Transfer Principles for Non Standard Analysis

Paul-Elliot Anglès d'Auriac

June 1, 2012

Abstract

Non standard analysis is a model of analysis, reals, functions and also infinitesimals, which differs from the usual interpretation of those but satisfies the same class of first order formulas. In this situation, we say that there is a *transfer principle*. Here we study variations of this principle, in constructive non standard analysis. We also study an application to the modelling of hybrid systems, and see that our weaker principle is not powerful enough. Then we generalize the transfer in a categorical setting.

Contents

This report is the synthesis of what I have done during my three months internship in the LMeASI team, in CEA, under the direction of Samuel Mimram. First of all I would like to thank all the team for their scientific discussion and especially Samuel Mimram for making me discover research and it's particular approach of problems.

1 Introduction

Mathematicians have studied reals or other very well known structures for a very long time. But with the progress made in logic and model theory, we have access to lots of other models, such as by doing the "ultraproduct".

The ultraproduct gives us a new model from any previous model[4]. The model obtained is interesting because lots of properties valid in the first model are also valid in the model generated by the ultraproduct. This fact, relating validity of the same formula in two different models, is called a transfer principle. The variety of formulas on which we can apply the transfer relates how close the two models are. In this report we will present the transfer principle between a model and its ultraproduct, but also for another (weaker) construction of model: the Fréchet product.

The hyperreals are the ultraproduct of the reals. They are interesting because they integrate elements that behave exactly like infinitesimals, so hyperreals allows us to make analysis with actual infinitesimals and to define derivation in a new way[3]. Should we redemonstrate everything in this new model, or can we take advantage of the previous results on the reals? We will see an example where infinitesimals are useful and in which the transfer principle give us directly the result we want[2].

But now the Category Theory had developped a lot[5], we have a generalization of the interpretation of formulas[1]. In the last part of this report we try to see if there is a transfer theorem adapted to this notion of validity of formulas.

Then we show another kind of transfer, with a really different meaning. It uses an embedding of logic in category theory. The idea remains the same, but the fact that we embed the logic in category completly change the meaning of the transfer.

2 Transfer principle

Transfer principles are theorems relating the validity of formulas between two models. This is useful to prove efficiently lots of statement in one exotic model, if there is a transfer principle from a very well known model, such as $\mathbb{R}$ as in the next example. In this section we see two transfer principles and an example in which they are useful.

2.1 Non standard analysis

Infinitesimals have largely been used in analysis, to simplify notations. But they are not part of the reals so we should be careful when doing calculus with them. Non standard analysis, also called the hyperreals, is a model of the reals which includes infinitesimals. To do that, instead of reals we speak about sequences of reals. An infinitesimal would then be a sequence converging towards 0. But we still should be able to inverse these "new reals", we have to consider quotient of the sequences: two sequences are in the same equivalence class if the set of indexes on which they coincide is "large enough". It turns out that the proper definition of "large set" is given by the notion of filter.

After having defined this, how can we take advantage of the infinitesimal? For example, we can redefine the derivative of a function f, as: $f'(x) = \frac{f(x+\delta)-f(x)}{\delta}$ where δ is an infinitesimal. But to be sure that it correspond to the usual notion, we will need a theorem: the transfer principle, that reals and hyperreals behave the same way.

Definition 1. A *filter* $\mathcal{F}$ over a set I is a (non-empty) subset of $\mathcal{P}(I)$, such that :

1. if A and B are in $\mathcal{F}$ then $A \cap B$ is in $\mathcal{F}$
2. if $A \subseteq B$ and A is in $\mathcal{F}$ then B is in $\mathcal{F}$

A filter is *proper* if it does not contain $\emptyset$. It is *principal* if there exists $x \in I$ such that $A \in \mathcal{F} \Leftrightarrow x \in A$.

An example of filter which we will be particularly interested in is the Fréchet filter.

Definition 2. The *Fréchet filter* on I is the filter of cofinite subsets of I (the sets whose complement is finite).

Proof. We prove that the Fréchet filter is a filter. We will write $\overline{A}$ for the complementary of A. If A and B are cofinite, then $\overline{A \cap B} = \overline{A} \cup \overline{B}$, which is finite because it is the union of two finite sets, so $A \cap B$ is cofinite. If A is cofinite and $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$ so $\overline{B}$ is finite, that is B is cofinite. □

A filter reflects the property of being "large". Suppose fixed a filter I, we and say that a set is *large* if it belongs to the filter, *small* otherwise. Under this interpretation, the axioms of filters say that a larger set than a large set is large, and the intersection of two large sets is large. We remark that in this case, a set and its complementary can be both small, such as even numbers in the Fréchet filter on $\mathbb{N}$: both $2\mathbb{N}$ and $2\mathbb{N}+1$ are not cofinite.

Definition 3. An *ultrafilter* $\mathcal{U}$ is a filter which satisfies also : for all $A \in \mathcal{P}(E)$, either A or $\overline{A}$ is in $\mathcal{U}$

We will be interested only in proper nonprincipal ultrafilters, because otherwise the induced notion of hyperreals is degenerated. The existence of such an ultrafilter can be shown assuming the axiom of choice.

Hyperreals are equivalence classes of sequences of reals, those equal on a large set, that is (x_i) and (y_i) are in the same equivalence class if $\{i : x_i = y_i\} \in \mathcal{U}$. So the ultrafilter decides on wich set of indexes the value of the sequence is meaningful. On a principal ultrafilter, the only relevant index is the one generating the ultrafilter, n, and two sequences of reals are equal over this ultrafilter if and only if they are equal on the index n. With a non principal ultrafilter, the situation is more interesting.

Proposition 1. *If $\mathcal{F}$ is a filter over I, and A_i are sets indexed by I, then the relation on $\Pi_I A_i$ written $\sim_\mathcal{F}$ and defined by: $(x_i) \sim_\mathcal{F} (y_i)$ if and only if $\{i \in I : x_i = y_i\} \in \mathcal{F}$, is an equivalence relation.*

Proof. It is reflexive because $\{i \in I : x_i = x_i\} = I$ and I is in every filter over I. Symmetry is immediate. The transitivity is because if $A = \{i \in I : x_i = y_i\}$ and $B = \{i \in I : y_i = z_i\}$ then since $A \cap B \subseteq \{i \in I : x_i = z_i\}$, A and B are in $\mathcal{F}$ which is upward closed and closed under intersection, $\{i \in I : x_i = z_i\}$ is in $\mathcal{F}$. □

Definition 4. The *$\mathcal{F}$-product* of a collection $(A_i)_{i\in I}$ with $\mathcal{F}$ a filter over I is $(\Pi_{i\in I}A_i)/\sim_{\mathcal{F}}$ also written $(\Pi_{i\in I}A_i)/\mathcal{F}$. A *Fréchet product* over I is a product over the Fréchet filter on I. An *ultraproduct* is a product over an ultrafilter. The set of *hyperreals* $\mathbb{R}^*$ over an ultrafilter $\mathcal{U}$ is the ultraproduct $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$.

For example, consider the sequence $\langle 0101010101\ldots\rangle$. If the set of odd indexes is in the ultrafilter, then the sequence is equal to the sequence valued 1 everywhere (written 1^*), otherwise it is equal to 0^*: the ultrafilter decides which of odd and even members of the sequence are relevant.

We write x^* for the hyperreal $\langle(x)_{i\in I}\rangle$ and $x \leq y$ for $\{i : x_i \leq y_i\} \in \mathcal{U}$, $x+y$ for $\langle x_i + y_i\rangle$ (where (x_i) and (y_i) are any representants of x and y, it does not depend on this choice). We say that a positive (resp. negative) hyperreal h is *unlimited* if $x^* \leq h$ (resp. $x^* \geq h$) for all real x, and *limited* otherwise. Similarly, h is be *infinitesimal* if it is lower than every positive real and greater than every negative one.

Theorem 2. *For all limited h, there exists a unique real r such that $h - r$ is infinitesimal. This r is called the* standard part *of h and is written* $\mathrm{St}(h)$.

Sketch of the proof. The existence is given by the compactness of $\mathbb{R}$: we take the least upper bound of the up-bounded (since h is limited) set $\{r \in \mathbb{R} : r^* \leq h\}$. The unicity can be verified easily by doing the difference of the two potential reals, which will be 0. □

This theorem is useful to perform computations in hyperreals and then "go back" into reals, as in the computation of a derivative.

In order to formulate the transfer principle in full generality in the next section, we will make use of the following notions:

Definition 5. An *internal set* A is a set equal to $\langle(A_i)_{i\in I}\rangle$ for some (A_i), where $\langle x_i\rangle \in \langle(A_i)_{i\in I}\rangle$ if and only if $\{i : x_i \in A_i\} \in \mathcal{F}$. Similarly, an *internal function* $f : A \to B$ is a function such that there exist $(f_i : A_i \to B_i)$ with $A = \langle A_i\rangle$, $B = \langle B_i\rangle$ and $f(\langle x_i\rangle) = \langle f_i(x_i)\rangle$.

It can be checked that internal sets are well defined: if $\langle x_i\rangle = \langle y_i\rangle$, then $\{i \in I : x_i \in A_i\} \in \mathcal{F}$ if and only if $\{i \in I : y_i \in A_i\} \in \mathcal{F}$. True, because each set contains the other set intersected with $\{i : x_i = y_i\}$ which is in $\mathcal{F}$. We can remark that the definition of internal functions coincides with internal sets, if we see a function as the set $\{(x, f(x))\}$.

We write A^* for $\langle A\rangle$, f^* for $\langle f\rangle$. We can see that the function $\cdot +^* \cdot$ is equal to $\cdot + \cdot$ for hyperreals as defined previously. The definition of internal sets allows us to extend relation similarly to how we extended functions.

2.2 The Transfer Principle

The *transfer principle* states that a formula is valid on $\mathbb{R}$ if and only if the corresponding formula (obtained by replacing involved sets and functions by their internal counterparts) is valid on $\mathbb{R}^*$. For instance, the fact that each real not equal to zero has an inverse implies that each hyperreal not equal to zero have an inverse. We can express this fact by the following formulas: $\forall x \in \mathbb{R}, x \neq 0 \Rightarrow \exists y \in \mathbb{R} : xy = 1$ is valid (over the reals), so $\forall x \in \mathbb{R}^*, x \neq 0 \Rightarrow \exists y \in \mathbb{R}^* : xy = 1$ is also valid (over the hyperreals).

We now define exactly to which formulas the transfer can be applied. We define the formulas themselves, containing terms, and their interpretation.

Definition 6. A *term* is built from symbols of functions and symbols of constants in an at least countable set $\mathcal{T}$, respecting arity (the number of arguments of a function). A *first order formula* is built from the following connectives:

$$\bot,\ \wedge\ ,\ \vee\ ,\ \Rightarrow\ ,\ \neg,\ \exists x,\ \forall x \text{ and predicates } \mathrm{R}(t_1, \ldots, t_n)$$

It is *closed* if it has no free variables.

Definition 7. An *interpretation function* $\mathcal{I}$ on a domain $\mathbb{K}$ is a mapping from symbols of functions to functions of $\mathbb{K}$, respecting arities, from variables and symbols of constant to constant of $\mathbb{K}$ (then defined by induction on terms); and from predicates to relations of $\mathbb{K}$. We say that $\mathcal{I} \models \phi$ if ϕ is true when we interpret the relations and terms with $\mathcal{I}$, using the natural semantics (Tarski's one) of the connectives and $\mathbb{K}$ for quantification ($\mathcal{I} \models \forall x : \phi(x)$ iff for all $x \in \mathbb{K}$, $\mathcal{I} \models \phi(x)$).

If $\mathcal{I}$ is an interpretation function, we write $\mathcal{I}^*$ for the interpretation function which maps _ to $\mathcal{I}(\_)^*$. First it is well defined: $(f(x_1, \ldots, x_n))^* = f^*(x_1^*, \ldots, x_n^*)$. We remark that it looks correct: if $\mathcal{I}$ maps a predicates to the equality relation, then $\mathcal{I}^*$ maps to the "equality on a large set" which is exactly the equality on hyperreals. The same remark can be made for $\leq$ or for the function $+$ for reals and hyperreals. Thanks to $\mathcal{I}$ and $\mathcal{I}^*$ we can interpret a formula in two different models.

Theorem 3 (Transfer principle)**.** *For all interpretation function $\mathcal{I}$ on $\mathbb{R}$, we have:*

$$\mathcal{I} \models \phi \qquad \textit{if and only if} \qquad \mathcal{I}^* \models \phi$$

The proof is by induction. We have to prove a stronger result to make it works.

Theorem 4 (Lòs theorem)**.** *For all interpretation function $\mathcal{I}$ and formula ϕ with free variables $x_1, \ldots, x_n$, and $r^1, \ldots, r^n \in \mathbb{R}^{\mathbb{N}}$, we have:*

$$\mathcal{I}^*{}_{\tilde{x} \mapsto \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} \phi \qquad \Longleftrightarrow \qquad \{i : \mathcal{I}_{\tilde{x} \mapsto \tilde{r_i}} \models_{\mathbb{R}} \phi\} \in \mathcal{U}$$

where $\mathcal{I}_{x \mapsto r}$ is the interpretation function $\mathcal{I}$ completed by the fact that the variable x is interpreted by r.

Before we start the induction on formulas, we need to prove three little lemmas on ultrafilters.

Lemma 5. *$\forall A, B \subseteq I$ we have $A \cup B \in \mathcal{U} \Longleftrightarrow A \in \mathcal{U}$ or $B \in \mathcal{U}$*

Proof. 1. If $A \cup B \in \mathcal{U}$ and $A \notin \mathcal{U}$,then $\overline{A} \in \mathcal{U}$, and $(A \cup B) \cap \overline{A} = B \cap \overline{A} \in \mathcal{U}$. As $B \cap \overline{A} \subseteq B$, we have $B \in \mathcal{U}$.

2. If $A \in \mathcal{U}$, then as $A \subseteq A \cup B$ we have $A \cup B \in \mathcal{U}$. Symmetrically for B. $\square$

Lemma 6. *$\forall A, B \subseteq I$ we have $A \cap B \in \mathcal{U} \Longleftrightarrow A \in \mathcal{U}$ and $B \in \mathcal{U}$*

Proof.

1. If $A \cap B \in \mathcal{U}$ then as $A \cap B \subseteq A$ we have $A \in \mathcal{U}$. Symmetrically for B.
2. If A and B are in $\mathcal{U}$, then $A \cap B$ is in $\mathcal{U}$, by definition of a filter.

$\square$

Lemma 7. *$\forall A, B \subseteq I$ we have $A \cup \overline{B} \in \mathcal{U} \Longleftrightarrow$ if $B \in \mathcal{U}$ then $A \in \mathcal{U}$*

Proof. $A \cup \overline{B} \in \mathcal{U} \Longleftrightarrow A \in \mathcal{U}$ or $\overline{B} \in \mathcal{U} \Longleftrightarrow A \in \mathcal{U}$ or $B \notin \mathcal{U} \Longleftrightarrow$ if $B \in \mathcal{U}$ then $A \in \mathcal{U}$. $\square$

We write $[\![\phi(\tilde{r})]\!]$ for $\{i : \mathcal{I}_{\tilde{x} \to \tilde{r}_i} \models_{\mathbb{R}} \phi\}$.

Proof of Theorem 4.

1. if ϕ is a predicate, we have:
$\mathcal{I}^*_{\tilde{x} \to \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} R(\tilde{x})$ iff $\widetilde{\langle r \rangle} \in R^*$ iff $\{i : \tilde{r_i} \in R\} \in \mathcal{U}$ iff $\{i : \mathcal{I}_{\tilde{x} \to \tilde{r_i}} \models_{\mathbb{R}} R(\tilde{x})\} \in \mathcal{U}$

2. $\mathcal{I}^*_{\tilde{x} \to \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} \phi \wedge \psi$
$\Longleftrightarrow \mathcal{I}^*_{\tilde{x} \to \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} \phi$ and $\mathcal{I}^*_{\tilde{x} \to \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} \psi$
$\Longleftrightarrow [\![\phi]\!] \in \mathcal{U}$ and $[\![\psi]\!] \in \mathcal{U}$ (by induction hypothesis)
$\Longleftrightarrow [\![\phi]\!] \cap [\![\psi]\!] \in \mathcal{U}$ (by Lemma 6)
$\Longleftrightarrow [\![\phi \wedge \psi]\!] \in \mathcal{U}$.

3. $\mathcal{I}^*_{\tilde{x} \to \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} \phi \vee \psi$
$\Longleftrightarrow \mathcal{I}^*_{\tilde{x} \to \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} \phi$ or $\mathcal{I}^*_{\tilde{x} \to \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} \psi$
$\Longleftrightarrow [\![\phi]\!] \in \mathcal{U}$ or $[\![\psi]\!] \in \mathcal{U}$ (by induction hypothesis)
$\Longleftrightarrow [\![\phi]\!] \cup [\![\psi]\!] \in \mathcal{U}$ (by Lemma 5)
$\Longleftrightarrow [\![\phi \vee \psi]\!] \in \mathcal{U}$.

4. $\mathcal{I}^*_{\tilde{x} \to \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} \phi \Rightarrow \psi$
$\Longleftrightarrow$ if $\mathcal{I}^*_{\tilde{x} \to \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} \phi$ then $\mathcal{I}^*_{\tilde{x} \to \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} \psi$
$\Longleftrightarrow$ if $[\![\phi]\!] \in \mathcal{U}$ then $[\![\psi]\!] \in \mathcal{U}$ (by induction hypothesis)
$\Longleftrightarrow \overline{[\![\phi]\!]} \cup [\![\psi]\!] \in \mathcal{U}$ (by Lemma 7)
$\Longleftrightarrow [\![\phi \Rightarrow \psi]\!] \in \mathcal{U}$.

5. $\mathcal{I}^*_{\tilde{x} \to \langle \tilde{r} \rangle} \models_{\mathbb{R}^*} \neg\phi$
$\Longleftrightarrow [\![\phi]\!] \notin \mathcal{U}$ (by induction hypothesis)
$\Longleftrightarrow \overline{[\![\phi]\!]} \in \mathcal{U}$
$\Longleftrightarrow [\![\neg\phi]\!] \in \mathcal{U}$

6. Let $A_{(x_i)} = \{i : \mathcal{I}_{x \to x_i} \models_{\mathbb{R}} \phi\}$ and $B = \{i : \forall x, \mathcal{I}_{x \to x} \models_{\mathbb{R}} \phi\}$. Then we will show that:
$$B \in \mathcal{U} \Longleftrightarrow \forall (x_i) \in \mathbb{R}^{\mathbb{N}}, A_{(x_i)} \in \mathcal{U}$$
If $B \in \mathcal{U}$, then let $x_i \in \mathbb{R}^{\mathbb{N}}$. $B \subseteq A_{(x_i)}$ so $A_{(x_i)} \in \mathcal{U}$
If $B \notin \mathcal{U}$, then $\overline{B} \in \mathcal{U}$. But $\overline{B} = \{i : \exists x_i, \mathcal{I}_{x \to x_i} \not\models_{\mathbb{R}} \phi\}$. So $\exists (x_n) \in \mathbb{R}^{\mathbb{N}}$ such that $\{i : \mathcal{I}_{x \to x_i} \not\models_{\mathbb{R}} \phi\} \in \mathcal{U}$ which is by the induction hypothesis:

$\exists (x_n) \in \mathbb{R}^{\mathbb{N}}$ such that $\{i : \mathcal{I}_{x\to x_i} \models_{\mathbb{R}} \phi\} \notin \mathcal{U}$ or $\exists (x_n) \in \mathbb{R}^{\mathbb{N}}$ such that $A_{(x_n)} \notin \mathcal{U}$.

This is exactly $[\![\forall x : \phi(x)]\!] \iff \forall x : [\![\phi(x)]\!]$.

7. Let $A_{(x_i)} = \{i : \mathcal{I}_{x\to x_i} \models_{\mathbb{R}} \phi\}$ and $B = \{i : \exists x, \mathcal{I}_{x\to x} \models_{\mathbb{R}} \phi\}$. Then we will show that:

$$B \in \mathcal{U} \iff \exists (x_i) \in \mathbb{R}^{\mathbb{N}}, A_{(x_i)} \in \mathcal{U}$$

If $A_{(x_i)} \in \mathcal{U}$, then since $A_{(x_i)} \subseteq B$, $B \in \mathcal{U}$.

If $B \in \mathcal{U}$, then we take the (x_i) given by B, $A_{(x_i)} = B \in \mathcal{U}$.

This is exactly $[\![\exists x : \phi(x)]\!] \iff \exists x : [\![\phi(x)]\!]$.

□

The transfer principle is just the particular case of Lòs theorem when ϕ has no free variables. If ϕ is valid in $\mathcal{I}$ then $\{i : \mathcal{I} \models_{\mathbb{R}} \phi\}$ is equal to $I \in \mathcal{U}$, and if ϕ is not valid in $\mathcal{I}$, it is equal to $\emptyset \notin \mathcal{U}$. So $\{i : \mathcal{I} \models_{\mathbb{R}} \phi\} \in \mathcal{U} \iff \mathcal{I} \models_{\mathbb{R}} \phi$

This gives a precise meaning to the transfer principle: be careful when using it! It is important to see that the semantics of the connectives are preserved, the only thing which changes is the range of the quantifiers and the meaning of the predicates (they change in a natural way).

This theorem can be extended (with a very similar proof) to second order formulas.

Theorem 8 (Claim)**.** *We have the transfer principle over second order formulas, where quantifying over sets is quantifying over internal sets.*

The proof works the same way as the previous one. We really need to quantify only over internal sets and not over all sets of hyperreals, otherwise the transfer is false. An example of formula on which the transfer would not work (quantifying over every sets): the compactness property. It is true in $\mathbb{R}$, but false in $\mathbb{R}^*$: $\mathbb{R}$ (seen as hyperreals) is bounded (by any unlimited) in $\mathbb{R}^*$ but has no least upper bound, since if h is unlimited then $h - 1$ is also unlimited, and any bound of $\mathbb{R}$ is unlimited. The transfer seen here allows us to prove that any bounded *and internal* subset of $\mathbb{R}^*$ has a least upper bound.

We can deduce from this property that $\mathbb{R} \subseteq \mathbb{R}^*$ is not internal.

2.3 Constructive non standard analysis

The problem with the ultraproduct construction is that the notion of ultrafilter is not constructive, and an infinitesimal would not always be expressed by a sequence converging towards 0. An infinitesimal is a sequence converging towards 0 *wrt. the ultrafilter* $\mathcal{U}$, whereas we would like our usual definition of convergence. Our notion of convergence corresponds to the one of the Fréchet filter, so we will try to replace $\mathcal{U}$ by $\mathcal{F}$ the Fréchet filter.

It is only a filter so we do not have anymore $A \in \mathcal{F} \vee \overline{A} \in \mathcal{F}$. But we will still try to have a transfer. Using $(\cdot)^*$ for the Fréchet product, we have:

Theorem 9. *If ϕ is made from predicates, $\forall$, $\exists$, $\wedge$ (so does not contains $\neg$, $\vee$, $\Rightarrow$), then*

$$\mathcal{I} \models_{\mathbb{R}} \phi \Longrightarrow \mathcal{I}^* \models_{\mathbb{R}^*} \phi$$

If ϕ is made from predicates, $\neg$, $\exists$, $\vee$, $\wedge$ (so does not contains $\forall$, $\Rightarrow$), then

$$\mathcal{I} \models_{\mathbb{R}} \phi \Longleftarrow \mathcal{I}^* \models_{\mathbb{R}^*} \phi$$

Proof. The proof is similar to the ultraproduct case. However, some lemmas are weakened.

1. Lemma 5: $A \cup B \in \mathcal{F} \Longleftarrow A \in \mathcal{F}$ or $B \in \mathcal{F}$
2. Lemma 6: $A \cap B \in \mathcal{F} \Longleftrightarrow A \in \mathcal{F}$ and $B \in \mathcal{F}$
3. Lemma 7: $A \cup \overline{B} \in \mathcal{U} \Longrightarrow$ if $B \in \mathcal{U}$ then $A \in \mathcal{U}$

Also the induction does only work on some connectives:

1. Predicates work similarly,
2. Conjunction work similarly as Lemma 6 is inchanged,
3. Disjunction lose the converse way as Lemma 5 lose the converse way,
4. Implication: even if we lose only one way in Lemma 7, the induction does not work here, because in the previous proof, we used the two ways of the induction hypothesis to prove one way.
5. Negation: Only the converse way works here. □

We can see that we lose the direct way for implication, disjunction and negation. Intuitively it seems correct because the semantics of the three connectives are different in intuitionistic and classical logic. Here are some examples of properties true and false in the Fréchet product of $\mathbb{R}$:

1. $\forall x : x \neq 0 \Rightarrow \exists y : xy = 1$ is valid in $\mathbb{R}^*$ (and in $\mathbb{R}$).
2. $\forall x : \neg(x = 0) \Rightarrow \exists y : xy = 1$ is not valid in $\mathbb{R}^*$ (but valid in $\mathbb{R}$).

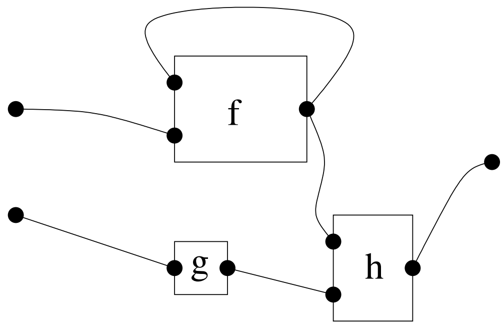
In the first formula the $\neq$ is the predicate of being not equal. Its interpretation on the Frechet product is to be different in a big set, that is equal to zero on a finite number of values.

The second formula says that a sequence not (equal to zero on a cofinite) would be inversible. But the hyperreal $\langle 010101\ldots\rangle$ is not equal to zero on a cofinite set, but for any y, xy will be equal to zero on at least the even numbers, so xy will be different from 1^*.

2.4 Nets, semantics and CPO

Now we give an example where we can use the transfer principle. Instead of having to prove a difficult thing, we show that it can be expressed with formulas on which we can apply the transfer principle. Then, as it has already been proved in $\mathbb{R}$, it is also true in $\mathbb{R}^*$.

Definition 8. A *net* is a kind of electrical circuit used to model the programming languages “of boxes and wires”.



a net

The aim of Samuel Mimram and Romain Beauxis' paper is to give a semantics to Nets wich represent continous time. For that purpose, instead of giving an ad-hoc semantic, they give a way to interpret nets for every fixpoint category.

Theorem 10. *If we have a fixpoint category and way to interpret boxes, we can interpret every nets using these boxes (for a formal claim see ?? Romain Beauxis and Samuel Mimram).*

Here we do not give the details, not even the definition of a fixpoint category but they can be found in [5]. We could guess what would be a semantic of nets according to how we represent them as electrical circuits, with boxes with input, output, and wires connecting them. But here the main point is that it gives a context to show that a transfer principle can be useful.

2.4.1 Sampling with non standard analysis

CPO is a fixpoint category. If we interpret boxes as continuous function from and to $\mathbb{R}^{\leq\mathbb{N}}$, the cpo of possibly finite sequences of $\mathbb{R}$ ordered by the length of the sequences, then we recover the usual "Kahn-semantic": discrete time, and at each step values are propagated over the wires.

But we would like to be able to do it and recover a "continuous time" semantic: as in electrical circuits, data flow over the wire continuously, such that we can integrate or derivate it.

To do that we can think of interpret boxes in $\mathbb{R}^{\leq\mathbb{R}_+}$ but we also want to be able to express derivation and integration. With this Cpo and continuous functions, we could not do that. But we can achieve it with infinitesimals: $f'(x)$ would then be $\frac{f(x+\delta)-f(x)}{\delta}$ which is expressible with boxes and wires. A natural candidate, close to $\mathbb{R}^{\leq\mathbb{R}_+}$ but with access to infinitesimals, is ${\mathbb{R}^*}^{\leq\mathbb{N}^*}$. We can relate these two sets by a kind of sampling: δ will represent a discrete but infinitesimal step in time.

But we need to show that ${\mathbb{R}^*}^{\leq\mathbb{N}^*}$ is an object of an interesting fixpoint category, which would give us a continuous semantic of nets. To do that we use a transfer. Our set is member of **ICPO** the collection of ultraproduct of Cpos. So if we prove that **ICPO** is a fixpoint category, we have achieved our goal.

We only need one way in our transfer principle: we have the result in the standard model, and we want to have it in the product. But we still need the formulas to be simple enough to be expressed in the language of the transfer. Here is a non exhaustive list of what we had to verify:

1. Axioms of Cpos:
   (a) $\text{Directed}(A) \equiv \forall x, y \in A : \exists z \in A : x \leq z \wedge y \leq z$
   (b) $\text{Supremum}(m, A) \equiv \forall x \in A : x \leq m \wedge \forall m' \in A : (\forall x \in A : x \leq m') \Rightarrow m \leq m'$
   (c) $\text{CPO}(E) \equiv \forall A \subseteq E : \text{Directed}(A) \Rightarrow \exists m \in E : \text{Supremum}(m, A)$
   (d) $\text{Scott}(f) \equiv \forall A \subseteq E : \forall m \in E : \text{Supremum}(m, A)$
   $\Rightarrow \text{Supremum}(f(m), f(A))$
2. Axioms of Category:
   (a) $\forall A, B, f : A \to B, g : B \to C : \text{CPO}(A, B) \wedge \text{Scott}(f, g) \Rightarrow \text{Scott}(g \circ f)$
   (b) $\text{Identity}(f, A) \equiv \forall B : \text{CPO}(B), \forall g : A \to B : \text{Scott}(g) \Rightarrow g \circ f = g \wedge \forall g : B \to A : \text{Scott}(g) \Rightarrow f \circ g = g$
   (c) $\forall A, \exists id_A : A \to A : \text{CPO}(A) \wedge \text{Identity}(id_A, A)$
3. Axioms of fixpoint

We need to be careful: after applying the transfer, it is not really the same property that is true, because the semantic of quantification has changed. So Internal Cpos are not always Cpos. But we have chosen our category carefully: the property of being in **ICPO** is exactly the transfer of the property of being in **CPO**. So by transfer we have that **ICPO** is a fixpoint category !

2.4.2 Conclusion of the example

Even if it does not add new possibilities, transfer principle allows us to simplify some proofs, and to take benefit of a very well known set, to prove things that would otherwise be very unpractical.

Here we have not been clear about which $(\cdot)^*$ we were talking about. As only the transfer principle for ultraproducts is powerful enough, it is the $(\cdot)^*$ of ultraproduct. The first aim of the internship was to find a transfer principle to have a semantic of net with a constructive object, the Fréchet filter. But, even if we don't have any disjunction in our formulas, we still need the implication. So the transfer principle I proved is not powerful enough. But I next moved to another transfer principle to see what it can offers.

3 Categorical Logic

Here we study a more abstract way to define a semantic for formulas, which is more general in the sense that the usual semantic is an instance of this construction (but the most natural one). We then study to what extent the transfer principle holds in this semantics both for the ultraproduct and the Frechet product, and how it differs from the previous transfer. The particularity is that the semantic of every connectives depends on which model we are interpreting the formula. By doing the ultraproduct of a model, we change the semantic of every connectives, instead of only the quantifiers as in the previous transfer.

It requires some definitions. It is not important to understand all of them in full generality, but it is important to see what they do mean in **Set**. An advice would be to have a first overview of the definitions, then to read how we embed logic in category, refering back to the definition when needed. The idea is to embed the semantic in a category, which will be our model, using categorical property of existence, such that the interpretation in **Set** corresponds to the usual semantic.

3.1 Definitions

We often use notions of category theory that we recall here.

Definition 9. A *category* consists of a collection of objects and a collection of morphisms between objects, together with an associative composition $\circ$ of morphisms, such that for all objects A there exists an identity morphism Id_A which is neutral for right and left composition.

Example 1. The category of sets and functions between sets,

2. The category of groups and morphisms of groups,

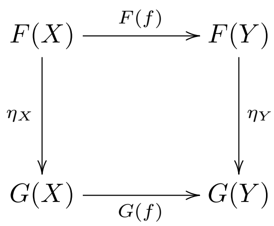
3. The category of open spaces of a topological space with the inclusion as arrow relation,

4. ...

Definition 10. A *functor* $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is a mapping of objects (and resp. morphisms) of $\mathcal{C}$ to objects (and resp. morphisms) of $\mathcal{D}$ such that if $f : A \to B$ then $Ff : FA \to FB$ and compatible with composition and identity: $F(f \circ g) = Ff \circ Fg$ and $F(Id_A) = Id_{FA}$.

Functors are the canonical morphisms of categories. **Cat** is the category of (small) categories with functor as arrows.

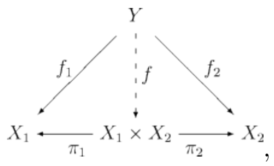
Definition 11. A *natural transformation* η from F to G, two functors from $\mathcal{C}$ to $\mathcal{D}$, is a collection of morphisms of $\mathcal{D}$ indexed by objects of $\mathcal{C}$ such that the following diagram:

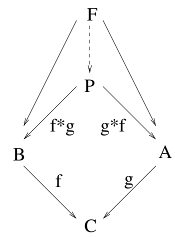


commutes.

Definition 12. A *limit* for a diagram in $\mathcal{C}$ (diagram as the one above) is an object L, and an arrow from L to every object of the diagram, such that the diagram commutes and universal in the sense that for every other object L' and arrows making the diagram commutes, there exists a unique arrow from L' to L making the whole diagram commute. Cartesian product and pullbacks are examples of limits. Finite limits are limits from finite diagrams.

Definition 13. A *cartesian product* is a limit for the diagram of two objects without arrows. A *pullback* is a limit of $B \longrightarrow C \longleftarrow A$:

A cartesian product: 

, and a pullback: 

.
Note that the way we denote the morphisms (π_1 and $f * g$) will be used.

Definition 14. An *adjunction* from $\mathcal{C}$ to $\mathcal{D}$ is two functors: $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and for each $c \in \mathcal{C}$, $d \in \mathcal{D}$, a bijection $\phi_{c,d}$ between $\mathcal{D}(Fc, d)$ and $\mathcal{C}(c, Gd)$ which is natural in c and d.

Definition 15. A *mono* m is an arrow which is left invertible: for every morphisms f, g, $g \circ m = f \circ m$ implies $g = f$.

In **Set**, monos are injective functions.

Definition 16. A *subobject* of an object X is an object A together with a mono $A \rightarrowtail X$.

In **Set**, subobjects modulo bijections are subsets. Subobjects can be view as a preorder, (or a poset when quotienting cycles), with $i_1 \leq i_2$ iff there is mono m such that $i_1 = i_2 \circ m$. If A is an object of the category, we write $\mathrm{Sub}(A)$ for the category of subobjects

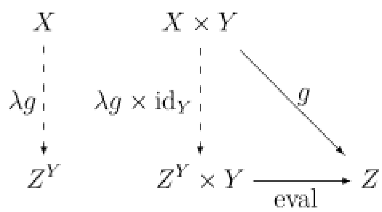
Definition 17. An object Z^Y, together with a morphism

$$\mathrm{eval}\colon (Z^Y \times Y) \to Z$$

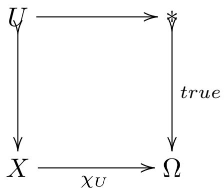
is an *exponential object* if for any object X and morphism $g : (X \times Y) \to Z$ there is a unique morphism

$$\lambda g\colon X \to Z^Y$$

such that the following diagram commutes:



Definition 18. In a category $\mathcal{C}$ with finite limits, a *subobject classifier* is an object Ω together with a monomorphism *true* $: * \rightarrowtail \Omega$ out of the terminal object, such that for every monomorphism $U \rightarrowtail X$ in $\mathcal{C}$ there is a unique morphism $\chi_U : X \to \Omega$ such that there is a pullback diagram



Definition 19. A *topos* is a category with

1. Finite limits and colimits
2. Exponential objects
3. A subobject classifier

Set is a topos. Topoi have a lots of categorical property, and are quite similar to **Set**.

Definition 20. A *Heyting algebra* H is a bounded lattice such that for all a and b in H there is a greatest element x of H such that

$$a \wedge x \leq b$$

x is be written $a \Rightarrow b$. We can also say that a Heyting algebra is a poset cartesian closed category.

Proposition 11. *If $\mathcal{C}$ is a topos, then for all A object of $\mathcal{C}$,* Sub(A) *is a Heyting algebra, and has finite limits (so has binary meets $\wedge$).*

$f \mapsto g * f$ is a functor, written $g*$ (see the definition of the pullback).

Proposition 12. *If $\mathcal{C}$ is a topos, then for all projection $\pi : A \times B \to A$, the functor $\pi*$ has both a right and a left adjoint.*

3.2 Interpreting logic in categories

The language on which we work is sligthly different from the one of section 2.2. It is be more expressive: variables are typed in quantification. We fix a category $\mathcal{C}$ as model for formulas, an interpretation of a type T will be an object of the category, (in **Set** the set of elements of type T). An interpretation of an environment will be the cartesian product of the types of the environment. And finally the interpretation of a formula with free variables in a context typing its free variables will be a subobject of the interpretation of the environment. In **Set** it will be a subset, the elements of the environment which satisfies the formula.

Now we define the language. It is sorted: terms will have a type, predicates a signature (types of its argument).

Definition 21. A *language* $\mathcal{L}$ consists of:

- a set of types,
- terms: constants together with their types, symbols of functions with their signature $(A_1, \cdots, A_n; B)$ (the type of its argument and the type of the resulting term)

- predicates: symbols together with signatures.

The formulas of $\mathcal{L}$ are defined inductively with:
$\wedge$, $\vee$, $\Rightarrow$, $\top$, $\bot$, $\exists x$, $\forall x$, $R(t_1, \ldots, t_n)$

Now we define the notion of categorical model. Most of the several properties used here are defined in the previous section, but here we will discuss their meaning. We will often see what does it means in **Set** to have an intuition of the soundness of the definitions.

Definition 22. An interpretation consists of a category $\mathcal{C}$ and a mapping $[\![\cdot]\!]$ such that:

1. The category $\mathcal{C}$ is a topos (think of **Set**)
2. A type A is interpreted by an object of $\mathcal{C}$
3. An environment $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$ is interpreted by $[\![A_1]\!] \times \cdots \times [\![A_n]\!]$.
4. A function symbol with signature $(A_1, \ldots, A_m; B)$ will be interpreted as a morphism $[\![f]\!] : [\![A_1]\!] \times \ldots \times [\![A_m]\!] \to [\![B]\!]$
5. A term in a context $\Gamma = x_0 : A_0, \cdots, x_n : A_n$ is interpreted as a functions from the interpretation of the environment to the interpretation of the type of the term, as follows:
    (a) A variable $[\![\Gamma \mid x_i]\!]$ is the ith projection.
    (b) A composite term $[\![\Gamma \mid f(t_1, \cdots, t_m)]\!]$ is the composition: $[\![f]\!] \circ \langle [\![t_1]\!], \cdots, [\![t_m]\!] \rangle$
    Here if the environment is empty, $[\![\Gamma]\!]$ will be the terminal object, in **Set** a singleton. And a function from this singleton to a set of a type is exactly an element of this type! A "closed" term of type T is, as expected, an element of $[\![T]\!]$.
6. A basic relation symbol R with signature $(A_1, ..., A_n)$ is interpreted as a subobject $[\![R]\!] \in \mathbf{Sub}([\![A_1]\!] \times \ldots \times [\![A_n]\!])$
7. A formula in a context $\Gamma \mid \phi$ will be interpreted as a subobject $[\![\Gamma \mid \phi]\!] \in \mathbf{Sub}([\![\Gamma]\!])$ in the way described below.
8. We will say that we have a logical entailement $\Gamma \mid \phi \vdash \psi$ verified if $[\![\phi]\!] \leq [\![\psi]\!]$ ($\leq$ is on subobjects).

    In **Set**, $\leq$ is the inclusion order. It means that the subset of Γ verifying ψ is included in the subset verifying ψ. That sounds correct in regard to our usual meaning of $\phi \vdash \psi$.

It remains to explain how formulas are interpreted as subobjects. What we define should be reflecting our common sense of the semantics, but we will have a discussion about soundness later. So, formulas are interpreted such that:

1. $\Gamma \mid \bot$ will be interpreted as the initial object (and its unique morphism to $[\![\Gamma]\!]$).

    In **Set** the initial object corresponds to the empty set. For every given environment, the subset verifying false is empty.

2. $\Gamma \mid \top$ by the maximal subobject: $1_{[\![\Gamma]\!]} : [\![\Gamma]\!] \to [\![\Gamma]\!]$

   It means that the subobject verifying $\top$ is the object itself: all elements of an environment verify $\top$.

3. An atomic formula $\Gamma \mid R(t_1, \cdots, t_m)$ is interpreted as the pullback :

$$\begin{array}{ccc} [\![\Gamma \mid R(t_1, \cdots, t_m)]\!] & \longrightarrow & [\![R]\!] \\ \downarrow & & \downarrow {\scriptstyle [\![R]\!]} \\ [\![\Gamma]\!] & \xrightarrow[\langle [\![t_1]\!], \cdots, [\![t_m]\!] \rangle]{} & [\![A_1]\!] \times \cdots [\![A_m]\!] \end{array}$$

   In **Set**, this pullback selects the elements of Γ such that the corresponding $(t_1, \cdots, t_n)$ belongs to R.

4. $[\![\Gamma \mid \phi \wedge \psi]\!] = [\![\Gamma \mid \phi]\!] \wedge [\![\Gamma \mid \psi]\!]$. The first $\wedge$ is syntactic, whereas the second is the meet on **Sub**($[\![\Gamma]\!]$), and its existence is given by the fact that our category is a topos. The meet in **Set** is the biggest subset included in the two subset: the intersection. That correspond to our intuition of the conjunction.

5. $[\![\Gamma \mid \phi \Rightarrow \psi]\!] = [\![\Gamma \mid \phi]\!] \Rightarrow [\![\Gamma \mid \psi]\!]$ where the first $\Rightarrow$ is syntactic whereas the second is the Heyting operation on **Sub**($[\![\Gamma]\!]$) (which is a Heyting algebra because $\mathcal{C}$ is a topos)

6. $[\![\Gamma \mid \exists x : A.\phi]\!] = \exists_A [\![\Gamma, x : A \mid \phi]\!]$ where $\exists_A$ is the right adjoint of π_A*, and where π_A is the projection $[\![\Gamma]\!] \times [\![A]\!] \longrightarrow [\![\Gamma]\!]$. Cf. Proposition 12.

7. $[\![\Gamma \mid \forall x : A.\phi]\!] = \forall_A [\![\Gamma, x : A \mid \phi]\!]$ where $\forall_A$ is the left adjoint of π_A*, and where π_A is the projection $[\![\Gamma]\!] \times [\![A]\!] \longrightarrow [\![\Gamma]\!]$.

An important remark is that an interpretation is fully defined by its value on the types, terms, relations and by the domain topos. The values are what define the interpretation itself: what sense do we give to the syntactic relation, symbols of functions... and the topos gives us the notion of logic: what sense do we give to conjunction, implication, ... But are our definitions interesting and suitable for our purpose? This can be answered in a way by the soundness theorem.

Theorem 13 (Soundness 1)**.** *An interpretation on* **Set** *validates a formula iff the corresponding interpetation validates the formula in Tarski's semantics (the one of section 2.2).*

Theorem 14 (Soundness 2)**.** *If an interpretation validates each axiom of a theory, then it validates each formula that can be derived from these axioms.*

The next step is to define the Fréchet product of categories and interpretation, and to show the Categorical Lòs Theorem to have the transfer. But here the transfer actually has a very different meaning.

3.3 Ultraproducts of categories and interpretation

Definition 23. Let $\mathcal{F}$ be a filter on I and $(\mathcal{C}_i, [\![\cdot]\!]_i)_{i\in I}$ be categorical interpretations. We call *Frechet product* the couple $(\Pi_I \mathcal{C}_i, [\![\cdot]\!]_{\Pi_I})$ where:

1. $\Pi_I \mathcal{C}_i$ is (the category of the) collections $\langle A_i \rangle$, $A_i \in Ob(\mathcal{C}_i)$ and $\langle f_i : A_i \to b_i \rangle : \langle A_i \rangle \to \langle B_i \rangle$, $f_i \in Arrow(\mathcal{C}_i)$. The composition is $\langle f_i \rangle \circ \langle g_i \rangle = \langle f_i \circ g_i \rangle$. We will then prove that it is well defined.

2. $[\![\cdot]\!]_{\Pi_I}$ is (the interpretation) $\langle [\![\cdot]\!]_i \rangle$.

Now we are almost done with the definitions. The transfer theorem in this case would be : If a logical entailment is verified for almost all interpretation $[\![\cdot]\!]_i$, then it is for $[\![\cdot]\!]_{\Pi_I}$. We first prove that the above definition makes sense, i.e. that $\Pi_I \mathcal{C}_i$ is a category and that $[\![\cdot]\!]_{\Pi_I}$ is an interpretation. It is most of the work as the interpretation embed the logic.

Theorem 15. *If I is a set and $\mathcal{F}$ a filter over I, $([\![\cdot]\!]_i)_{i\in I}$ interpretations, and $[\![\cdot]\!]_\Pi$ the $\mathcal{F}$-product of the $[\![\cdot]\!]_i$, then $[\![\cdot]\!]_\Pi$ is an interpretation.*

Proof. We will prove each of the conditions of the definition.

1. *Topos:* We will admit the fact that the ultraproduct of toposes is a topos, but we will need some more powerful things, such as the commutation of operations with the equivalence relation.

   A morphism $\langle f_i : A_i \to B_i \rangle$ is of domain $\langle A_i \rangle$ and codomain $\langle B_i \rangle$. If we have $\langle \tilde{f}_i : \tilde{A}_i \to \tilde{B}_i \rangle = \langle f_i : A_i \to B_i \rangle$, then $\{i : f_i = \tilde{f}_i\} \in \mathcal{F}$, but as it is bigger, $\{i : A_i = \tilde{A}_i\} \in \mathcal{F}$ and $\langle A_i \rangle = \langle \tilde{A}_i \rangle$ and the domain is well defined. Codomain is well defined as well.

   The composition defined by $\langle f_i \rangle \circ \langle g_i \rangle = \langle f_i \circ g_i \rangle$ is well defined : if the domain of $\langle f_i \rangle$ and codomain of $\langle g_i \rangle$ are equal, they are equal on a set of $\mathcal{F}$ and $f_i \circ g_i$ is defined on a set of $\mathcal{F}$. So $\langle f_i \circ g_i \rangle$ is well defined and has good domain and codomain. And if $\langle f_i \rangle = \langle \tilde{f}_i \rangle$, then $\langle f_i \rangle \circ \langle g_i \rangle = \langle f_i \circ g_i \rangle = \langle \tilde{f}_i \circ g_i \rangle = \langle \tilde{f}_i \rangle \circ \langle g_i \rangle$ so it does not depend of the representant and is well defined on equivalence classes.

   For an object $\langle A_i \rangle$, the morphism $\langle Id_{A_i} : A_i \to A_i \rangle$ is well defined and is the identity of $\langle A_i \rangle$.

2. *Commutation with finite limits:* We have to prove that we have binary cartesian product, terminal object and equalizer.

   - The terminal object is the equivalence class of the constant terminal object.
   - Binary cartesian product. Let's show that $\langle A_i \rangle \times \langle B_i \rangle = \langle A_i \times B_i \rangle$, with the two projections $\pi_1 = \langle (\pi_1)_i \rangle$ and $\pi_2 = \langle (\pi_2)_i \rangle$. First of all it is well defined: if $\langle \tilde{A}_i \rangle = \langle A_i \rangle$ then $\langle \tilde{A}_i \times B_i \rangle = \langle A_i \times B_i \rangle$. Then, if we have C and two morphism $\langle f_i \rangle : C \to \langle A_i \rangle$ and $\langle g_i \rangle : C \to \langle B_i \rangle$, $\langle f_i \times g_i \rangle$ will be a good candidate for $\langle f_i \rangle \times \langle g_i \rangle$.
   - Equalizer: the equalizer $\mathrm{Eq}(\langle f_i \rangle, \langle g_i \rangle)$ is $\langle \mathrm{Eq}(f_i, g_i) \rangle$. It is well defined because $\mathrm{Eq}(f_i, g_i)$ is defined on a set of $\mathcal{F}$ and is the equalizer because we have commutation with $\circ$.

As every limit is generated by cartesian product and equalizer, and we have proved that they commute with equivalence classes, we have proved that the limit of a diagram of equivalence class is the equivalence class of the limit of the diagram.

3. *Meets and Heyting algebra:* As the meet is a cartesian product on the category of subobject, which is a pullback on the category, we have commutation: $\langle A_i \rangle \wedge \langle B_i \rangle = \langle A_i \wedge B_i \rangle$.

   We have that $\langle A \Rightarrow B \rangle$ is equal to $\langle A \rangle \Rightarrow \langle B \rangle$. The proof is very similar to the previous ones.

   We have shown that each operations (Heyting,...) are compatible with equivalences classes : $\langle A \rangle \star \langle B \rangle = \langle A \star B \rangle$ where $\star$ is any previous operation.

4. Now we will show that $\Pi\mathcal{M}$ is an interpretation :

   - $[\![1]\!]_{\Pi\mathcal{M}} = \langle [\![1]\!]_{\Pi\mathcal{M}_i} \rangle = \langle 1_{\mathcal{M}_i} \rangle = 1_{\Pi\mathcal{M}}$
   - $[\![x_1 : A_1, ..., x_n : A_n]\!]_{\Pi\mathcal{M}} = [\![A_1]\!]_{\Pi\mathcal{M}} \times ... \times [\![A_n]\!]_{\Pi\mathcal{M}}$ because we have already proved the stability of products throught equivalence class
   - $[\![f]\!]_{\Pi\mathcal{M}} : [\![A_1]\!]_{\Pi\mathcal{M}} \times ... \times [\![A_m]\!]_{\Pi\mathcal{M}} \to [\![B]\!]_{\Pi\mathcal{M}}$ as proved throught the fact that $\Pi\mathcal{M}$ is a category (for *dom* and *codom* of $[\![f]\!]_{\Pi\mathcal{M}}$).
   - $[\![\Gamma \mid t : B]\!]_{\Pi\mathcal{M}} : [\![\Gamma]\!]_{\Pi\mathcal{M}} \to [\![B]\!]_{\Pi\mathcal{M}}$, with
     - Projections remain projections
     - Composition is stable through equivalence classes
   - $[\![R]\!]_{\Pi\mathcal{M}} = \langle [\![R]\!]_{\mathcal{M}_i} \rangle = \langle [[\![R]\!]_{\mathcal{M}_i} \to [\![A]\!]_{\mathcal{M}_i}] \rangle = [\langle [\![R]\!]_{\mathcal{M}_i} \to [\![A]\!]_{\mathcal{M}_i} \rangle]$ because equivalence classes of *Sub* and $\Pi\mathcal{M}$ commutes (to do)
   - A formula in a context : $[\![\Gamma \mid \phi]\!]_{\Pi\mathcal{M}}$:
     - $[\![\Gamma \mid \top]\!]_{\Pi\mathcal{M}} = 1$,
     - Weakening, substitution: pullback is stable through equivalence classes,
     - Conjunction: meet is stable through equivalence classes,
     - Implication: $\Rightarrow$ (of Heyting algebras) is stable throught equivalence classes,
     - Adjoints: same argument.

Here we gave the details of the proof only for the first of the several things to prove, but each of them works the same way. We showed that product of an interpretation is still an interpretation because it can still be considered as defined by induction. □

Theorem 16. *If* $\{i : \mathcal{M}_i \models \Gamma \mid \phi \vdash \psi\} \in \mathcal{F}$, *then* $\Pi\mathcal{M} \models \Gamma \mid \phi \vdash \psi$

Proof. Suppose $E = \{i : \mathcal{M}_i \models \Gamma \mid \phi \vdash \psi\} \in F$.

By expanding definition, $E = \{i : [\![\Gamma \mid \phi]\!]_{\mathcal{M}_i} \leq [\![\Gamma \mid \psi]\!]_{\mathcal{M}_i}\}$.

So there exist f_i for almost all i such that the $[\![\Gamma \mid \psi]\!]_{\mathcal{M}_i} \circ f_i = [\![\Gamma \mid \phi]\!]_{\mathcal{M}_i}$.

By choosing any f_i for others i, we have: $\langle [\![\Gamma \mid \psi]\!]_{\mathcal{M}_i} \circ f_i \rangle = \langle [\![\Gamma \mid \phi]\!]_{\mathcal{M}_i} \rangle$, so $[\![\Gamma \mid \psi]\!]_{\Pi\mathcal{M}} \circ \langle f_i \rangle = [\![\Gamma \mid \phi]\!]_{\Pi\mathcal{M}}$ which is exactly $[\![\Gamma \mid \phi]\!]_{\Pi M} \leq [\![\Gamma \mid \psi]\!]_{\Pi M}$, or $\Pi\mathcal{M} \models \Gamma \mid \phi \vdash \psi$ □

3.4 A new transfer principle

The usual transfer theorem asserts that validity of first order formulas is the "same" in standard and non standard (wrt. an ultrafilter) models. For instance, since the formula $\forall x, x \in \{0;1\} \Rightarrow x = 0 \vee x = 1$ is valid in $\mathbb{R}$, we know that the formula $\forall x, x \in \{0;1\}^* \Rightarrow x = 0^* \vee x = 1^*$.

However, we have seen that the transfer for the Fréchet product is not necessarily valid for formulas containing $\Rightarrow$ and $\vee$. In fact, we actually do not have that $\forall x, x \in \{0;1\}^* \Rightarrow x = 0^* \vee x = 1*$ is valid in the Fréchet-hyperreals: think of $\langle 010101 \ldots \rangle$.

So there is an apparent contradiction: in the transfer theorem 17 since validity is preserved for *all* first order formulas, even if the non standard construction is obtained by a Fréchet product (ie. modulo a *filter*).

The exotic thing of embedding logic in categories is that the structure of the category define the semantic of the connectives. Here, when we do the product over the filter, we change the structure of the category, so the meaning of the connectives. It was already the case in our previous transfer theorem with the quantification which applied only for internal sets. But now even the simpler connectives have changed.

Let's study our example. The formula $\forall x : x \in \{0;1\} \Rightarrow x = 0 \vee x = 1$, with the natural meaning of $x \in \{0;1\}$, is true in **Set**. When we do the product of the interpretation in **Set**, we obtain an interpretation in **InternalSet**, interpreting $x \in \{0;1\}$ by $x \in \{0;1\}^*$, 0 by 0^* and 1 by 1^*. The statement seems to become false: $\langle 010101 \ldots \rangle$ is equal neither to 0^* nor to 1^* but is in $\{0;1\}^*$. What is wrong?

This paradox is due to the fact that the interpretation of $\vee$ has changed. The interpretation of $x = 0$ is the internal set $\{0\}$. The interpretation of $x = 1$ is the internal set $\{1\}$. But the interpretation of $x = 0 \vee x = 1$ is the meet of $\{0\}$ and $\{1\}$ which is $\{0;1\}^*$ in **InternalSet**, whereas it is only $\{0;1\}$ in **Set**.

Notice that there is an obvious forgetful functor U from **InternalSet** to **Set**: an internal set is simply a set and an internal function is a function. Now, in order for the transfer theorem to correspond with the situation described in section 2.2, the interpretations of connectives should commutes with U: $U(A \wedge B) = U(A) \wedge U(B)$ and the same for every connectives. More generally, we can consider an injecting functor instead of U, to try to have transfers in other categories on which there is no canonical forgetful functor.

4 Conclusion

We have shown that there is a number of possible variations on transfer principles, all related to ultraproduct or Frechet product, with different power and usefulness. The first one is about the hyperreals and is an equivalence, the second is about Fréchet's hyperreals but is weaker, the third change the meaning of the connectives in addition to changing the meaning of the predicates. Hyperreals are very useful because they act as the reals but we have access to actual infinitesimals. But the difficulties are hidden in the ultrafilter, as we want to see hyperreals as limits of reals. The Fréchet filter is more about our notion of limit but the corresponding transfer principle is not very powerful.

Because we want to see hyperreals as limits of reals, we can think of the "(ultra)filter monad": instead of considering $A^I/\mathcal{F}$, we consider the set of (ultra)filter over a A. In a way, the ultrafilter monad add the limits to every sequence. A monad is a kind of completion, as: Cauchy completion, free monoid over a set, and many others. If you apply the completion two times you won't get anything more: the Cauchy completion of a complete set is in bijection with the set itself. In the case of the ultrafilter monad, we add for every sequence a unique limit. It seems possible to construct a kind of hyperreals from that and to prove a transfer theorem. It would be a new interesting way to hyperreals as it is a monad, whereas the current construction of hyperreals add limits but not all of them: it is not a monad!

References

[1] S. Awodey and A. Bauer. Introduction to categorical logic. 2009.

[2] R. Beauxis and S. Mimram. A non-standard semantics for kahn networks in continuous time. *Arxiv preprint arXiv:1108.5062*, 2011.

[3] R. Goldblatt. *Lectures on the hyperreals: an introduction to nonstandard analysis*, volume 188. Springer Verlag, 1998.

[4] A. Halper. Ultraproducts and model theory. 2010.

[5] S. Mac Lane. *Categories for the working mathematician*, volume 5. Springer verlag, 1998.