Computation of Lyapunov functions and contraction metrics for dynamical systems

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Overview

1. Dynamical Systems: Basin of attraction and Lyapunov functions
2. Two construction methods for Lyapunov functions
   - Continuous and piece-wise affine functions (CPA)
   - Meshfree collocation with Radial Basis Functions (RBF)
3. Complete Lyapunov functions
   - Meshfree collocation
   - Quadratic programming
   - New optimisation problem
4. Contraction metrics

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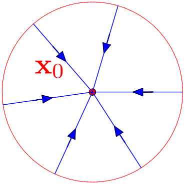
System of autonomous ordinary differential equations



$$(1) \qquad \begin{cases} \frac{d}{dt}\mathbf{x}(t) = \dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t)) \\ \mathbf{x}(0) & = & \boldsymbol{\xi} \end{cases}$$

$\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$
Solution of (1) is called flow and denoted $S_t\boldsymbol{\xi} := \mathbf{x}(t)$

Assumptions

- $\mathbf{x}_0$ is equilibrium $(\mathbf{f}(\mathbf{x}_0) = \mathbf{0})$
- $\mathbf{x}_0$ is asymptotically stable (eigenvalues of $D\mathbf{f}(\mathbf{x}_0)$)

Definition (Basin of attraction) The basin of attraction of $\mathbf{x}_0$ is

$$A(\mathbf{x}_0) := \{\boldsymbol{\xi} \in \mathbb{R}^n \mid \|S_t\boldsymbol{\xi} - \mathbf{x}_0\| \stackrel{t\to\infty}{\longrightarrow} 0\}.$$

$\mathbf{x}_0$ is called globally stable, if $A(\mathbf{x}_0) = \mathbb{R}^n$

In general difficult to determine.

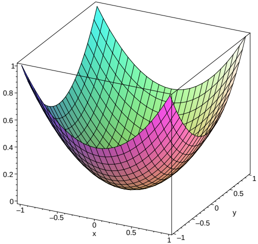
Goal: Determine basin of attraction $A(\mathbf{x}_0)$ using a Lyapunov function

Lyapunov function: idea

Idea

- Lyapunov function is like energy in dissipative system
- It implies stability of equilibrium and gives lower bound on basin of attraction
- Has minimum at equilibrium
- Is strictly decreasing along solutions

$v(x, y)$



Lyapunov function: definition

Definition

Let $\mathbf{x}_0$ be an equilibrium. Let

- $v \in C^1(\mathbb{R}^n, \mathbb{R})$
- $U \subset \mathbb{R}^n$ neighborhood of the equilibrium $\mathbf{x}_0$
- v has strict minimum at equilibrium:
  $v(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in U$ and $v(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{x}_0$
- v is strictly decreasing along trajectories:
  $\dot{v}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in U$ and $\dot{v}(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{x}_0$

Then v is called a strict Lyapunov function and $\mathbf{x}_0$ is asymptotically stable.

$\dot{v}$ derivative along solutions or orbital derivative

Orbital derivative

Definition (Orbital derivative)

Let $v \in C^1(\mathbb{R}^n, \mathbb{R})$. The derivative of v along solutions $S_t\mathbf{x}$ of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, the orbital derivative, is defined as

$$\dot{v}(\mathbf{x}) = \frac{d}{dt} v(S_t\mathbf{x})\Big|_{t=0} = \nabla v(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}(\mathbf{x}) f_i(\mathbf{x})$$

Lyapunov function determines basin of attraction

Theorem

Let $\mathbf{x}_0$ be equilibrium, U open neighborhood of $\mathbf{x}_0$ and $v\colon \mathbb{R}^n \to \mathbb{R}$ strict Lyapunov function.
Let $S_R = \{\mathbf{x} \in \mathbb{R}^n \mid v(\mathbf{x}) \leq R\}$ for $R \in \mathbb{R}_0^+$ be a sublevel set of v and assume that

- *S_R is compact*
- *$S_R \subset U$*

Then $S_R \subset A(\mathbf{x}_0)$ and S_R is positively invariant.

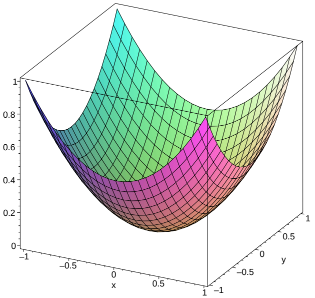
Remark: Positive invariance is true if $\dot{v}(\mathbf{x}) < 0$ holds for all $\mathbf{x} \in \partial S_R = \{\mathbf{x} \in \mathbb{R}^n \mid v(\mathbf{x}) = R\}$

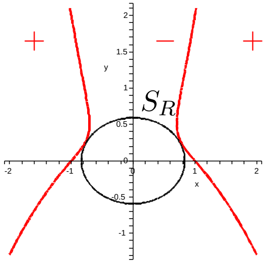
$$\begin{cases} \dot{x} &= -x + x^3 \\ \dot{y} &= -\frac{1}{2}y + x^2 \end{cases}$$

$v(x,y) = \frac{1}{2}x^2 + y^2$

sign of $\dot{v}(x,y) = \nabla v(x,y) \cdot \mathbf{f}(x,y)$

$$= \begin{pmatrix} x \\ 2y \end{pmatrix} \cdot \begin{pmatrix} -x + x^3 \\ -\frac{1}{2}y + x^2 \end{pmatrix}$$





2. Existence and construction of Lyapunov functions

- “converse Theorems” (Massera 1949) etc. – but not constructive!
- explicit construction possible for linear equations, special cases
- used in applications (engineering, biology)

We will present two general construction methods:

- construct continuous and piece-wise affine (CPA) Lyapunov functions
- solve a partial differential equation by meshless collocation with Radial Basis Functions (RBF)

For more methods, see review (Giesl, Hafstein 2015)

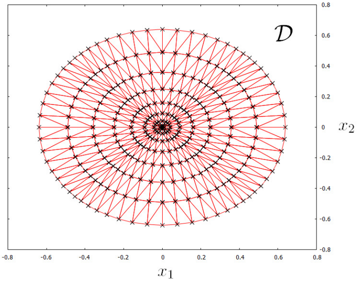
Reminder: for a Lyapunov function we require

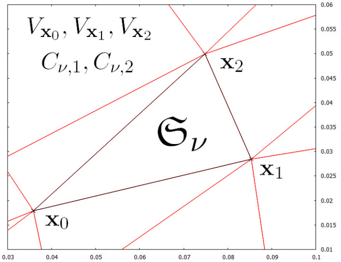
- $V(\mathbf{x}) \geq 0$
- $\dot{V}(\mathbf{x}) \leq 0$

Continuous piece-wise affine function, affine on each simplex

- define triangulation: collection $\mathcal{T}$ of simplices $\mathcal{S} = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$
- vertex set $\mathcal{V}_\mathcal{T}$
- h_C: largest distance of vertices in a simplex

Example of triangulation





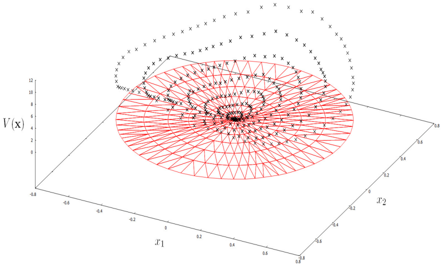
CPA function

CPA function defined by values on vertices:

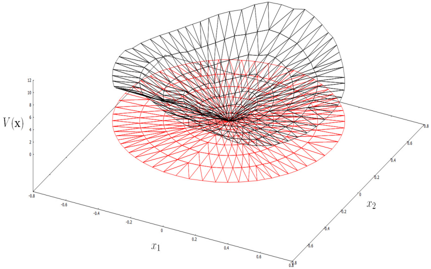
i) Fix $V_{\mathbf{x}}$ for every $\mathbf{x} \in \mathcal{V}_{\mathcal{T}}$ (vertex set)

ii) V is affine on every simplex $\mathcal{S}_\nu \in \mathcal{T}$, i.e. $V(\mathbf{x}) = \mathbf{a}_\nu^T \mathbf{x} + b_\nu$ for $\mathbf{x} \in \mathcal{S}_\nu$ with $\mathbf{a}_\nu \in \mathbb{R}^n$, $b_\nu \in \mathbb{R}$

Values at vertices



CPA function



CPA conditions at vertices

Translate Lyapunov function conditions

1. $V(\mathbf{x}) \geq \|\mathbf{x}\|$
2. $\dot{V}(\mathbf{x}) \leq -\|\mathbf{x}\|$

into sufficient conditions on values at vertices

For every $\mathcal{S}_\nu = \mathrm{co}(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ and every vertex $\mathbf{x}_i \in \mathcal{S}_\nu$,

1. $V(\mathbf{x}_i) \geq \|\mathbf{x}_i\|$
2. $\dot{V}(\mathbf{x}_i) + B_\nu h_C^2 \|\nabla V_\nu\|_1 \leq -\|\mathbf{x}_i\|$ where
   $B_\nu \geq \max_{m,r,s=1,\ldots,n} \max_{\mathbf{x}\in\mathcal{S}_\nu} \left|\frac{\partial^2 f_m}{\partial x_r \partial x_s}(\mathbf{x})\right|$, h_C size of simplex

Remarks

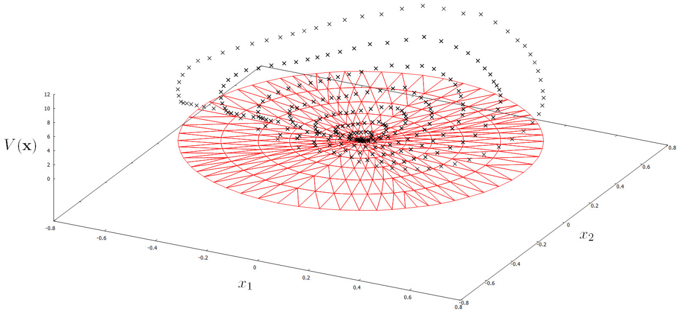
- B_ν need to be input by hand – only upper bounds necessary
- V is not differentiable, but smooth on each simplex
- Constraints are linear in $V(\mathbf{x}_i)$

Solution to the LP problem $\Longrightarrow$ CPA Lyapunov function

- Write conditions as constraints of Linear Programming (LP) problem with variables $V_{\mathbf{x}_i}$
- If the LP problem has a solution, then the CPA function is a Lyapunov function
- Note: not a numerical approximation, V is a Lyapunov function!
- Moreover, if the triangulation is sufficiently fine, then the method always finds a Lyapunov function

Example

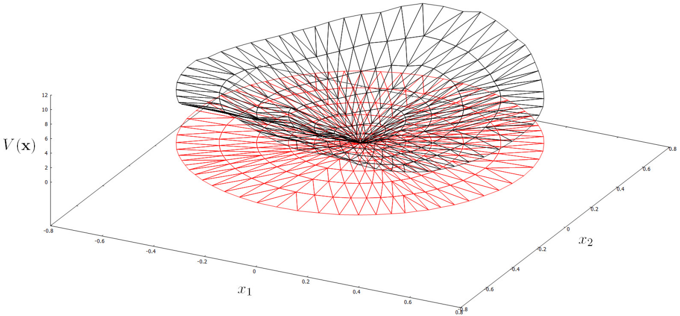
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 + x_1^3/3 - x_2 \end{pmatrix}, \quad B_\nu = 2 \max_{\mathbf{x} \in \mathcal{S}_\nu} |x_1|$$



$V_{\mathbf{x}}$ solution to the LP problem

Example

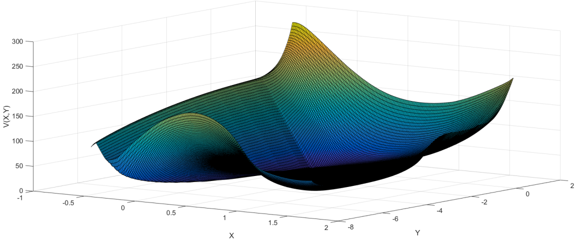
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 + x_1^3/3 - x_2 \end{pmatrix}, \quad B_\nu = 2 \max_{\mathbf{x} \in \mathcal{S}_\nu} |x_1|$$



CPA Lyapunov function

Computing sublevel sets

- find connected component which includes equilibrium
- increase level
- until boundary of admissible area reached
- results in subset of basin of attraction
- algorithm for CPA functions



2.2 Meshfree collocation with Radial Basis Functions (RBF)

Converse Theorem

Theorem (Existence of V)

Let $\mathbf{f} \in C^\sigma(\mathbb{R}^n, \mathbb{R}^n)$, $\sigma \geq 1$, $\mathbf{0}$ exponentially stable equilibrium. Then there exists $V \in C^\sigma(A(\mathbf{0}), \mathbb{R})$ with

(2) $$\dot{V}(\mathbf{x}) := \nabla V(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) = -\|\mathbf{x}\|^2 \text{ for all } \mathbf{x} \in A(\mathbf{0}).$$

Proof: $V(\mathbf{x}) = \int_0^\infty \|S_t\mathbf{x}\|^2 \, dt$

Goal: explicit construction of Lyapunov function

Idea: approximate solution of first-order linear PDE (2)

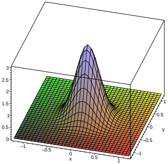
Overview

- Consider linear PDE

(PDE) $$LV(\mathbf{x}) = \dot{V}(\mathbf{x}) = \sum_{i=1}^{n} f_i(\mathbf{x}) \frac{\partial V}{\partial x_i}(\mathbf{x}) = -\|\mathbf{x}\|^2$$

- Approximation V_R of V using Meshless collocation, in particular Radial Basis Functions (RBF)
- Approximation V_R itself is a Lyapunov function

Radial Basis Functions: approximate solution of PDE



- PDE: $LV(\mathbf{x}) = -\|\mathbf{x}\|^2$, L linear differential operator (orbital derivative)
- $\psi_k(\|\mathbf{x}\|)$ (Radial Basis Function), here: ψ_k Wendland's function (compact support)
- Corresponds to Reproducing Kernel Hilbert space H of functions with kernel $\Phi(\mathbf{x}, \mathbf{y}) := \psi_k(\|\mathbf{x} - \mathbf{y}\|)$ (Sobolev space)
- Collocation points $X_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{R}^n$, $\lambda_i := (\delta_{\mathbf{x}_i} \circ L) \in H^*$
- Solution of problem

$$\begin{cases} \text{minimise} & \|V\|_H \\ \text{subject to} & LV(\mathbf{x}_i) = -\|\mathbf{x}_i\|^2, \ \forall \mathbf{x}_i \in X_N \end{cases}$$

  is $V_R(\mathbf{x}) = \sum_{j=1}^N \alpha_j \lambda_j^{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y})$
- $\alpha \in \mathbb{R}^N$ determined by: $\dot{V}_R(\mathbf{x}_j) = -\|\mathbf{x}_j\|^2$ for all $j = 1, \ldots, N$, i.e.
- $A\alpha = r$, where $a_{ij} = \lambda_i^{\mathbf{x}} \lambda_j^{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y})$, $r_i = -\|\mathbf{x}_i\|^2$
- A is symmetric and positive definite $\Rightarrow$ non-singular

Error Estimate

$$|\dot V(\mathbf{x}) - \dot V_R(\mathbf{x})| \le C h_R^{k-1/2} \text{ for all } \mathbf{x} \in K$$

where

- k smoothness of Radial Basis Function
- $h_R := \sup_{\mathbf{y}\in K} \min_{\mathbf{x}\in X_N} \|\mathbf{x}-\mathbf{y}\|$: fill distance, measuring how dense collocation points are in K

Estimate

V_R is Lyapunov function: if $C h_R^{k-1/2} \le \varepsilon$, then

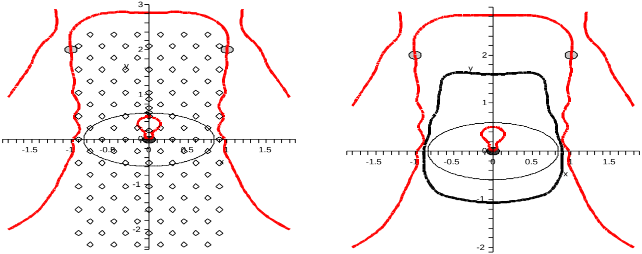
$$\dot V_R(\mathbf{x}) \le \dot V(\mathbf{x}) + \varepsilon \le -\|\mathbf{x}\|^2 + \varepsilon < 0$$

for $\|\mathbf{x}\|^2 > \varepsilon$ (local problem)

Example

$$\begin{cases} \dot{x} &= -x + x^3 \\ \dot{y} &= -\frac{1}{2}y + x^2 \end{cases}$$

Grid, $\dot{v} = 0$, sublevel set (thick black), previous sublevel set (thin black)



Problem: verification

Problem:

- How to verify $\dot{V}_R(\mathbf{x}) < 0$ for all $\mathbf{x} \in K$ (infinitely many)?
- Error estimate depends on V and is not known in practice

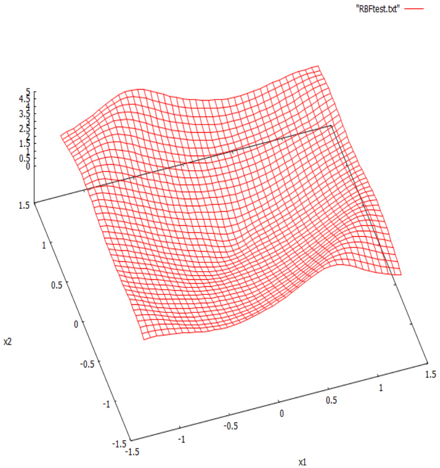
Two methods:

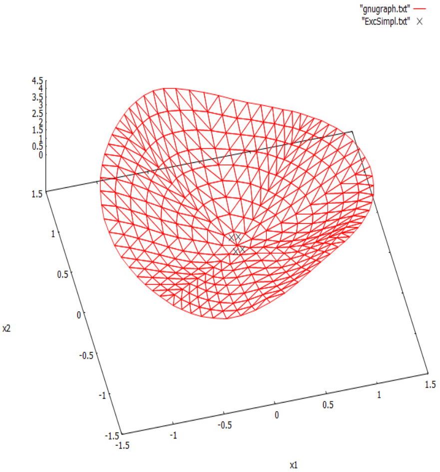
1. Evaluate computed function at finitely many points, apply Taylor approximation with explicit bounds on derivatives.
   Many evaluation points, but verifies computed function
2. Use CPA (continuous piecewise affine) interpolation V_C of V_R and use verification as discussed earlier.
   Much faster, but verifies different function

Both methods can be shown to always work if evaluation/interpolation is sufficiently fine.

Example 1: $\dot{x} = -y$, $\dot{y} = x + y(x^2 - 1)$

RBF approximation 19×19 – CPA interpolation, x – inequality violated 4/648



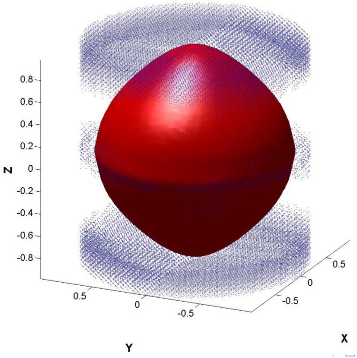


Example 2

System:

$$\dot{\mathbf{x}} = \begin{pmatrix} x(x^2+y^2-1) - y(z^2+1) \\ y(x^2+y^2-1) + x(z^2+1) \\ 10z(z^2-1) \end{pmatrix}.$$

Level set (red); orbital derivative (of CPA interpolation) is not negative (blue dots)



Summary

Lyapunov function

- minimum at equilibrium ($v(\mathbf{x}) \geq 0$)
- decreasing along solutions ($\dot{v}(\mathbf{x}) \leq 0$, orbital derivative)
- gives information about basin of attraction/positively invariant sets through (sub)level sets

Construction methods

- CPA: affine function on simplices, conditions as constraints of Linear Programming problem
- RBF: smooth function, conditions as solution of linear Partial Differential Equation

Comparison: CPA vs. RBF

- CPA and RBF work on compact set and have problems close to equilibrium
- CPA: inequalities, RBF needs equation
- CPA slow but delivers a true (nonsmooth) Lyapunov function
- RBF (comparatively) fast, smooth function, but separate verification is necessary
- CPA and RBF are guaranteed to succeed if sufficiently fine simplices/dense collocation points

References

Review

- P. Giesl & S. Hafstein, Review on computational methods for Lyapunov functions, Discrete Cont. Dyn. Syst. Ser. B, **20** (2015), 2291–2331.

RBF

- P. Giesl, *Construction of Global Lyapunov Functions Using Radial Basis Functions*, Lecture Notes in Math. 1904, Springer, 2007.
- P. Giesl & H. Wendland, *Meshless Collocation: Error Estimates with Application to Dynamical Systems*, SIAM J. Numer. Anal. 45 No. 4 (2007), 1723–1741.

CPA

- S. Hafstein, "An algorithm for constructing Lyapunov functions", Electron. J. Differential Equ. Monogr., **8** (2007).

RBF-CPA

- P. Giesl & S. Hafstein, Computation and Verification of Lyapunov functions, SIAM J. Applied Dyn. Syst., **14** (2015), 1663–1698.

3. Complete Lyapunov functions

Classical Lyapunov functions: for one attractor, e.g. equilibrium or periodic orbit. Now consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x} \in \Omega \subseteq \mathbb{R}^n$.

- Phase space can be split into chain-recurrent set $\mathcal{R}$ (containing equilibria, periodic orbits, attractors, repellers, etc.) and the complement with gradient-like flow
- Complete Lyapunov function (Conley) $V : \Omega \to \mathbb{R}$ satisfies
  - $\dot{V}(\mathbf{x}) < 0$ (decreasing along solutions) on gradient-like part (transient behaviour)
  - $\dot{V}(\mathbf{x}) = 0$ on chain-recurrent set, has distinct values on distinct chain-transitive components of $\mathcal{R}$

(Conley 1978, 1988), (Hurley 1991, 1998), (Osipenko 2007), (Patrao 2011)

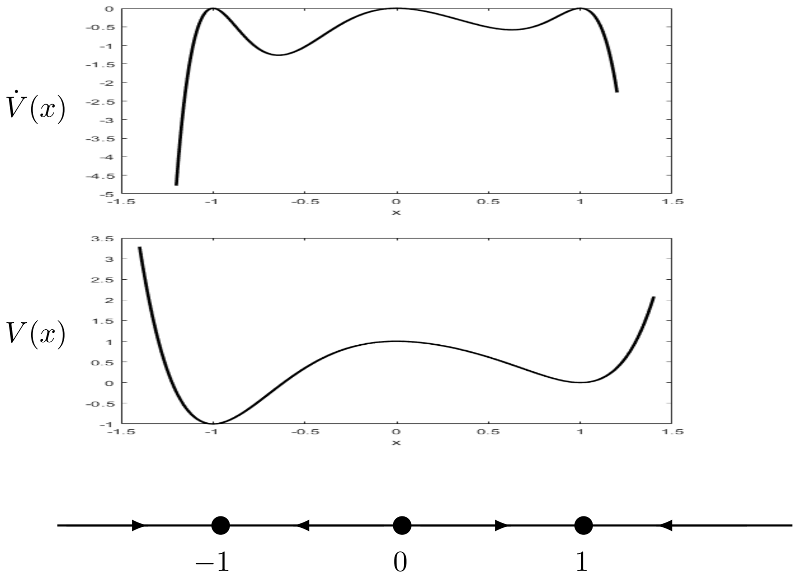
Construction of complete Lyapunov functions

- Goal to find a (candidate) complete Lyapunov function satisfying $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Omega$
- with large area $\{x \in \Omega \mid \dot{V}(\mathbf{x}) < 0\}$

Then

- $\{\mathbf{x} \in \Omega \mid \dot{V}(\mathbf{x}) = 0\}$ contains chain-recurrent set
- Maxima/minima of V indicate stability

Example: $\dot{x} = -x(x^2 - 1)$

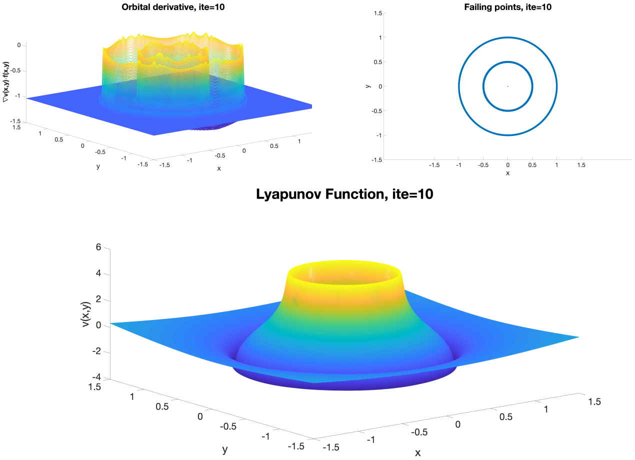


3.1 Numerical construction: Meshfree collocation RBF (equation)

Solve $\dot{V}(\mathbf{x}) = -1$ via meshless collocation.

- the PDE has no solution on chain-recurrent set
- meshless collocation has solution v
- set where $\dot{V}_R(\mathbf{x}) \approx 0$ approximates chain-recurrent set $\mathcal{R}$
- iterations for better approximations
- no proof of convergence
- software LyapXool

Complete Lyapunov function: equation



More appropriate: differential equation/inequality

solve

$$\begin{cases} \text{minimise} & \|V\|_H \\ \text{subject to} & \dot{V}(\mathbf{x}) = -1 \text{ for all } \mathbf{x} \in \Gamma \\ & \dot{V}(\mathbf{x}) \leq 0 \text{ for all } \mathbf{x} \in \Omega(\setminus\Gamma) \end{cases}$$

Remarks:

- We need to ensure Γ lies in the gradient-flow part
- How large does Γ need to be?
- Can we assume $\dot{V}(\mathbf{x}) = -1$ in the gradient-flow part; does such a function exist?
- How do we find a (numerical) solution?

Existence of complete Lyapunov function with prescribed derivative

Theorem (Giesl, Suhr, Hafstein (2022))

Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ define a dynamical system on an open set $\Omega \subset \mathbb{R}^d$ with $\mathbf{f} \in C^l(\Omega, \mathbb{R}^d)$, where $l \in \mathbb{N} \cup \{\infty\}$.
Then for every compact set $K \subset \Omega \setminus \mathcal{R}$ and every C^l-function $g \colon \Omega_K \to (-\infty, 0)$ defined on a neighborhood $\Omega_K \subset \Omega$ of K there exists a complete C^l-Lyapunov function $V : \Omega \to \mathbb{R}$ with

- $\dot{V}(\mathbf{x}) = g(\mathbf{x})$ *for all* $\mathbf{x} \in K$ *and*
- $\dot{V}(\mathbf{x}) < 0$ *for all* $\mathbf{x} \in \Omega \setminus \mathcal{R}$.

Proof is based on a modification of a C^∞ complete Lyapunov function from (Hafstein, Suhr 2021)

Hence, we can set $g(\mathbf{x}) = -1$

3.2 Discretising differential inequalities

- Let $\Gamma \subset \Omega \subset \mathbb{R}^d$
- Goal: solve

$$\begin{cases} Lv(\mathbf{x}) = -1, \ \forall \mathbf{x} \in \Gamma, \\ Lv(\mathbf{x}) \le 0, \ \forall \mathbf{x} \in \Omega \setminus \Gamma \end{cases}$$

- L is a linear (differential) operator
- Consider (Reproducing Kernel) Hilbert space H of functions $v\colon \Omega \to \mathbb{R}$ with kernel $\Phi(\mathbf{x}, \mathbf{y})$
- Optimisation problem for $v \in H$

$$\begin{cases} \text{minimise} & \|v\|_H \\ \text{subject to} & Lv(\mathbf{x}) = -1, \ \forall \mathbf{x} \in \Gamma, \\ & Lv(\mathbf{x}) \le 0, \ \forall \mathbf{x} \in \Omega \setminus \Gamma. \end{cases}$$

Discretising differential inequalities: convergence

- Continuous problem:

$$(3)\quad \begin{cases} \text{minimise} & \|v\|_H \\ \text{subject to} & Lv(\mathbf{x}) = -1, \ \forall \mathbf{x} \in \Gamma, \\ & Lv(\mathbf{x}) \le 0, \ \forall \mathbf{x} \in \Omega \setminus \Gamma. \end{cases}$$

- Meshfree collocation: discretise problem. Given discrete (regular) points $X_\Gamma \subset \Gamma$, $X_\Omega \subset \Omega \setminus \Gamma$, solve

$$(4)\quad \begin{cases} \text{minimise} & \|v\|_H \\ \text{subject to} & Lv(\mathbf{x}_i) = -1, \ \forall \mathbf{x}_i \in X_\Gamma, \\ & Lv(\mathbf{x}_i) \le 0, \ \forall \mathbf{x}_i \in X_\Omega. \end{cases}$$

Results:

- (3) and (4) have unique solution
- (4) can be solved by quadratic optimisation
- Strong convergence in H of solutions of discretised problem (4) to solution of continuous system (3)

Discretised version: quadratic optimisation

H RKHS with kernel Φ, $M, N \in \mathbb{N}$, $\lambda_i = (\delta_{x_i} \circ L) \in H^*$, $i = 1, \ldots, M+N$ linearly independent

$$(5) \qquad \begin{cases} \text{minimise} & \|v\|_H \\ \text{subject to} & \lambda_i(v) = -1, \quad i = 1, \ldots, M, \\ & \lambda_{M+i}(v) \leq 0, \quad i = 1, \ldots, N. \end{cases}$$

Then

- (5) has unique minimiser $v^*(\mathbf{x}) = \sum_{j=1}^{M+N} \alpha_j \lambda_j^{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y})$
- coefficients α_j are the unique solution of the minimisation problem

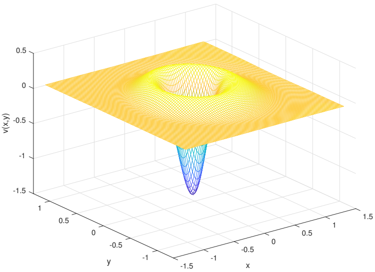
$$(6) \qquad \begin{cases} \text{minimise} & \alpha^T A \alpha \\ \text{subject to} & A_1\,\alpha \quad = -\mathbf{1} \quad \in \mathbb{R}^M \\ \text{and} & A_2\,\alpha \quad \leq \mathbf{0} \quad \in \mathbb{R}^N. \end{cases}$$

$A = (a_{ij}) = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, $A_1 \in \mathbb{R}^{M \times (M+N)}$, $A_2 \in \mathbb{R}^{N \times (M+N)}$ and $a_{ij} = \lambda_i^{\mathbf{x}} \lambda_j^{\mathbf{y}} \Phi(\mathbf{x}, \mathbf{y})$

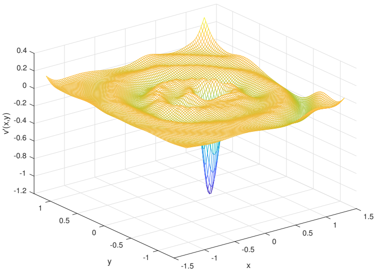
Example 1: two periodic orbits

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x(x^2+y^2-1/4)(x^2+y^2-1)-y \\ -y(x^2+y^2-1/4)(x^2+y^2-1)+x \end{pmatrix}$$

$v(x,y)$



$\dot{v}(x,y)$

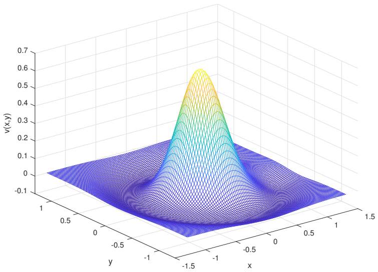


$\dot{v}(0.1846, 0) = -1$ by the equality constraint

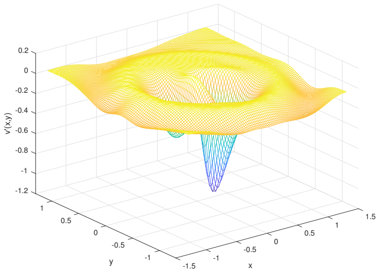
Example 2: homoclinic orbit

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(1-x^2-y^2) - y((x-1)^2 + (x^2+y^2-1)^2) \\ y(1-x^2-y^2) + x((x-1)^2 + (x^2+y^2-1)^2) \end{pmatrix}$$

$v(x,y)$



$\dot{v}(x,y)$



$\dot{v}(0.1846, 0) = -1$ by the equality constraint

3.3 New optimisation problem

- Drawback of previous approach: some knowledge of chain-recurrent set (for equality condition)
- $$\begin{cases} \text{minimise} & \|V\|_H \\ \text{subject to} & \dot{V}(\mathbf{x}) \le 0 \text{ for all } \mathbf{x} \in \Omega \end{cases}$$

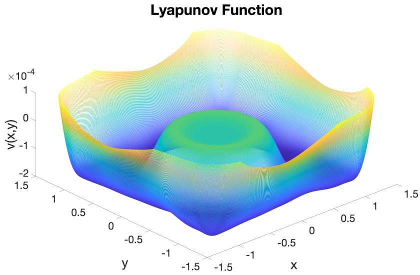
  has trivial solution $V \equiv 0$

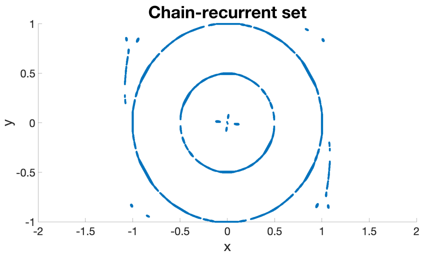
New idea (Giesl, Argáez, Hafstein, Wendland 2021): consider

$$\begin{cases} \text{minimise} & \|V\|_H^2 + \int_\Omega \dot{V}(\mathbf{x})\, d\mathbf{x} \\ \text{subject to} & \dot{V}(\mathbf{x}) \le 0 \text{ for all } \mathbf{x} \in \Omega \end{cases}$$

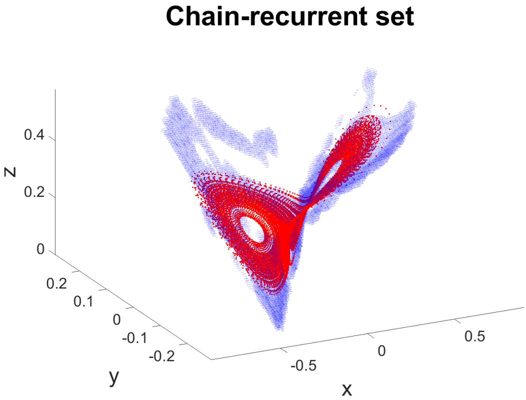
- Cost function rewards areas with negative orbital derivative
- No knowledge of gradient-like flow required
- Still leads to quadratic optimisation problem

Example 1: two periodic orbits





Example 2: Lorenz attractor



blue: computed set containing attractor, red: attractor

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Equation

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Differential inequalities

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Existence of complete Lyapunov functions

- S. Hafstein & S. Suhr: Smooth complete Lyapunov functions for ODEs. J. Math. Anal. Appl. **499** (2021), Article 125003.
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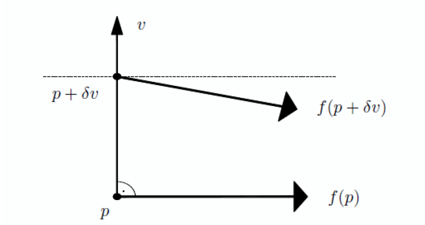
4. Contraction metrics

- Consider $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$
- Adjacent solutions contract with respect to contraction metric
- Can be used to show existence, uniqueness, stability and basin of attraction of equilibria/periodic orbits
- Robust with respect to perturbations of the system



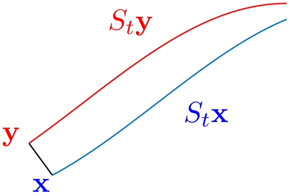
Problem Find Riemannian metric $M \in C^1(\Omega; \mathbb{S}^{n\times n})$ (symmetric matrices) with scalar product $\langle v, w\rangle_M = v^T M(\mathbf{x}) w$ such that

- $M(\mathbf{x}) \succ 0$ (positive definite)
- $LM(\mathbf{x}) := M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + \dot{M}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) \prec 0$ (negative definite)

Idea of contraction metric

Idea

- Solutions $S_t\mathbf{x}$ and $S_t\mathbf{y}$, $\mathbf{y}$ near $\mathbf{x}$
- Time-dependent distance (squared)



$$d^2(t) := (S_t\mathbf{y} - S_t\mathbf{x})^T M(S_t\mathbf{x})(S_t\mathbf{y} - S_t\mathbf{x})$$

- Derivative, denoting $\mathbf{v} = S_t\mathbf{y} - S_t\mathbf{x}$: exponential decay of $d(t)$

$$\begin{aligned}
\frac{d}{dt}d^2(t) &\approx (S_t\mathbf{y} - S_t\mathbf{x})^T D\mathbf{f}(S_t\mathbf{x})^T M(S_t\mathbf{x})(S_t\mathbf{y} - S_t\mathbf{x}) \\
&\quad +(S_t\mathbf{y} - S_t\mathbf{x})^T \dot{M}(S_t\mathbf{x})(S_t\mathbf{y} - S_t\mathbf{x}) \\
&\quad +(S_t\mathbf{y} - S_t\mathbf{x})^T M(S_t\mathbf{x}) D\mathbf{f}(S_t\mathbf{x})(S_t\mathbf{y} - S_t\mathbf{x}) \\
&= \mathbf{v}^T [\underbrace{M(S_t\mathbf{x})D\mathbf{f}(S_t\mathbf{x}) + \dot{M}(S_t\mathbf{x}) + D\mathbf{f}(S_t\mathbf{x})^T M(S_t\mathbf{x})}_{=LM(S_t\mathbf{x}) \prec -2\nu M(S_t\mathbf{x})}]\mathbf{v} \\
&\leq -2\nu\, d^2(t)
\end{aligned}$$

Contraction metric and basin of attraction

Theorem

- $\varnothing \neq K \subset \mathbb{R}^n$ *positively invariant, compact and connected*
- *Riemannian metric* $M \in C^1(\mathbb{R}^n, \mathbb{S}^{n\times n})$ *(*$M(\mathbf{x}) \succ 0$*)*
- $LM(\mathbf{x}) = M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + \dot{M}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) \prec -2\nu M(\mathbf{x})$ *for all* $x \in K$ *with* $\nu > 0$

Then

- *Existence and uniqueness of an exponentially asymptotically stable equilibrium* $\mathbf{x}_0 \in K$
- $-\nu$ *is upper bound on rate of exponential attraction*
- $K \subset A(\mathbf{x}_0)$ *(basin of attraction)*

Remark: On compact set, it is sufficient to have

$$LM(\mathbf{x}) = M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + \dot{M}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) \prec 0$$

Idea of numerical construction

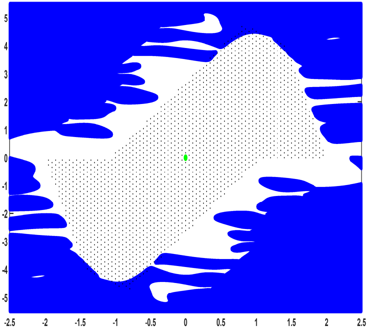
- There exists specific contraction metric satisfying

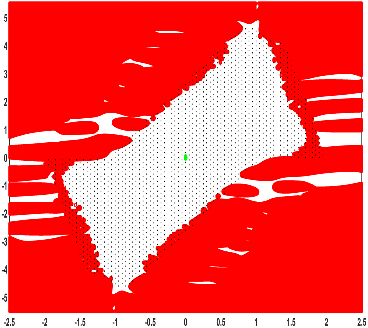
$$LM(\mathbf{x}) := M(\mathbf{x})D\mathbf{f}(\mathbf{x}) + \dot{M}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})^T M(\mathbf{x}) = -C \prec 0$$

  for all $\mathbf{x} \in A(\mathbf{x}_0)$
- Approximate M satisfying equation above using meshless collocation (of matrix-valued functions)
- Interpolation with CPA metric to verify conditions

Examples: Van der Pol (time-reversed, equilibrium)

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x - 3(1-x^2)y \end{aligned}$$



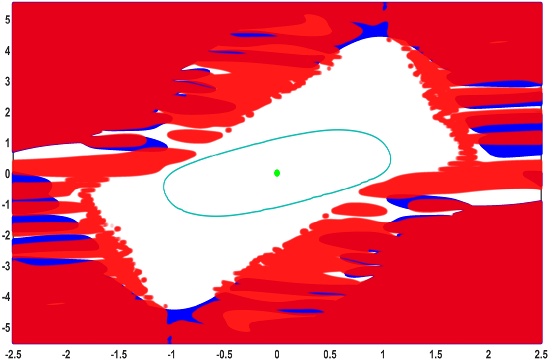


Black: 1926 collocation points
Blue: $M(\mathbf{x})$ not positive definite

Green: equilibrium
Red: $LM(\mathbf{x})$ not negative definite

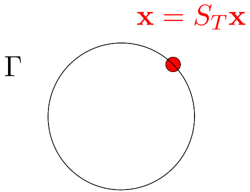
Example: Van der Pol (time-reversed, equilibrium)

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x - 3(1 - x^2)y \end{aligned}$$



Dark green: positively invariant set (using Lyapunov-like function)

Contraction metric for periodic orbit

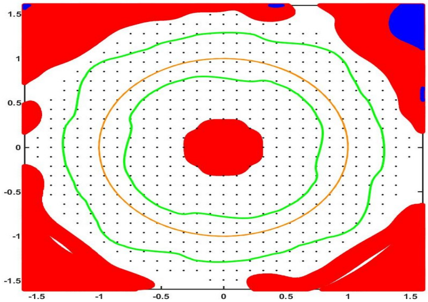


- periodic orbit $\Gamma = \{S_t\mathbf{x} \mid t \in [0, T)\}$ with $\mathbf{x} = S_T\mathbf{x}$
- basin of attraction
  $A(\Gamma) = \{\boldsymbol{\xi} \in \mathbb{R}^n \mid \text{dist}(S_t\boldsymbol{\xi}, \Gamma) \stackrel{t\to\infty}{\longrightarrow} 0\}$
- similar method for periodic orbits: contraction only in $(n-1)$-dimensional hyperplane perpendicular to $\mathbf{f}(\mathbf{x})$

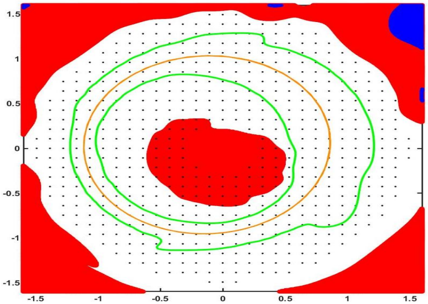
Example: unit circle (periodic orbit)

$$\begin{aligned}\dot{x} &= (x+\varepsilon)(1-x^2-y^2)-(y+\varepsilon)\\ \dot{y} &= (y+\varepsilon)(1-x^2-y^2)+(x+\varepsilon)\end{aligned}$$

$\varepsilon = 0$



$\varepsilon = 0.2$ (same metric)



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- P. Giesl, S. Hafstein & I. Mehrabinezhad, *Computation and verification of contraction metrics for periodic orbits*. J. Math. Anal. Appl. **503** (2021), Article 125309.

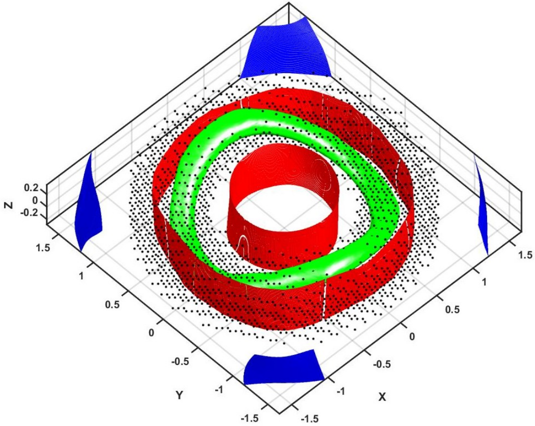
Summary

- Analytical tools:
  - (complete) Lyapunov function $V\colon \mathbb{R}^n \to \mathbb{R}$
  - contraction metric $M\colon \mathbb{R}^n \to \mathbb{S}^{n\times n}$,
    robust with respect to perturbations, no information about equilibrium/periodic orbit required
- Numerical methods:
  - RBF (Radial Basis Functions) – meshless collocation (solve system of linear equations or quadratic optimisation for differential inequalities)
  - CPA (continuous piecewise affine) – linear optimisation (triangulation of compact phase space, verification)

Extensions

- Discrete systems $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$
- Periodic time $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$, $\mathbf{f}(t+T, \mathbf{x}) = \mathbf{f}(t, \mathbf{x})$
- Finite time $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$, $t \in [0, T]$
- Non-smooth systems
- Stochastic systems
- Dimension of attractors, entropy

QUESTIONS?



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