An Introduction to Homotopy Type Theory

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Part I

Type Theory

The Setup of Classical Mathematics

- The mathematical universe consists of abstract collections called sets.
- Logic (connectives, rules of inference, ...) exists prior to the definition of the theory of sets.
- Properties of sets are axiomatized using this logic

- Law of Excluded Middle

  Connectives determined by their truth tables.

$\vee$	T	F
T	T	T
F	T	F

$\wedge$	T	F
T	T	F
F	F	F

- Proofs are "external" to the theory

  The theory is proof-irrelevant.

Some Criticisms of Set Theory

- Non-sensical but well-formed assertions:

  Is $7 \in \pi$?

- Properties of objects depend on implementation:

  $\mathbb{N} = \{\phi, \{\phi\}, \{\{\phi\}\}, \ldots\}$ $\quad \mathbb{N}' = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}, \ldots\}$

- Reasonable disagreement about axioms:

  AC? CH?

- Non-constructive by default

Type Theory

- The mathematical universe consists of Statements and their Proofs
- Proofs are gathered into collections based on what they prove.
- We write this as:

$$x : A$$

Term (proof)

Type (statement)

The Brouwer-Heyting-Kolmogorov (BHK) Interpretation

- A proof of $A \vee B$ is either a proof of A or a proof of B
- A proof of $A \wedge B$ is a pair of a proof of A and a proof of B
- A proof of $A \Rightarrow B$ is a function which assigns to any proof of A a proof of B.
- There is no proof of $\perp$
- A proof of $\neg A$ is a proof of $A \Rightarrow \perp$

Logical connectives explained by <u>evidence</u>.

Type Theory as an Implementation of BHK

- We make this idea precise by providing explicit syntax for constructing statements and their proofs.

$$\frac{a : A \quad b : B}{a, b : A \times B}$$

$$\frac{a : A}{\mathrm{inl}\ a : A \sqcup B} \qquad \frac{b : B}{\mathrm{inr}\ b : A \sqcup B}$$

$$\frac{x : A \vdash b : B}{\lambda x. b : A \Rightarrow B}$$

The Natural Numbers

- A proof $n : \mathbb{N}$ can be thought of as the proof of the statement:

  "I know a natural number"

$$\frac{}{0 : \mathbb{N}} \qquad \frac{n : \mathbb{N}}{Sn : \mathbb{N}}$$

Ex:

$0 : \mathbb{N}$

$S0 : \mathbb{N}$

$SS0 : \mathbb{N}$

Dependent Types

- So far we have only seen simple types.

$$\vee \quad \wedge \quad \Rightarrow \quad \neg \quad \mathbb{N} \quad \mathrm{Bool}$$

- But if types are to be a rich enough language for mathematics, we must also allow them to mention terms.

$$4 \leq 7 \qquad \forall n : \mathbb{Z}.\ n^2 \geq 0$$

- We call these dependent types.

Example

$$\frac{n : \mathbb{N} \quad m : \mathbb{N}}{n \leq m : \mathrm{Type}}$$

Formation Rule

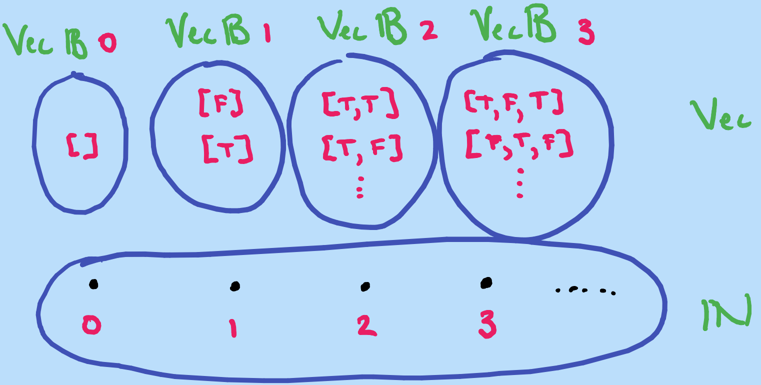
$$\frac{n : \mathbb{N}}{lt_0\, n : 0 \leq n} \qquad \frac{n : \mathbb{N} \quad m : \mathbb{N} \quad p : n \leq m}{lt_s\, p : Sn \leq Sm}$$

Introduction Rules

Ex: $lt_s(lt_s(lt_0\, 2)) : 2 \leq 4$

Dependent Types as Fibrations

- Formation for vectors: $\dfrac{A : \mathrm{Type} \quad n : \mathbb{N}}{\mathrm{Vec}\ A\ n : \mathrm{Type}}$



$\mathrm{Vec}\ \mathbb{B} : \mathbb{N} \to \mathrm{Type}$

Quantifiers

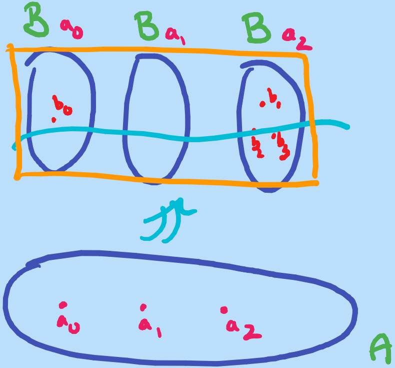
- Dependent Product (Forall)

$$\frac{A : \mathrm{Type} \qquad x : A \vdash B : \mathrm{Type}}{\prod_{x:A} B : \mathrm{Type}} \qquad \frac{x : A \vdash b : B}{\lambda x. b : \prod_{x:A} B}$$

- Dependent Sum (Exists)

$$\frac{A : \mathrm{Type} \qquad x : A \vdash B : \mathrm{Type}}{\sum_{x:A} B : \mathrm{Type}} \qquad \frac{a : A \qquad b : B[a/x]}{(a,b) : \sum_{x:A} B}$$

"Geometric" Interpretation of Quantifiers



Σ = Total space

Π = Space of sections

Martin-Löf's Methodology

- Formation – In what context is a type well-formed?
- Introduction – How do I construct terms of the type?
- Elimination – How do I use the terms of my type?
- Computation – How do introduction and elimination interact?

Elimination + Computation

$$\frac{p : \sum_{x:A} B}{\mathsf{fst}\ p : A} \qquad \frac{p : \sum_{x:A} B}{\mathsf{snd}\ p : B[\mathsf{fst}\ p / x]}$$

$$\mathsf{fst}\ (a, b) = a$$
$$\mathsf{snd}\ (a, b) = b$$

$$\frac{f : \prod_{x:A} B \qquad a : A}{f\,a : B[a/x]}$$

$$(\lambda x . b)\ a = b[a/x]$$

Normalization and Canonicity

- The combination of these rules let's us reduce intro/elim pairs:

$$\begin{aligned} &\mathsf{fst}\,((\lambda x.x)(4,7)) &&: \mathbb{N} \\ \rightsquigarrow\ &\mathsf{fst}\,(4,7) &&: \mathbb{N} \\ \rightsquigarrow\ &4 &&: \mathbb{N} \end{aligned}$$

- This equips type theory with a notion of computation
- A meta-theorem (canonicity) asserts that all closed terms reduce to introduction forms.

Part II

Homotopy Theory

Martin-Löf Identity Types

$$\frac{A : \mathrm{Type} \quad a : A \quad b : A}{\mathrm{Id}_A\, a\, b : \mathrm{Type}}$$

$$\frac{a : A}{\mathrm{refl}\ a : \mathrm{Id}_A\, a\, a}$$

- The only way to prove equality is reflexivity.
- This works modulo the computation rules

$$\mathrm{refl}\ 4 : \mathrm{Id}_{\mathbb{N}}\ 4\ (3+1)$$

Curious Features

1. Because the formation rule is stated for <u>any</u> A, it can be iterated:

$$\mathrm{Id}_{\mathrm{Id}_A\, a\, b}\, p\, q \qquad\qquad \mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A\, a\, b}\, p\, q}\, \alpha\, \beta$$

2. We cannot assume proofs of identity are unique.

What are we to make of this?

The Homotopy Interpretation

- Hoffman–Streicher ('94–95)

  Axiom K : $\prod_{x,y:A} \prod_{p,q:\mathrm{Id}_A\,x\,y} \mathrm{Id}_{\mathrm{Id}_A\,x\,y}\; p\; q$

  is **not** provable.

- Awodey–Warren (2008)

  Type theory can be interpreted in (certain) Quillen Model Categories

- Lumsdaine / Garner–Van den Berg (2008–9)

  Types give rise to weak ∞-groupoids

Groupoid Laws

$$\begin{tikzcd} x \arrow[r, "p"] & y \arrow[r, "q"] & z \arrow[r, "r"] & w \end{tikzcd}$$

- Can construct composition operation $p \cdot q : \mathrm{Id}_A\, x\, z$
- Can show various <u>laws</u> up to higher cells:

$$\mathsf{unit\text{-}l} : \mathrm{Id}_{\mathrm{Id}_A\, x\, y}\ p\ (p \cdot \mathsf{refl}_y)$$

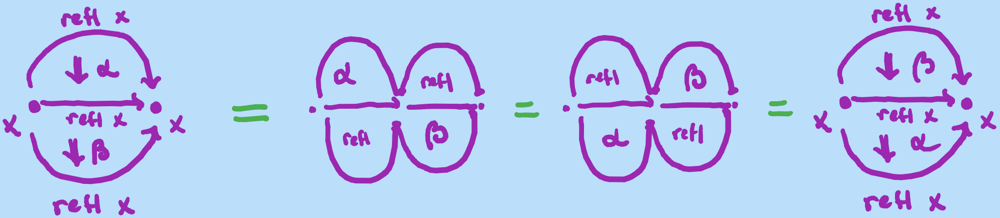
$$\begin{tikzcd} x \arrow[r, bend left, "p"] \arrow[r, bend right, "p \cdot \mathsf{refl}_y"'] \arrow[r, phantom, "\Downarrow \mathsf{u\text{-}l}"] & y \end{tikzcd}$$

$$\mathsf{assoc} : \mathrm{Id}_{\mathrm{Id}_A\, x\, w}\ ((p \cdot q) \cdot r)\ (p \cdot (q \cdot r))$$

$$\begin{tikzcd} x \arrow[r, bend left, "(p \cdot q) \cdot r"] \arrow[r, bend right, "p \cdot (q \cdot r)"'] \arrow[r, phantom, "\Downarrow \alpha"] & w \end{tikzcd}$$

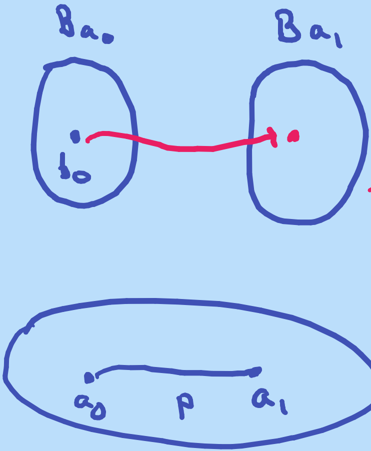
Eckmann-Hilton

- Some laws are non-obvious and come directly from topology



$$\alpha \cdot_2 \beta = \beta \cdot_2 \alpha$$

Fibrations Revisited



$\text{transp}\ B\ a_0\ p : B\ a_1$

H-Level

- We can use identity types to stratify the universe
- First, define contractible types

$$\text{is-contr } X := \sum_{x:X} \prod_{y:X} \mathrm{Id}_X\, x\, y$$

- Now define h-level by induction

$$\text{has-level } (-2)\ X := \text{is-contr } X$$
$$\text{has-level } (S\ n)\ X := \prod_{x,y:X} \text{has-level } n\ (\mathrm{Id}_X\, x\, y)$$

Low Dimensions

$\boxed{-2}$ • Contractible types

- If and only if equivalent to $\mathbf{1}$
- Implies Identity types also contractible

$\boxed{-1}$ • Propositions

- Types with "at most one" element
- Play the role of truth values

$\boxed{0}$ • Sets

- Elements are equal in at most one way

$\mathbb{N}, \mathbb{R}, \mathbb{Z}, \mathbb{B}$

$\boxed{1}$ • Groupoids

- Elements can have symmetries

FinType

Equivalences

- Homotopically correct notion of isomorphism
- Define the homotopy fiber of a map $f : X \to Y$

$$\mathrm{hfib}\, f\, y := \sum_{x : X} \mathrm{Id}_Y\, (f\, x)\, y$$

- Say a map f is an equivalence if all its homotopy fibers are contractible

$$\text{is-equiv}\, f := \prod_{y : Y} \text{is-contr}\, (\mathrm{hfib}\, f\, y)$$

- Being an equivalence is a proposition!

$$\mathrm{Equiv}\, A\, B := \sum_{f : A \to B} \text{is-equiv}\, f$$

Extensionality Principles

- One defect of Martin-Löf's identity type is that it fails to correctly reproduce the "natural" equality for some types

- Function Extensionality

$$\mathrm{Id}_{A \to B}\, f\, g \cong \prod_{a:A} \mathrm{Id}_B\, (f\,a)\, (g\,a)$$

is <u>not</u> provable.

- It is often assumed as an axiom.

But this breaks canonicity!

Univalence

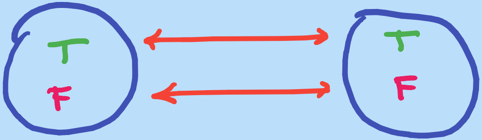
- Another type which Martin-Löf's identity types fail to determine is Type
- What is the natural notion here?
- Voevodsky:

$$\mathrm{Id}_{\mathrm{Type}}\ A\ B \cong \mathrm{Equiv}\ A\ B$$

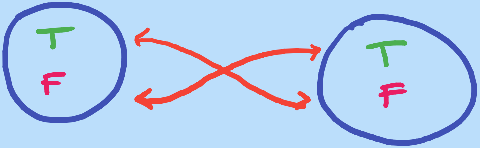
Univalence and Paths

- We can use univalence to produce examples of equalities which are not themselves equal.

$$\text{Univalence} \Rightarrow \neg(\text{Axiom K})$$



$$\mathrm{Id}_{\mathrm{Type}}\ \mathbb{B}\ \mathbb{B}$$



$$\mathrm{Id}_{\mathrm{Type}}\ \mathbb{B}\ \mathbb{B}$$

Higher Inductive Types

- Type theory has long struggled from the absence of a reasonable theory of quotients.

- Higher inductive types generalize inductive types by allowing introduction rules to return not only elements of the type being defined, but also its identity types.

Examples

- S^1

  $\mathrm{base} : S^1$

  $\mathrm{loop} : \mathrm{Id}_{S^1}\ \mathrm{base}\ \mathrm{base}$

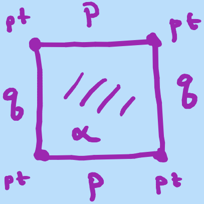
- T

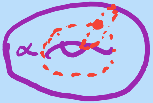
  $\mathrm{pt} : T$

  $p : \mathrm{Id}_T\ \mathrm{pt}\ \mathrm{pt}$

  $q : \mathrm{Id}_T\ \mathrm{pt}\ \mathrm{pt}$

  $\alpha : \mathrm{Id}_{\mathrm{Id}_T\ \mathrm{pt}\ \mathrm{pt}}\ (p \cdot q)\ (q \cdot p)$





Results from Homotopy Theory

- Homotopy Groups $\pi_n(S^n) = \mathbb{Z}$ $\quad \pi_3(S^2) = \mathbb{Z}/2$
- Fibration Sequences

  $F \to E \to B$ $\quad \cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$
- Eilenberg-MacLane Spaces (Cohomology)
- Spectral Sequences
- (Generalized) Blakers-Massey Theorem

  $\to$ Freudenthal Suspension Thm

Cubical Type Theories

- Inspired by homotopy interpretation
- Extensionality principles provable!
- Native HIT's
- Implementation in Agda

THANKS!