Approximating Discrete Dynamical Systems

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URJC

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Outline

- Preliminaries and motivation.
- Vietoris-like maps and multivalued maps.
- Lefschetz fixed point theorem.
- Approximating Discrete Dynamical Systems.
- Localization of finite spaces at Vietoris-like maps.

Preliminaries and motivation

Definition

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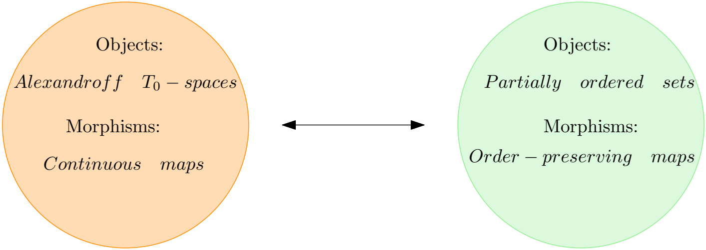
Theorem (Alexandroff, 1937)

The category of Alexandroff T_0-spaces is isomorphic to the category of partially ordered sets.



Preliminaries and motivation

Given an Alexandroff space X and $x \in X$, U_x denotes the intersection of all the open sets which contain x. Let $x, y \in X$, $x \leq y$ if and only if $U_x \subseteq U_y$ ($U_y \subseteq U_x$).

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Example. Finite topological T_0-spaces. We call them from now on just finite spaces.

Example. Let $X = \{A, B, C, D\}$ and $\tau = \{X, \emptyset, \{A\}, \{B\}, \{A, B\}, \{C, A, B\}, \{D, A, B\}\}$. Then $U_A = \{A\}$, $U_B = \{B\}$, $U_C = \{C, A, B\}$ and $U_D = \{D, A, B\}$, which yields $A < C, D$ and $B < C, D$.

Proposition

Let $f, g : X \to Y$ be continuous maps between finite spaces. Then f is homotopic to g if and only if there exists a finite sequence of continuous maps $f_1, ..., f_n : X \to Y$ such that $f(x) = f_1(x) \leq f_2(x) \geq ... \leq f_n(x) = g(x)$ for every $x \in X$.

Hasse diagrams. Let X be a finite space. The Hasse diagram of X is a directed graph. The vertices are the points of X and there is an edge between two points x and y if and only if $x < y$ and there is no z satisfying $x < z < y$.

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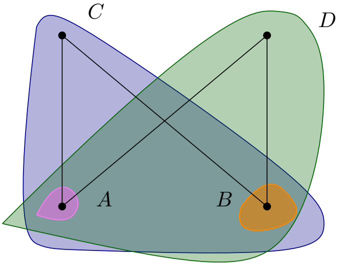
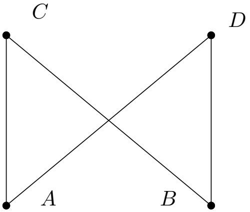
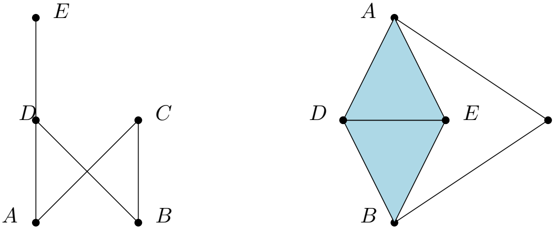
Example.

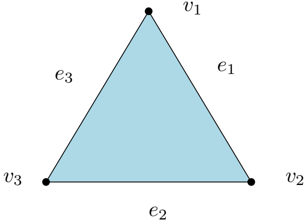


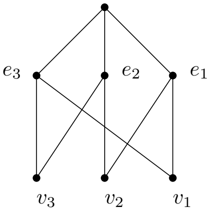


Order complex. Given a finite space X, the order complex of X, denoted by $\mathcal{K}(X)$, is the simplicial complex whose simplices are the non-empty chains of X.



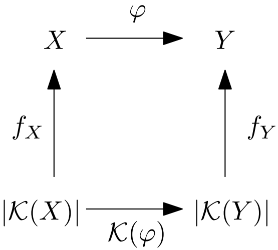
Face poset. Given a simplicial complex L, the face poset of L, denoted by $\mathcal{X}(L)$, is the poset of simplices of K ordered by inclusion.





Theorem (McCord, 1966)

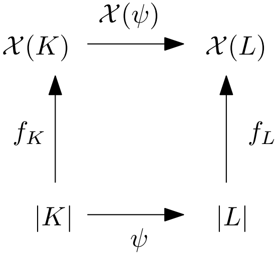
There exists a correspondence that assigns to each Alexandroff T_0-space a simplicial complex $\mathcal{K}(X)$ and a weak homotopy equivalence $f_X : |\mathcal{K}(X)| \to X$. Each continuous map $\varphi : X \to Y$ of Alexandroff T_0-spaces is also a simplicial map $\mathcal{K}(\varphi) : \mathcal{K}(X) \to \mathcal{K}(Y)$, and $\varphi \circ f_X = f_Y \circ \mathcal{K}(\varphi)$.



Preliminaries and motivation

Theorem (McCord, 1966)

There exists a correspondence that assigns to each simplicial complex K an Alexandroff T_0-space $\mathcal{X}(K)$ and a weak homotopy equivalence $f_K : |K| \to \mathcal{X}(K)$. Furthermore, to each simplicial map $\psi : K \to L$ is assigned a continuous map $\mathcal{X}(\psi) : \mathcal{X}(K) \to \mathcal{X}(L)$ such that $\mathcal{X}(\psi) \circ f_K$ is homotopic to $f_L \circ |\psi|$.



Preliminaries and motivation

Finite barycentric subdivision. Given a finite space X, the finite barycentric subdivision of X is defined as $\mathcal{X}(\mathcal{K}(X))$. We denote by X^n the n-th finite barycentric subdivision of X.

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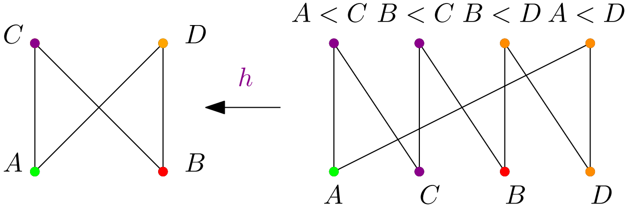

There is a natural map $h : X^1 \to X$ given by $h(x_1 < ... < x_n) = x_n$. Then, we can consider $h_{n,m} : X_m \to X_n$ for every $m \geq n$.

Given a simplicial complex K, X^0 denotes $\mathcal{X}(K)$. Therefore, there is a natural inverse sequence of finite spaces.

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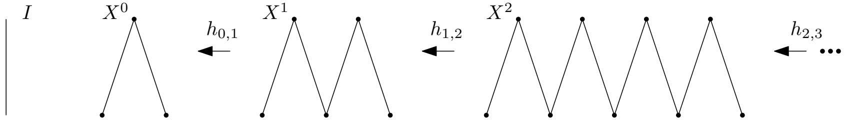
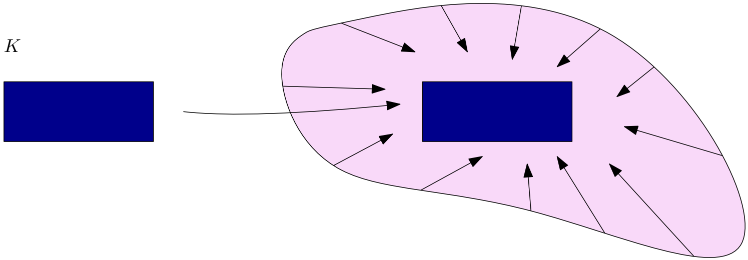
Example. Let us consider the unit interval I.



Theorem (Clader, 2009)

Let K be a compact simplicial complex. The inverse limit of $(X^n, h_{n,n+1})$ contains a homeomorphic copy of K, which is a strong deformation retract.



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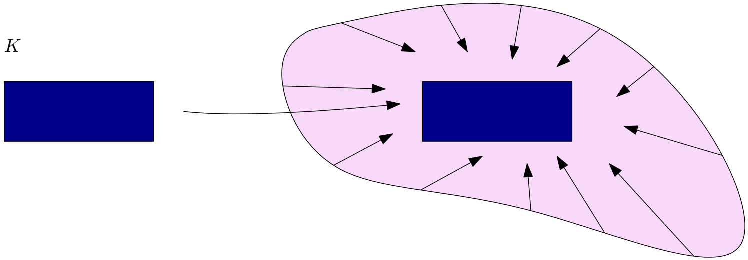


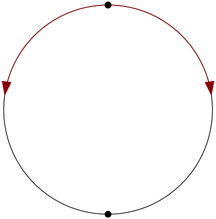
Remark. The same result also holds for compact metric spaces.

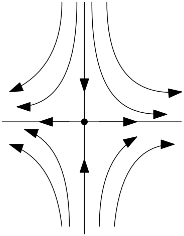
Preliminaries and motivation

Definition

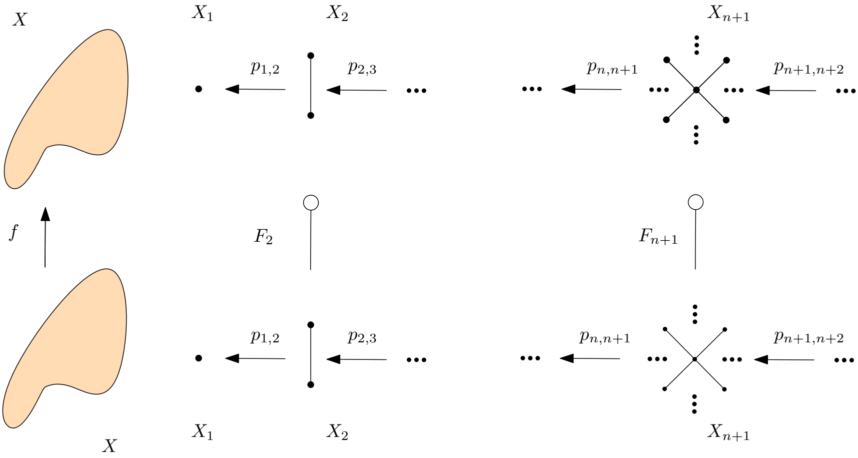
A dynamical system for a topological space X consists of a triad $(\mathbb{T}, X, \varphi)$, where $\mathbb{T}$ is usually $\mathbb{Z}$ or $\mathbb{R}$ and $\varphi : \mathbb{T} \times X \to X$ is a continuous function satisfying

1. $\varphi(0, x) = x$ for every $x \in X$.
2. $\varphi(t + s, x) = \varphi(t, \varphi(s, x))$ for all $s, t \in \mathbb{T}$ and $x \in X$.





Main Idea:


X
X1
X2
Xn+1
p1,2
p2,3
pn,n+1
pn+1,n+2
f
F2
Fn+1
p1,2
p2,3
pn,n+1
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X
X1
X2
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Preliminaries and motivation

Proposition

Let A be a finite space.

- If $(\mathbb{R}, A, \varphi)$ is a continuous dynamical system, then φ is trivial.
- If $(\mathbb{Z}, A, \varphi)$ is a discrete dynamical system, there exists $n \in \mathbb{N}$ such that $\varphi^n = id$.

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Consider **Multivalued maps** to define dynamical systems.

Vietoris-like maps and multivalued maps

We say that a topological space X is **acyclic** if the homology groups in all dimensions of X are isomorphic to the corresponding homology groups of a point.

Definition

Given a continuous map $f : X \to Y$ between two finite spaces, we say that f is a Vietoris-like map if for every chain $y_1 < y_2 < ... < y_n$ in Y we get that $\bigcup_{i=1}^{n} f^{-1}(y_i)$ is acyclic.

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Example. Every homeomorphism is a Vietoris-like map. Indeed, $f : X \to X$ is a Vietoris-like map if and only if f is a homeomorphism.

Vietoris-like maps and multivalued maps

Theorem

If $f : X \to Y$ is a Vietoris-like map, then f induces isomorphisms in all homology groups.

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Some properties of Vietoris-like maps

Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between finite spaces.

- If f and g are Vietoris-like maps, then $g \circ f : X \to Z$ is a Vietoris-like map.
- If f and $g \circ f$ are Vietoris-like maps, then g is a Vietoris-like map.
- The 2-out-of-3 property does not hold for Vietoris-like maps.

Vietoris-like maps and multivalued maps

Definition

Let $F : X \multimap Y$ be a mutlivalued map between finite spaces. We say that F is a Vietoris-like multivalued map if the projection p onto the first coordinate from the graph of $\Gamma(F)$ is a Vietoris-like map.

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Remark. $F_* : H_*(X) \to H_*(Y)$ is given by $q_* \circ p_*^{-1}$, where $q : \Gamma(F) \to Y$ is the projection onto the second coordinate.

Vietoris-like maps and multivalued maps

Examples

- Let $f : X \to Y$ be a continuous map. If we consider f as a multivalued map, then f is a Vietoris-like multivalued map since $p : \Gamma(f) \to X$ is a homeomorphism. Moreover, $f_* = q_* \circ p_*^{-1}$.
- If $f : X \to Y$ is a Vietoris-like map, then $F : Y \multimap X$ given by $F(y) = f^{-1}(y)$ is a Vietoris-like multivalued map.

Lefschetz number. Let $f : X \to X$ be a continuous map, where X is a finite space. The lefschetz number of f is given by

$$\Lambda(f) = \sum_{i=0} (-1)^i tr(f_* : H_i(X) \to H_i(X)),$$

where tr denotes the trace and f_* denotes the linear map induced by f on the torsion-free part of the homology of X.

Theorem

Let $f, g : X \to Y$ be continuous maps between finite spaces, where f is a Vietoris-like map. If $\Lambda(g_* \circ f_*^{-1}) \neq 0$, then there exists $x \in X$ such that $f(x) = g(x)$

Lefschetz fixed point theorem

Lefschetz fixed point theorem for multivalued maps

Let X be a finite space. If $F : X \multimap X$ is a Vietoris-like multivalued map and $\Lambda(F_* = q_* \circ p_*^{-1}) \neq 0$, then there exists $x \in X$ with $x \in F(x)$.

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Theorem

Let $F : X \multimap X$ be a multivalued map, where X is a finite space. Suppose that $F = G_n \circ \cdots \circ G_0$, where $G_i : Y_i \multimap Y_{i+1}$, $Y_0 = Y_{n+1} = X$, Y_i is a finite space and G_i is a Vietoris-like multivalued map. If $\Lambda(G_{n*} \circ \cdots \circ G_{0*}) \neq 0$, then there exists a point $x \in X$ such that $x \in F(x)$.

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Remark. Not every multivalued map may be expressed as a composition of Vietoris-like multivalued maps.

Approximating Discrete Dynamical Systems

Recall that given a finite space X^0 we may consider the following inverse sequence

$$X^0 \xleftarrow{h_{0,1}} X^1 \xleftarrow{h_{1,2}} X^2 \xleftarrow{h_{2,3}} X^3 \xleftarrow{h_{3,4}} \dots X^n \xleftarrow{h_{n,n+1}} X^{n+1} \xleftarrow{h_{n+1,n+2}} \dots$$

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Proposition

Let X be a finite space and $m \geq n$. Then $h_{n,m} : X^m \to X^n$ is a Vietoris-like map which induces the identity in homology.

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Corollary

Let X be a finite space and $m \geq n$. Then $H_{m,n} : X^n \multimap X^m$, defined by $H(x) = h^{-1}(x)$, is a Vietoris-like multivalued map which induces the identity in homology.

Approximating Discrete Dynamical Systems

Given a continuous map $f : |K| \to |K|$, there is a natural inverse sequence induced by f (use simplicial approximation theorem).

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Therefore, we have

$$\begin{array}{ccccccccc}
X^0 & \xleftarrow{f_{0,1}} & X^1 & \xleftarrow{f_{1,2}} & X^2 & \xleftarrow{f_{2,3}} & X^3 & \longleftarrow & \cdots \\
 & & \big\uparrow{\scriptstyle\circ}\, F_1 & & \big\uparrow{\scriptstyle\circ}\, F_2 & & \big\uparrow{\scriptstyle\circ}\, F_3 & & \\
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\end{array}$$

where $F_{n+1} = H_{n+1,n} \circ f_{n,n+1}$.

Proposition

If $\Lambda(f) \neq 0$, then there exists a point $x_{n+1} \in X^{n+1}$ such that $x_{n+1} \in F_{n+1}(x_{n+1})$ for every $n \in \mathbb{N}$.

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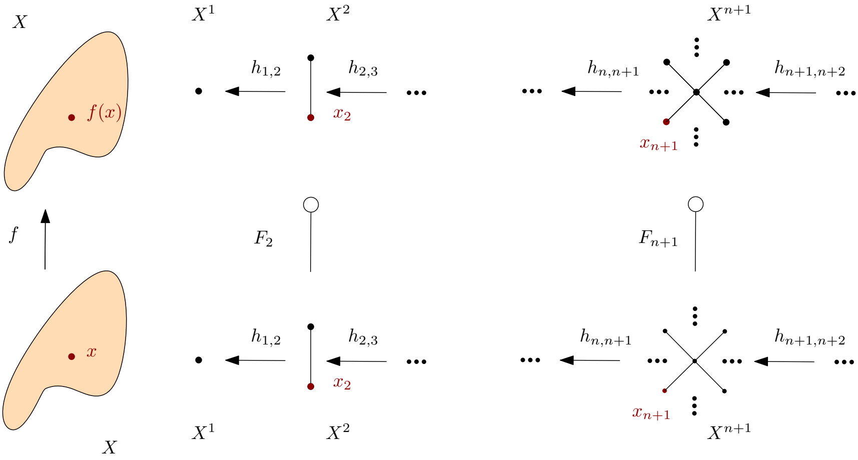
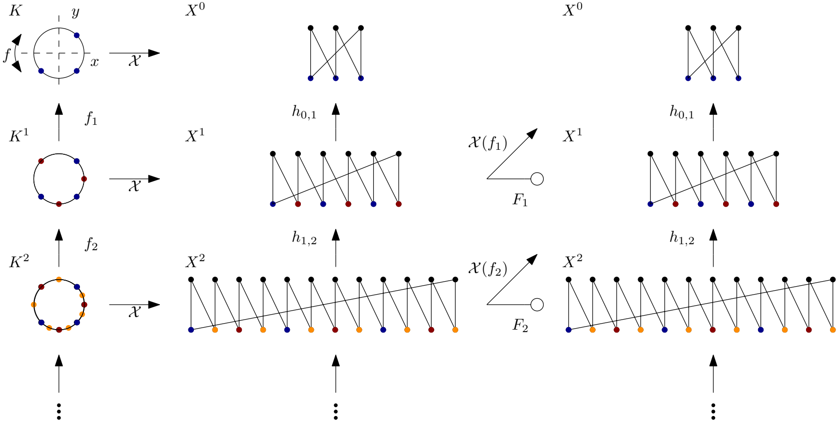
Theorem

If $f : |K| \to |K|$ is a continuous map, where K is a simplicial complex, then f has a fixed point if and only if there exist a finite approximative sequence for f, $(X^n, h_{n,n+1})$, a sequence $\{x_{n+1}\}_{n\in\mathbb{N}}$ and $m \in \mathbb{N}$ such that $x_{n+1} \in X^{n+1}$, $x_n = h_{n,n+1}(x_{n+1})$ for every $n \in \mathbb{N}$ and $x_{n+1} \in F_{n+1}(x_{n+1})$ for every $n+1 \geq m$.

Approximating Discrete Dynamical Systems



Example. Let $f : S^1 \to S^1$ be given by $f(x, y) = (x, -y)$.



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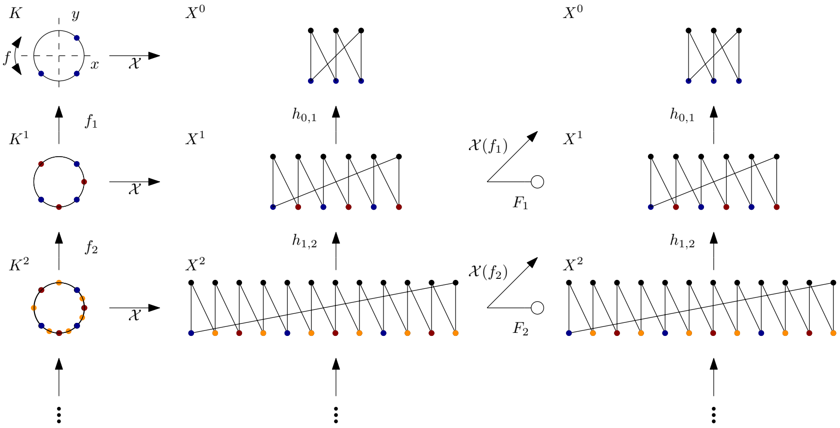

Goal. Generalize these results to compact metric spaces using more geometrical constructions.

Localization of finite spaces at Vietoris-like maps

Main Idea: Enclose Vietoris-like multivalued maps in a category to get other dynamical invariants.

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Definition

Let X and Y be finite spaces. We say that $X \xleftarrow{p} Z \xrightarrow{q}$ is a span or a diagram if p is a Vietoris-like map.

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Examples. Continuous maps and Vietoris-like multivalued maps.

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Examples. Continuous maps and Vietoris-like multivalued maps.

Steps

1. Define the composition of spans. Solution: pull-backs.
2. Define an equivalence relation between spans. Solution: define a new notion of homotopy that generalizes the usual notion of homotopy for single valued maps in the category of finite spaces.

Localization of finite spaces at Vietoris-like maps

Category of finite spaces

Objects: finite topological T_0-spaces

Morphisms: continuous maps

Homotopical category of finite spaces

Objects: finite topological T_0-spaces

Morphisms: homotopy classes of continuous maps

Localization of the category of finite topological spaces at the class of Vietoris-like maps

Objects: finite topological T_0-spaces

Morphisms: homotopy classes of spans

Topological degree

1. Let X and Y be finite models of S^n. In the usual category it is not possible to get that every integer number may be realized as the topological degree of a continuous map $f : X \to Y$.
2. In the localized category of finite spaces the above result is possible.

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Thanks for your attention! Any questions?