Asynchronous Games
Innocence without Alternation

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Groupe de travail concurrence
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Part I

Game semantics
Game semantics

- An *interactive* and *trace* semantics for proofs and programs
- A successful series of models:
  - PCF
  - PCF + control
  - references (Idealized Algol)
  - linear logic
  - ...

Can we reflect the *concurrency of proofs in games*?
Mixing different points of view

Mixing ideas from

• game semantics
• concurrency theory
• linear logic
Mixing different points of view

Mixing ideas from

- game semantics
- concurrency theory
- linear logic

We relate here

1. sequential games (traces)
2. event structures
3. relational model
4. concurrent games (closure operators)
Mixing points of view

<table>
<thead>
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Game semantics

- Formulas are interpreted as games

\[
[B] = \frac{q}{V \# F}
\]

consisting of and event structure \((M, \leq, \#)\) where

- \(M\) is a set of moves
- causal dependencies \((\leq)\) and incompatibilities \((\#)\) between these moves
- a polarization of moves \(\lambda : M \rightarrow \{O, P\}\)
Game semantics

- Formulas are interpreted as **games**

\[
\begin{align*}
[\mathcal{B}] &= q \\
V &\quad \# &\quad F
\end{align*}
\]

...consisting of and event structure \((M, \leq, \#)\) where

- \(M\) is a set of **moves**
- causal dependencies \((\leq)\) and incompatibilities \((\#)\) between these moves
- a **polarization** of moves \(\lambda : M \rightarrow \{O, P\}\)
- **plays** are paths between configurations game
Game semantics

- Formulas are interpreted as **games**

\[
[B] = \begin{array}{c}
q \\
V & \# & F
\end{array}
\]

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- a **polarization** of moves \(\lambda : M \rightarrow \{O, P\}\)
- **plays** are paths between configurations game
Game semantics

- Formulas are interpreted as games

\[ \mathcal{B} = \ast \]

consisting of and event structure \((M, \leq, \#)\) where

- \(M\) is a set of moves
- causal dependencies \((\leq)\) and incompatibilities \((\#)\) between these moves
- a polarization of moves \(\lambda : M \rightarrow \{O, P\}\)
- plays are paths between configurations game
• Proofs are interpreted as strategies

\[ [\text{true}] = \{ \varepsilon, q, q \cdot V \} \]
The strategy not
The strategy not

\[ B \xrightarrow{\text{not}} B \]

\[ B \]
\[ q \]
\[ q \]
\[ F \]
\[ V \]
Here,

- we only consider formulas of MALL:

\[
\Gamma, A, B \vdash \Gamma, A \otimes B (\otimes) \\
\Gamma, A \vdash \Gamma, A \otimes B (\otimes) \\
\Gamma, A \vdash \Gamma, B (\&) \\
\Gamma, A \& B \vdash \Gamma, A \& B (\&) \\
\Gamma, A \vdash \Gamma, A \& B (\&) \\
\Gamma, A \vdash \Gamma, B (\&) \\
\Gamma, A \vdash \Gamma, A \& B (\&) \\
\]

\[
\Gamma_1, A \vdash \Gamma_2, B (\ominus) \\
\Gamma_1, \Gamma_2, A \otimes B \vdash \Gamma_1, \Gamma_2, A \otimes B (\otimes) \\
\Gamma, A \vdash \Gamma, A \ominus B (\ominus) \\
\Gamma, A \vdash \Gamma, A \ominus B (\ominus) \\
\]
Game semantics

Here,

- we only consider formulas of MALL:

\[
\frac{}{\vdash \Gamma, A, B} \quad \frac{}{\vdash \Gamma, A \otimes B} (\otimes)
\]

\[
\frac{}{\vdash \Gamma, A, B} \quad \frac{}{\vdash \Gamma_1, \Gamma_2, A \otimes B}
\]

\[
\frac{}{\vdash \Gamma, A \otimes B} \quad \frac{}{\vdash \Gamma, A \oplus B} (\oplus)
\]

- with explicit moves:

\[
\frac{}{\vdash \Gamma, A} \quad \frac{}{\vdash \Gamma, \uparrow A} (\uparrow)
\]

\[
\frac{}{\vdash \Gamma, A} \quad \frac{}{\vdash \Gamma, \downarrow A} (\downarrow)
\]
Game semantics

Game semantics are usually:

• alternating

• sequential
Game semantics are usually:

- alternating
  - does not reflect the derivations!
- sequential
Game semantics are usually:

- alternating
  - does not reflect the derivations!
- sequential
  - conceals the concurrency of proofs!
Alternating game semantics

Left and
Alternating game semantics

Right and
Alternating game semantics

Parallel and
Alternating game semantics

Parallel and
Alternating game semantics

Parallel and
Can we characterize the *definable* strategies?
Can we characterize the *definable* strategies?

We have to restrict the space of strategies.
Can we characterize the \textit{definable} strategies?

We have to restrict the space of strategies.

\begin{center}
\verb+innocent strategy = strategy behaving like a proof+
\end{center}
Reformulating innocence

An **innocent** strategy is a strategy with partial memory which plays according to its *view*.

The original definition by Hyland and Ong

- is technical (pointers)
- relies on the fact that plays are alternating
From formulas to games

In linear logic, the formula corresponding to booleans is

$$B = \uparrow(\downarrow 1 \oplus \downarrow 1)$$

which is like of $1 \oplus 1$ with explicit changes of polarities.
From formulas to games

In linear logic, the formula corresponding to booleans is

\[ B = \uparrow(\downarrow 1 \oplus \downarrow 1) \]

It can be drawn as

```
    \uparrow
   /   \   /
/     \     /
1     ⊕     1
  \downarrow /   \downarrow
      1      1
```
In linear logic, the formula corresponding to booleans is

$$\mathbb{B} = \uparrow(\downarrow 1 \oplus \downarrow 1)$$

It can be drawn as
In linear logic, the formula corresponding to booleans is

\[ \mathbb{B} = \uparrow(\downarrow 1 \oplus \downarrow 1) \]

It can be drawn as

```
    q
   / \   \\
  V   \     F
```
From formulas to games

In linear logic, the formula corresponding to booleans is

$$B = \uparrow(\downarrow 1 \oplus \downarrow 1)$$

It can be drawn as

```
          q
         /   \
        /     \
       V       #
      / \     /  \n     F   #   V
```

From formulas to games

So, the game $\mathbb{B}$ is

```
q

V  #  F
```
So, the game $\mathbb{B} \otimes \mathbb{B}$ is
From formulas to games

So, the game $\mathbb{B} \otimes \mathbb{B}$ is

\[
\begin{array}{cc}
q & q \\
V & F
\end{array}
\]

Let’s consider strategies associated to the state true $\otimes$ false.
The strategy $\text{true} \otimes \text{false}$

The strategy $\text{true} \otimes \text{false}$.

\[
\begin{array}{cc}
B & \otimes & B \\
q & q \\
V & F
\end{array}
\]
The strategy true ⊗ false.

B ⊗ B

q q

V ⊗ * ~ q ⊗ q ~ * ⊗ F

qL qR

V ⊗ * ~ q ⊗ q ~ * ⊗ F

V ⊗ q ~ q ⊗ F

V ⊗ q ~ q ⊗ F
The strategy true $\otimes$ false

A biased variant.
The strategy $\text{true} \otimes \text{false}$

Another biased variant.
A game is an asynchronous graph $G$:
- vertices are positions (+ initial position $*$),
- edges are moves,

\[
\begin{aligned}
\left[ \uparrow(\downarrow 1 \otimes \downarrow 1) \right] &= \\
\uparrow(\downarrow * \otimes * ) &= \uparrow(\downarrow * \otimes * ) \\
\end{aligned}
\]
A **game** is an *asynchronous graph* $G$:

- vertices are **positions** (+ initial position $*$),
- edges are **moves**,
- 2-dimensional tiles generate **homotopy** between paths.

\[
[\uparrow(\downarrow 1 \otimes \downarrow 1)] = \uparrow(\downarrow \ast \otimes \downarrow \ast) \sim \uparrow(\downarrow \ast \otimes \downarrow \ast) \sim \uparrow(\ast \otimes \downarrow \ast) \sim \uparrow(\downarrow \ast \otimes \downarrow \ast)
\]
An approach to interferences

The Mazurkiewicz approach to true concurrency.

\[ a \parallel b \quad \text{vs.} \quad a \cdot b + b \cdot a \]
An approach to interferences

The Mazurkiewicz approach to *true concurrency*.

\[ a \parallel b \quad \text{vs.} \quad a \cdot b + b \cdot a \]

\[ \begin{array}{c}
  a \parallel b \\
  \sim \\
  b \quad a
\end{array} \quad \begin{array}{c}
  a \parallel b \\
  b \quad a
\end{array} \]

\[ x := 4 \parallel y := 5 \quad \text{vs.} \quad x := 4 \parallel x := 5 \]
An approach to interferences

The Mazurkiewicz approach to *true concurrency*.

\[ a \parallel b \quad \text{vs.} \quad a \cdot b + b \cdot a \]

\[ \sim \]

\[ x := 4 \parallel y := 5 \quad \text{multiplicatives} \]

\[ x := 4 \parallel x := 5 \quad \text{additives} \]
The game associated to $\uparrow A$ is of the form
The game associated to $\uparrow A \otimes \uparrow B = \uparrow A \triangleright \leftarrow \uparrow B$ is of the form

\begin{align*}
\uparrow \\
\downarrow \\
A \\
\uparrow \\
\downarrow \\
B
\end{align*}
The game associated to $\uparrow A \otimes \uparrow B = \uparrow A \otimes \uparrow B$ is of the form

\[
\begin{array}{c}
\uparrow \\
A
\end{array}
\quad \quad \quad \quad \quad \quad
\begin{array}{c}
\uparrow \\
B
\end{array}
\]

The corresponding asynchronous graph contains

\[
\begin{array}{c}
\ast, \ast \\
\uparrow_A \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quarter
The game associated to $\uparrow A \otimes \uparrow B = \uparrow A \otimes \uparrow B$ is of the form

The corresponding asynchronous graph contains

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+-----+-----+
|     |     |
|     |     |
+-----+-----+
```

```
A   B
```

```
+-----+-----+
|     |     |
|     |     |
+-----+-----+
```

```
* , *
```

```
\uparrow A \quad \uparrow B
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\uparrow *, *, \sim
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\uparrow *, *, \sim
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Non-alternation and asynchrony

Three proofs of \( \uparrow A \nleftrightarrow \uparrow B \):

\[
\begin{align*}
\vdash & \uparrow A, \uparrow B
\end{align*}
\]
Three proofs of $\uparrow A \nleftrightarrow \uparrow B$:
Non-alternation and asynchrony

Three proofs of $\uparrow A \not\leftrightarrow \uparrow B$:

$$
\vdash A, B \\
\vdash A, \uparrow B \quad (\uparrow) \\
\vdash \uparrow A, \uparrow B \quad (\uparrow)
$$
Non-alternation and asynchrony

Three proofs of $\uparrow A \nRightarrow \uparrow B$:

$$
\vdash A, B \\
\vdash A, \uparrow B \quad (\uparrow) \\
\vdash \uparrow A, \uparrow B \quad (\uparrow)
$$

$$
\vdash \uparrow A \\
\vdash \uparrow B \\
\vdash \uparrow \ast, \uparrow \ast
$$
Non-alternation and asynchrony

Three proofs of $\uparrow A \nleftrightarrow \uparrow B$:

$\ast, \ast$

\[ \vdash \uparrow A, \uparrow B \]
Non-alternation and asynchrony

Three proofs of $\uparrow A \not\leftrightarrow \uparrow B$:

\[
\frac{\vdash \uparrow A, B}{\vdash \uparrow A, \uparrow B} (\uparrow)
\]
Non-alternation and asynchrony

Three proofs of $\uparrow A \nrightarrow \uparrow B$:

$$
\frac{
\frac{

\frac{\vdash A, B}{\vdash A, B}}{\vdash \uparrow A, \uparrow B}
}{\vdash \uparrow A, \uparrow B}
$$
Non-alternation and asynchrony

Three proofs of $\uparrow A \not\iff \uparrow B$:

\[
\vdash \text{\ldots} \\
\vdash A, B \\
\vdash \uparrow A, B (\uparrow) \\
\vdash \uparrow A, \uparrow B (\uparrow)
\]

\[
\vdash \text{\ldots} \\
\uparrow*, \uparrow* \\
\vdash \text{\ldots}
\]

\[
\vdash *, * \\
\vdash \uparrow B \\
\vdash * , \uparrow * \\
\vdash \text{\ldots} \\
\vdash \uparrow A \\
\vdash * , \uparrow * \\
\vdash \text{\ldots}
\]
Non-alternation and asynchrony

Three proofs of $\uparrow A \not\equiv \uparrow B$:

$$
\vdash A, B \quad \vdash \uparrow A, \uparrow B
$$

Diagram: 

```
* , *  
  \uparrow \ 
  \uparrow A \quad \ 
  \uparrow * , *  \quad \ 
  \downarrow \ 
  \downarrow \uparrow A \quad \ 
  \downarrow \uparrow B
```

```
* , \uparrow *  
  \downarrow \ 
  \downarrow \uparrow *  \quad \ 
  \uparrow \ 
  \uparrow B \quad \ 
  \uparrow * , \uparrow *
```

```
* , *  
  \downarrow \ 
  \downarrow \uparrow *  \quad \ 
  \uparrow \ 
  \uparrow A \quad \ 
  \uparrow \uparrow *
```

...
Non-alternation and asynchrony

Three proofs of $\uparrow A \nleftrightarrow \uparrow B$:

\[
\begin{array}{c}
\vdash A, B
\
\vdash \uparrow A, \uparrow B (\uparrow, \uparrow)
\end{array}
\]
Non-alternation and asynchrony

| play       | = | exploration of the formula |
| proof      | = | strategy of exploration |

Proofs correspond to refinements of the partial order of the game.
Non-alternation and asynchrony

\[
\begin{align*}
\text{play} & \quad = \quad \text{exploration of the formula} \\
\text{proof} & \quad = \quad \text{strategy of exploration}
\end{align*}
\]

Proofs correspond to refinements of the partial order of the game.

\[
\begin{array}{c}
\vdash \; A, B \\
\vdash \; A, \uparrow B \quad (\uparrow) \\
\vdash \; \uparrow A, \uparrow B \quad (\uparrow)
\end{array}
\]
Non-alternation and asynchrony

\[
\begin{align*}
\text{play} & = \text{exploration of the formula} \\
\text{proof} & = \text{strategy of exploration}
\end{align*}
\]

Proofs correspond to refinements of the partial order of the game.

\[
\vdash A, B \\
\vdash \uparrow A, B \quad \uparrow \quad \text{(↑)} \\
\vdash \uparrow A, \uparrow B \quad \uparrow \quad \text{(↑)}
\]

\[
A \quad \cdots \quad B
\]
Non-alternation and asynchrony

| play       = exploration of the formula |
| proof      = strategy of exploration   |

Proofs correspond to refinements of the partial order of the game.

\[ \vdash A, B \Rightarrow \vdash \uparrow A, \uparrow B (\uparrow, \uparrow) \]

\[ \vdash A, \ldots, B \]

\[ A \quad \ldots \quad B \]
Part II

Traces vs. partial orders
Traces vs. partial orders

formula = partial order on the moves

proof = refinement of the partial order of the formula

How do we relate *sequential* and *causal* strategies?
Mixing points of view

<table>
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From causality to sequentiality

Every partial order defines an asynchronous graph.
Extracting causality from sequentiality

Conversely, one needs the 2-dimensional structure.
Theorem

*Paths modulo homotopy are characterized by partial order on their moves.*
Games

Definition
A **game** is an asynchronous graph satisfying the Cube Property.

Definition
A **play** is a path in A starting from the root.
**Definition**

A *strategy* $\sigma : A$ is a set of plays $A$.

**Definition**

A strategy $\sigma : A$ is *positiona* when its paths form a subgraph of the game $A$. 
We consider positional strategies which satisfy

\[ \sigma \ni m \quad n \in \sigma \]

\[ y_1 \sim y_2 \quad \text{implies} \quad y_1 \sim y_2 \]

\[ p \quad q \]

\[ z \]

\[ \sigma \ni p \quad q \in \sigma \]

\[ \sigma \ni m \quad n \in \sigma \]

\[ y_1 \sim y_2 \quad \text{implies} \quad y_1 \sim y_2 \]

\[ \sigma \ni p \quad q \in \sigma \]

\[ z \]

(this implies the Cube Property)
Unfortunately, the Cube Property is not compositional.
The strategy not:

\[
\begin{align*}
\mathbb{B} & \xrightarrow{\text{not}} \mathbb{B} \\
q & \\
q & \\
\lor & \\
F
\end{align*}
\]
A ⊸ B = A* ⊘ B = A* ⊗ B

The strategy **not**:

\[ B \rightarrow B \rightarrow q \rightarrow q \rightarrow F \rightarrow V \]
Traces compose by *parallel composition*

\[ B \rightarrow B \quad B \rightarrow B \]

\[ q \]

\[ q \]

\[ F \]

\[ V \]
Traces compose by *parallel composition*

\[
\begin{align*}
B & \rightarrow B \\
\quad & \quad q \\
\quad & \quad q \\
\quad & \quad V \\
\quad & \quad F
\end{align*}
\]
Traces compose by *parallel composition*

$$B \rightarrow B \quad B \rightarrow B$$

$q$

$q \quad q$

$q$

$V$

$F \quad F$

$V$
Traces compose by *parallel composition* + *hiding*.

\[ \mathcal{B} \rightarrow \rightarrow \mathcal{B} \]

\[ q \]

\[ q \]

\[ V \]
Definition
A strategy $\sigma : A$ is **deterministic** when

\[
\begin{align*}
\sigma \ni m & \rightarrow x \\
n \in \sigma & \rightarrow y_2 \\
y_1 & \rightarrow x \\
\end{align*}
\]

implies

\[
\begin{align*}
\sigma \ni m & \rightarrow x \\
n \in \sigma & \rightarrow y_2 \\
y_1 & \sim y_2 \\
\sigma \ni p & \rightarrow z \\
q \in \sigma & \rightarrow z \\
\end{align*}
\]

where $m$ is a Proponent move.
Deterministic strategies do compose!
Deterministic strategies do compose!

sequential strategies ⇔ causal strategies
Part III

Partial orders vs. concurrent games
## Mixing points of view

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Concurrent strategies

Abramsky and Melliès introduced the notion of concurrent games.

Definition

A closure operator $\sigma$ on a complete lattice $A$ is a function $\sigma : A \rightarrow A$ such that

1. $\sigma$ is increasing: $\forall x \in D, \quad x \leq \sigma(x)$,
2. $\sigma$ is idempotent: $\forall x \in D, \quad \sigma(x) = \sigma(\sigma(x))$,
3. $\sigma$ is monotone: $\forall x, y \in D, \quad x \leq y \Rightarrow \sigma(x) \leq \sigma(y)$.

A closure operator is **continuous** when

$$\sigma(\bigvee\limits_{i} x_i) = \bigvee\limits_{i} \sigma(x_i)$$
Composing concurrent strategies

Composing $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$.

\[
A \xrightarrow{\sigma} B \xrightarrow{\tau} C
\]

* * *
Composing concurrent strategies

Composing \( \sigma : A \rightarrow B \) and \( \tau : B \rightarrow C \).

\[
\begin{align*}
A \xrightarrow{\sigma} B \xrightarrow{\tau} C \\
* \quad * \quad * \quad & \quad \tau \\
* \quad y_1 \quad z_1 
\end{align*}
\]
Composing concurrent strategies

Composing $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$.

\[
\begin{array}{c}
A \xrightarrow{\sigma} B \xrightarrow{\tau} C \\
\downarrow \sigma & \downarrow \tau & \downarrow \tau \\
* & y_1 & * \\
\sigma & & \\
x_1 & y_2 & z_1
\end{array}
\]
Composing concurrent strategies

Composing $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$.

\[ A \xrightarrow{\sigma} B \xrightarrow{\tau} C \]

\[
\begin{array}{ccc}
\ast & \ast & \ast \\
\sigma & y_1 & z_1 \\
x_1 & y_2 & z_1 \\
\vdots & \vdots & \vdots \\
\end{array}
\]
Composing concurrent strategies

Composing $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$.

![Diagram showing the composition of strategies $\sigma$ and $\tau$.]
Composing concurrent strategies

Composing $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$.

\[
\begin{array}{c}
A \xrightarrow{\sigma} B \xrightarrow{\tau} C \\
\vdots \quad \vdots \quad \vdots \\
X \xleftarrow{\sigma} \quad \vdots \xleftarrow{\sigma} X \\
\therefore \quad \therefore \quad \therefore
\end{array}
\]
Closure operators as relations

closure operator  ⇐⇒  set of positions closed under meets
Closure operators as relations

closure operator $\iff$ set of positions closed under meets

$$\sigma \implies \text{fix}(\sigma) = \{ x \mid \sigma(x) = x \}$$
Closure operators as relations

Closure operator $\iff$ set of positions closed under meets

\[
\sigma \implies \text{fix}(\sigma) = \{x \mid \sigma(x) = x\}
\]

\[
x \mapsto \bigwedge \{y \in X \mid x \leq y\} \iff X
\]
Closure operators as relations

\[
\begin{align*}
\text{closure operator} & \iff \text{set of positions closed under meets} \\
\sigma & \Rightarrow \text{fix}(\sigma) = \{x \mid \sigma(x) = x\} \\
\sigma(x) & \leq y \Rightarrow \bigwedge\{y \in X \mid x \leq y\} \iff X
\end{align*}
\]

Extends to continuous closure operators.
Closure operators as relations

\[ \text{closure operator } \iff \text{set of positions closed under meets} \]

\[ \sigma \implies \text{fix(} \sigma \text{)} = \{ x \mid \sigma(x) = x \} \]

\[ x \mapsto \bigwedge \{ y \in X \mid x \leq y \} \iff X \]

A strategy \( \sigma : A \to B \) can be seen as a relation on \( A \times B \).
Definition
A position of a strategy $\sigma : A$ is **halting** when there is no Proponent move $m : x \rightarrow y$ in $\sigma$. 

Halting positions
Halting positions

The game $B \otimes B$. 

\[
\begin{array}{c}
V \otimes * \\
\downarrow V_L \quad q \otimes * \quad \downarrow q_R \\
q \otimes q \quad \sim \quad \ast \otimes q \\
\downarrow q_L \quad \sim \quad \ast \otimes F \\
V \otimes F \quad \downarrow V_L \quad \sim \quad q \otimes F \quad \downarrow q_R \\
F \otimes V \quad \downarrow F_R \quad \sim \quad \ast \otimes q \\
F \otimes V \quad \downarrow q_R \quad \sim \quad \ast \otimes q \\
\ast \otimes * \quad \downarrow q_L \\
\end{array}
\]
Halting positions

The parallel implementation of true and false.
Halting positions

The *left* implementation of true and false.

\[
\begin{array}{c}
\ast \otimes \ast \\
\downarrow \quad \quad \downarrow \\
q_L \quad \quad q_R \\
\downarrow \quad \quad \downarrow \\
q \otimes q \\
\downarrow \quad \quad \downarrow \\
q \otimes \ast \\
\downarrow \quad \quad \downarrow \\
V \otimes \ast \\
\downarrow \quad \quad \downarrow \\
q \otimes \ast \\
\downarrow \quad \quad \downarrow \\
V \otimes q \\
\downarrow \quad \quad \downarrow \\
q \otimes \ast \\
\downarrow \quad \quad \downarrow \\
V \otimes \ast \\
\downarrow \quad \quad \downarrow \\
q \otimes \ast \\
\downarrow \quad \quad \downarrow \\
V \otimes \ast \\
\end{array}
\]
Halting positions

The *right* implementation of true and false.
We would like strategies to be characterized by their *halting positions*. 
Ingenuous strategies

Definition
A strategy $\sigma : A$ is ingenuous when it is
1. positional,
2. deterministic,
3. courteous:

where $m$ is a Proponent move.
Ingenuous strategies as closure operators

Theorem

*Under suitable conditions, we have:*

\[ \sigma \iff \sigma^\circ \iff \text{Cl}(\sigma^\circ) \]

\[ \text{ingenuous strategies} \iff \text{ingenuous concurrent strategies} \]
Part IV

Innocence
Functoriality

There is a mismatch between sequential and concurrent games: we don’t have

\[(\sigma; \tau)^{\circ} = \sigma^{\circ}; \tau^{\circ}\]
The *livelock*:

\[(\sigma; \tau)^\circ \subseteq \sigma^\circ; \tau^\circ\]

\[A \xrightarrow{\sigma} B \xrightarrow{\tau} C\]

\[
\begin{array}{ccc}
* & * & * \\
/ & / & / \\
X & & Z \\
/ & / & / \\
\vdots & & \vdots
\end{array}
\]
Functoriality

The *livelock*:

\[(\sigma; \tau)^\circ \subseteq \sigma^\circ ; \tau^\circ\]

\[A \xrightarrow{\sigma} B \xrightarrow{\tau} C\]

Solution: handle infinite positions
The *deadlock*:

\[(\sigma; \tau)^{\circ} \supseteq \sigma^{\circ}; \tau^{\circ}\]

\[A \xrightarrow{\sigma} B \xrightarrow{\tau} C\]

\[\ast \quad \ast \quad \ast\]

\[\ast \quad \ast \quad \ast\]

\[x \quad y \quad z\]
Functoriality

The *deadlock*:

\[(\sigma; \tau)^* \supseteq \sigma^*; \tau^*\]

\[A \xrightarrow{\sigma} B \xrightarrow{\tau} C\]

Solution: add a scheduling criterion
The scheduling criterion

the left conjunction:

\[ B \otimes B \rightarrow B \]
The right boolean composed with the left conjunction:

\[ B \otimes B \rightarrow B \]
The scheduling criterion

Two kinds of tensors: $\otimes$ and $\bowtie$.

\[ B \otimes B \rightarrow B = B^* \bowtie B^* \bowtie B \]
The scheduling criterion

Two kinds of tensors: \( \otimes \) and \( \uplus \).

\[
\begin{array}{c}
\mathbb{B} \quad \otimes \quad \mathbb{B} \\
\text{q} \\
\left\{ \\
\text{F} \\
\text{q} \\
\left( \\
\text{V}
\right)
\end{array}
\]
The scheduling criterion

Two kinds of tensors: $\otimes$ and $\otimes$. 

\[
\begin{array}{ccc}
B & \otimes & B \\
\quad & q & \\
\quad & F & \\
\quad & q & \\
\quad & V & 
\end{array}
\]
The scheduling criterion

Two kinds of tensors: $\otimes$ and $\mathcal{B}$. 

\begin{align*}
\mathcal{B} & \otimes \mathcal{B} \\
q & \quad F \\
q & \quad V
\end{align*}
Functoriality

Definition
A strategy $\sigma : A$ is **receptive** when for every path $s : \ast \rightarrow x$ in $\sigma$ and for every Opponent move $m : x \rightarrow y$ the path $s \cdot m : \ast \rightarrow y$ is also in $\sigma$. 
Definition
A strategy $\sigma : A$ is **receptive** when for every path $s : * \rightarrow x$ in $\sigma$ and for every Opponent move $m : x \rightarrow y$ the path $s \cdot m : * \rightarrow y$ is also in $\sigma$.

Theorem
*Ingenuous strategies which satisfy the scheduling criterion and are receptive compose and satisfy*

$$(\sigma; \tau)^\circ = \sigma^\circ ; \tau^\circ$$
Towards innocence

The scheduling criterion detects directed cycles.
Towards innocence

The scheduling criterion does not detect non-directed cycles.
Towards innocence

The scheduling criterion does not detect non-directed cycles.

We need a more elaborate scheduling criterion.
Thanks for your attention

Any question?