?

3 – Link with directed topology



Theorem

There is a bijection between facets of the n-dimensional chromatic subdivision and cube chains in the $(n+1)$*-dimensional cube.*

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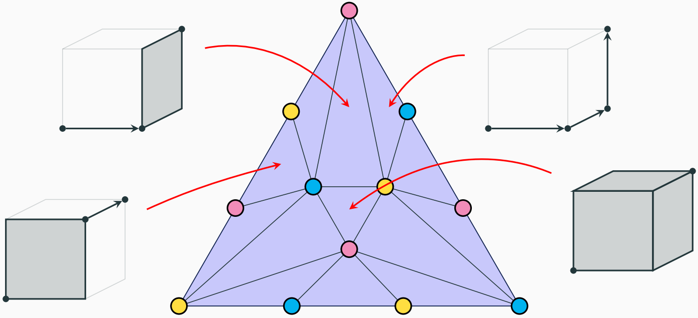

Theorem

There is an order isomorphism between the face poset of the n-dimensional chromatic subdivision and the poset of partial cube chains in the $(n+1)$-dimensional cube.

Thanks!