Coherence in cartesian theories using rewriting

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Coherence theorems

The goal of this work is

- to better understand **coherence theorems**
- to provide **tools** to show such theorem

The coherence theorem for monoidal categories

A **monoidal category** $(C, \otimes, e, \alpha, \lambda, \rho)$ comes equipped with

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z) \qquad \lambda_x : e \otimes x \xrightarrow{\sim} x \qquad \rho_x : x \otimes e \xrightarrow{\sim} x$$

satisfying axioms:

$$\begin{array}{ccccc}
((x \otimes y) \otimes z) \otimes w & \longrightarrow & (x \otimes (y \otimes z)) \otimes w & \longrightarrow & x \otimes ((y \otimes z) \otimes w) \\
\downarrow & & & & \downarrow \\
(x \otimes y) \otimes (z \otimes w) & & \longrightarrow & & x \otimes (y \otimes (z \otimes w))
\end{array}$$

$$\begin{array}{ccc}
(x \otimes e) \otimes y & \longrightarrow & x \otimes (e \otimes y) \\
& \searrow \quad \swarrow & \\
& x \otimes y &
\end{array}$$

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satisfying axioms.

The **coherence theorem** for monoidal categories states that every diagram whose morphisms are composites of α, λ and ρ commutes:

$$\begin{array}{ccccc}
(e \otimes x) \otimes y & \longrightarrow & e \otimes (x \otimes y) & & \\
 & & & \searrow & \\
\Big\downarrow & & & & e \otimes (x \otimes (y \otimes e)) \\
 & & & & \Big\downarrow \\
x \otimes y & & \longrightarrow & & x \otimes (y \otimes e)
\end{array}$$

The coherence theorems for monoidal categories

In fact, there are various ways of formulating the coherence theorem:

1. **Coherence**:
   every diagram in a monoidal category made up of α, λ and ρ commutes.
2. **Strictification**:
   every monoidal category is monoidally equivalent to a strict monoidal category.
3. **Global strictification**:
   the forgetful 2-functor from strict monoidal categories to monoidal categories has a left adjoint and the components of the unit are equivalences.

The coherence theorems for symmetric monoidal categories

A monoidal category is **symmetric** when equipped with

$$\gamma_{x,y} : x \otimes y \to y \otimes x$$

satisfying axioms, which do not imply the commutation of

$$x \otimes x \underset{\mathrm{id}_{x \otimes x}}{\overset{\gamma_{x,x}}{\rightrightarrows}} x \otimes x$$

1. **Coherence**:
   every generic diagram in a monoidal category made up of α, λ, ρ and γ commutes.
2. **Strictification**:
   every symmetric monoidal category is symmetric monoidally equivalent to a strict symmetric monoidal category.
3. **Global strictification**: ...

Global strictification

There is a monad T on **Cat** whose

- strict algebras are strict monoidal categories,
- pseudo algebras are unbiased monoidal categories.

Theorem (Power'89)
The canonical 2-functor

$$T\text{-StrAlg} \to T\text{-PsAlg}$$

admits a left 2-adjoint such that the components of the unit of the adjunctions are equivalences of T-pseudo-algebras.

A generic framework for coherence

Here, we investigate general coherence theorems which

- apply to biased notions of categories
- are partial, i.e. coherence holds with respect to part of the structure (e.g. α, λ and ρ but not γ)
- handle structural morphisms that can erase or duplicate variables:

$$\delta_{x,y,z} : x \otimes (y \oplus z) \to (x \otimes y) \oplus (x \otimes z)$$

- use rewriting theory.

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We begin by studying the situation in an abstract setting.

Part I

Abstract coherence

An abstract setting

Fix a category $\mathcal{C}$ which we think of as describing an **algebraic structure**.

For instance, we have a theory of symmetric monoidal categories:

- the objects of $\mathcal{C}$ are formal tensor expressions

$$e \otimes ((x \otimes e) \otimes y)$$

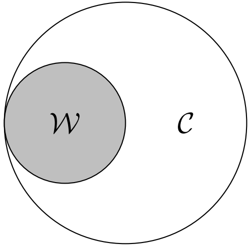
- morphisms are composites of α, λ, ρ and γ modulo axioms.

An abstract setting

Fix a category $\mathcal{C}$ which we think of as describing an **algebraic structure**.

We suppose fixed a subgroupoid $\mathcal{W} \subseteq \mathcal{C}$ with the same objects,
which we are interested in strictifying.

(for SMC, $\mathcal{W}$ would be the groupoid of composites of α, λ and ρ, but not γ)



Quotient of categories

The **quotient** $\mathcal{C}/\mathcal{W}$ is the universal way of making the elements of $\mathcal{W}$ identities

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & \nearrow_{\tilde{F}} & \\ \mathcal{C}/\mathcal{W} & & \end{array}$$

Question
When is the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{W}$ an equivalence of categories?

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Intuitively, when $\mathcal{W}$ does not contain non-trivial information!

Rigid groupoids

A groupoid $\mathcal{W}$ is **rigid** when either

(i) any two parallel morphisms $f, g : x \to y$ are equal

(ii) any automorphism $f : x \to x$ is an identity

(iii) $\mathcal{W}$ is equivalent to $\bigsqcup_I \mathbf{1}$

Quotienting by rigid groupoids

When $\mathcal{W}$ is rigid the **quotient** $\mathcal{C}/\mathcal{W}$ has a simple description:

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$$\begin{array}{ccc} x & \xrightarrow{\;f\;} & y \\ {\scriptstyle \mathcal{W}\ni v}\downarrow & & \downarrow{\scriptstyle w\in\mathcal{W}} \\ x' & \xrightarrow[\;g\;]{} & y' \end{array}$$

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- we compose $[f] : [x] \to [y]$ and $[g] : [y] \to [z]$ as

$$x \xrightarrow{f} y$$

$$y' \xrightarrow{g} z$$

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Rigidification

The **rigidification** $\mathcal{C}/\!\!/\mathcal{W}$ of $\mathcal{W}$ in $\mathcal{C}$ is obtained from $\mathcal{C}$ by identifying any two parallel morphisms in $\mathcal{W}$ (i.e. we make $\mathcal{W}$ rigid in a universal way).

Proposition
The quotient can be obtained is two steps:

$$\mathcal{C}/\mathcal{W} = (\mathcal{C}/\!\!/\mathcal{W})/\mathcal{W}$$

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In particular, the canonical functor

$$\begin{array}{ccc} \mathcal{C} & \dashrightarrow & \mathcal{C}/\mathcal{W} \\ & \searrow \quad \nearrow & \\ & \mathcal{C}/\!\!/\mathcal{W} & \end{array}$$

is surjective on objects and full.

Coherence for quotients

Theorem
The quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{W}$ is an equivalence of categories if and only if $\mathcal{W}$ is rigid.

$\mathcal{C}$	$\mathcal{C}/\mathcal{W}$
$x \overset{f}{\underset{g}{\rightrightarrows}} y$	$x \circlearrowleft f$
$x \overset{f}{\underset{g}{\rightrightarrows}} y$	x

Coherence for quotients

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The quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{W}$ is an equivalence of categories if and only if $\mathcal{W}$ is rigid.

Proof.
We need to show that it is faithful iff $\mathcal{W}$ is rigid.

- If the quotient functor is faithful, given $w, w' : x \to y$, we have $[w] = [w'] = \mathrm{id}$ and thus $w = w'$.
- If $\mathcal{W}$ is rigid, given $f, g : x \to y$ such that $[f] = [g]$, we have

$$\begin{array}{ccc} x & \xrightarrow[f]{} & y \\ {\scriptstyle \mathcal{W} \ni v}\downarrow & & \downarrow{\scriptstyle w \in \mathcal{W}} \\ x & \xrightarrow{g} & y \end{array}$$

  By rigidity, $v = \mathrm{id}_x$ and $w = \mathrm{id}_y$. □

Coherence for algebras

An **algebra** for $\mathcal{C}$ in $\mathcal{D}$ is a functor $\mathcal{C} \to \mathcal{D}$, we write $\mathsf{Alg}(\mathcal{C}, \mathcal{D})$ for the category of algebras.

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Theorem
A functor $\mathsf{F} : \mathcal{C} \to \mathcal{C}'$ is an equivalence iff $\mathsf{Alg}(\mathsf{F}, \mathcal{D}) : \mathsf{Alg}(\mathcal{C}, \mathcal{D}) \to \mathsf{Alg}(\mathcal{C}', \mathcal{D})$ is an equivalence natural in $\mathcal{D}$.

Proof.
Given a $\mathbf{2}$-category $\mathcal{K}$, the Yoneda functor

$$\begin{aligned} \mathsf{Y}_{\mathcal{K}} : \mathcal{K}^{\mathrm{op}} &\to [\mathcal{K}, \mathbf{Cat}] \\ \mathsf{C} &\mapsto \mathcal{K}(\mathsf{C}, -) \end{aligned}$$

is a local isomorphism. In particular, with $\mathcal{K} = \mathbf{Cat}$, we have $\mathsf{Y}_{\mathbf{Cat}}\mathcal{C} = \mathsf{Alg}(\mathcal{C}, -)$. □

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Proof.
Given a 2-category $\mathcal{K}$, the Yoneda functor

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is a local isomorphism. In particular, with $\mathcal{K} = \mathbf{Cat}$, we have $Y_{\mathbf{Cat}}\mathcal{C} = \mathsf{Alg}(\mathcal{C}, -)$. □

Conjecture (?)
The canonical functor $\mathsf{Alg}(\mathcal{C}/\mathcal{W}, \mathbf{Cat}) \to \mathsf{Alg}(\mathcal{C}, \mathbf{Cat})$ is an equivalence iff $\mathcal{W}$ is rigid.

Question
How do we show **rigidity** in practice?

In the following, we are interested in the case where $\mathcal{C}$ is a groupoid.

Abstract rewriting systems

An **abstract rewriting system P** is a graph

$$P_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} P_1$$

$$P \quad = \quad x \underset{g}{\overset{f}{\rightleftarrows}} y \xleftarrow{h} z$$

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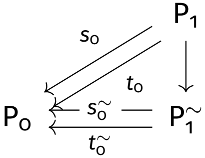


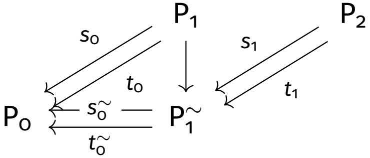
It generates a groupoid with $P_1^\sim$ as set of morphisms.

$$P \quad = \quad x \underset{g}{\overset{f}{\rightleftarrows}} y \xleftarrow{\;h\;} z$$

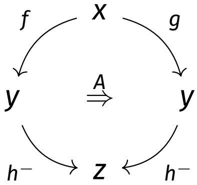
$$x \xrightarrow{\;f\;} y \xrightarrow{\;g\;} x \xrightarrow{\;f\;} y \xrightarrow{\;h^-\;} z$$

Abstract rewriting systems

An **extended abstract rewriting system** P is a graph



together with a set of 2-cells



such that

$$s_0^{\sim} \circ s_1 = s_0^{\sim} \circ t_1 \qquad\qquad t_0^{\sim} \circ s_1 = t_0^{\sim} \circ t_1$$

The case of monoidal categories

The prototypical situation we have in mind is the EARS P with

1. P_0: formal tensor expressions, e.g. $e \otimes ((x \otimes e) \otimes y)$
2. P_1: generated by α, λ, ρ (in context)
3. P_2: the coherences

Tietze equivalence

An extended abstract rewriting system $P = (P_0, P_1, P_2)$ **presents** the groupoid

$$\overline{\mathsf{P}} = \mathsf{P}^{\sim} / \sim$$

Tietze equivalence

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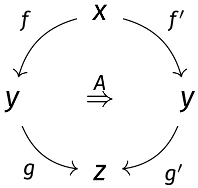
Two EARS P and Q are **Tietze equivalent** when $\overline{P} \cong \overline{Q}$.

Tietze transformations

Suppose given an EARS $P = (P_0, P_1, P_2)$ with a 2-cell



We have the following **Tietze transformations**:

- if A can be derived from other elements P_2, we can remove it,
- we can remove $f \in \mathsf{P}_1$ and $A \in \mathsf{P}_2$ replacing all occurrences of f by $f' \cdot g' \cdot g^-$.

Those transformations produce Tietze equivalent EARS.

Abstract rewriting systems

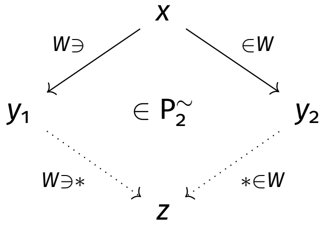
Suppose given an extended ARS P together with $W \subseteq \mathrm{P}_1$.

We say that P is ***W*-convergent** when it has

- termination: there is no infinite sequence of morphisms in W

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} x_2 \xrightarrow{f_2} \cdots$$

- local confluence:



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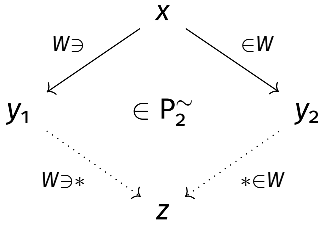
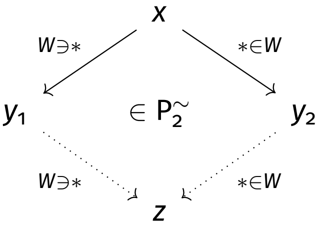


Note, by termination every element has a normal form: $x \xrightarrow[*\in W]{n_x} \hat{x}$.

Abstract rewriting systems

By adapting standard rewriting techniques,

Lemma (“Newman”)
If P *is* W*-convergent then it is* W*-confluent:*



Abstract rewriting systems

By adapting standard rewriting techniques,

Lemma ("Newman")
If P *is* W*-convergent then it is* W*-confluent:*

Lemma ("Church-Rosser")
If P *is* W*-convergent then for any two parallel* W*-morphisms in* $\overline{\mathrm{P}}$ *are equal.*

Proof.

$$
\begin{array}{ccccccccccccc}
x & \xrightarrow{p_1^-} & y_1 & \xrightarrow{q_1^+} & x_2 & \longrightarrow & \cdots & \longrightarrow & x_n & \xrightarrow{p_n^-} & y_n & \xrightarrow{q_n^-} & y \\
{\scriptstyle n_x}\downarrow & & \downarrow{\scriptstyle n_{y_1}} & & \downarrow{\scriptstyle n_{x_2}} & & & & \downarrow{\scriptstyle n_{x_n}} & & \downarrow{\scriptstyle n_{y_n}} & & \downarrow{\scriptstyle n_y} \\
\hat{x} & = & \hat{x} & = & \hat{x} & = & \cdots & = & \hat{x} & = & \hat{x} & = & \hat{x}
\end{array}
$$

□

Abstract rewriting systems

Corollary

If P *is* W*-convergent then the groupoid generated by* W *in* $\overline{\mathsf{P}}$ *is rigid.*

Abstract rewriting systems

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If P *is* W*-convergent then the groupoid generated by* W *in* $\overline{\mathsf{P}}$ *is rigid.*

Writing $N(\overline{\mathsf{P}})$ for the full subcategory of $\overline{\mathsf{P}}$ whose objects are normal forms (are not the source of a morphism in W),

Theorem
If (P, W) *is* W*-convergent then* $\overline{P}/W \cong N(\overline{\mathsf{P}})$.

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In the example of monoids, normal forms are expressions of the form

$$x_1 \otimes (x_2 \otimes (x_3 \otimes x_4))$$

A concrete description of normal forms

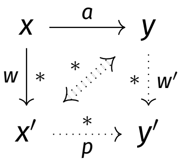
We have the intuition that the groupoid $N(\overline{P})$ is presented by the extended ars $\mathsf{P} \setminus W$ obtained by "restricting P to normal forms":

- $(\mathsf{P} \setminus W)_0$: the objects of $\mathsf{P} \setminus W$ are the those of P in W-normal form,
- $(\mathsf{P} \setminus W)_1$: the rewriting rules of $\mathsf{P} \setminus W$ are those of P whose source and target are both in $(\mathsf{P} \setminus W)_0$ (in particular, it does not contain any element of W, thus the notation),
- $(P \setminus W)_2$: the coherence relations are those of P_2 whose source and target both belong to $(\mathsf{P} \setminus W)_1^{\sim}$.

A concrete description of normal forms

Theorem
Suppose that

1. P *is* W*-convergent,*
2. *every rule* $a : x \to y$ *in* P_1 *with* x *is* W*-normal also has a* W*-normal target* y*,*
3. *for every coinitial rule* $a : x \to y$ *in* P_1 *and path* $w : x \xrightarrow{*} x'$ *in* W^**, there are paths* $p : x' \xrightarrow{*} y'$ *in* P_1^* *and* $w' : y \xrightarrow{*} y' \in W^*$ *such that* $a \cdot w' \overset{*}{\Leftrightarrow} w \cdot p$*:*



4. *for every coherence relation …*

Then $\mathsf{N}(\overline{\mathsf{P}})$ *is isomorphic to* $\overline{\mathsf{P} \setminus W}$*.*

Summing up

Given (P, W), we have shown that the following definitions of **coherence** of P wrt W are equivalent:

(i) Every parallel zig-zags with edges in W are equal (i.e. the subgroupoid of $\overline{P}$ generated by W is rigid).

(ii) The quotient map $\overline{\mathsf{P}} \to \overline{\mathsf{P}}/W$ is an equivalence of categories.

(iii) The inclusion $\mathsf{Alg}(\overline{\mathsf{P}}/W, -) \to \mathsf{Alg}(\overline{\mathsf{P}}, -)$ is an equivalence of categories.

(iv) The canonical morphism $N(\mathsf{P}) \to \overline{\mathsf{P}}$ is an equivalence.

Part II

Coherence from term rewriting systems

From ARS to TRS

In order to obtain result about actual categorical structures,
we need to go from ARS to term rewriting systems!

Term rewriting systems

A **term rewriting system** P consists of

- P_1: operations with arities
- P_2: equations between generated terms

Example
The TRS **Mon** for **monoids** is

$$\left\langle \begin{array}{r} m : 2 \\ e : 0 \end{array} \;\middle|\; \begin{array}{l} \alpha : m(m(x,y),z) = m(x,m(y,z)) \\ \lambda : \quad m(e,x) = x \\ \rho : \quad m(x,e) = x \end{array} \right\rangle$$

Term rewriting systems

An **extended term rewriting system** P consists of

- P_1: operations with arities
- P_2: 2-generators between generated terms
- P_3: equations between 2-generators

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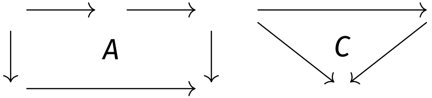

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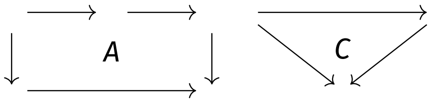

Remark
Fixing m and n, P induces an **abstract rewriting system** on terms $m \to n$.

Lawvere theories

A **Lawvere theory** $\mathcal{T}$ is a cartesian category such that objects are integers with cartesian product given by addition.

Lawvere theories

A **2-Lawvere theory** $\mathcal{T}$ is a cartesian 2-category with invertible 2-cells such that objects are integers with cartesian product given by addition.

Lawvere theories

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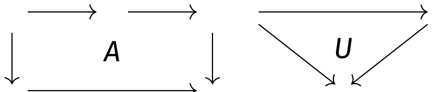
An **algebra** for $\mathcal{T}$ is a product-preserving 2-functor $\mathcal{T} \to \mathbf{Cat}$.

Example
An algebra for $\overline{\mathsf{Mon}}$ is a monoidal category.

Algebras for Mon

With **Mon** being

$$\left\langle \begin{array}{r} m : 2 \\ e : 0 \end{array} \;\middle|\; \begin{array}{l} \alpha : m(m(x,y),z) = m(x,m(y,z)) \\ \lambda : \qquad\quad m(e,x) = x \\ \rho : \qquad\quad m(x,e) = x \end{array} \;\middle|\; \cdots \right\rangle$$



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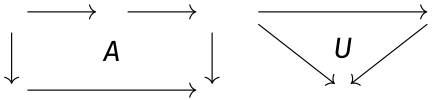

A functor $F : \overline{\mathsf{Mon}} \to \mathbf{Cat}$ consists of

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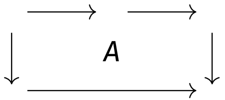

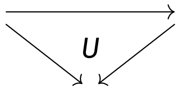

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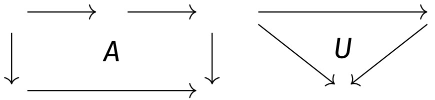

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- satisfying the axioms of monoidal categories

Quotienting by rigid subgroupoids

Fix a 2-TRS P with a subset $W \subseteq \mathsf{P}_2$ of 2-generators generating a $(2,1)$-category $\mathcal{W}$.

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A **critical branching** is a minimal non-trivial overlapping of the left member of two 2-generators.

Theorem
If P contains a 3-generator corresponding to the confluence of each critical W-branching then $\mathcal{W}$ is 2-rigid.

Coherence for monoids

Consider the TRS for monoids with W = all 2-cells.

$$\left\langle \begin{array}{r} m : 2 \\ e : 0 \end{array} \;\middle|\; \begin{array}{rl} \alpha : & m(m(x,y),z) \Rightarrow m(x,m(y,z)) \\ \lambda : & m(e,x) \Rightarrow x \\ \rho : & m(x,e) \Rightarrow x \end{array} \right.$$

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It is terminating.

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The critical branchings are

$$m(m(m(x_1,x_2),x_3)) \quad m(m(e,x_1),x_2) \quad m(m(x_1,e),x_2) \quad m(m(x_1,x_2),e) \quad m(e,e)$$

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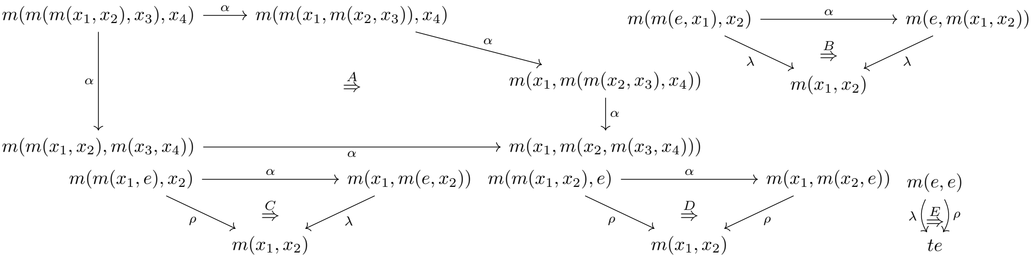

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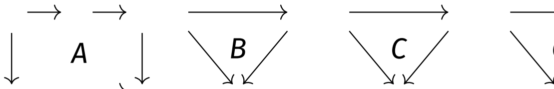

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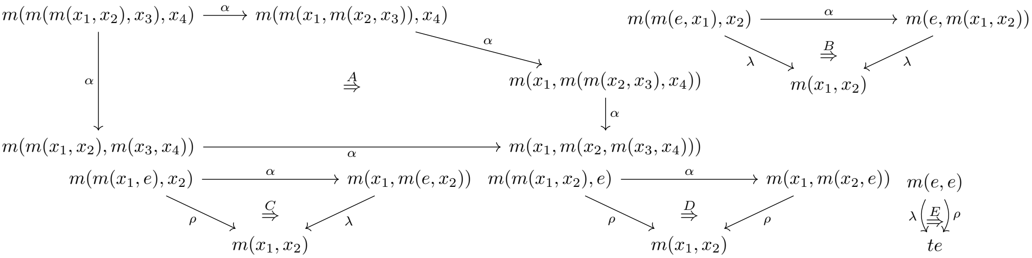

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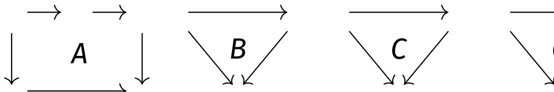

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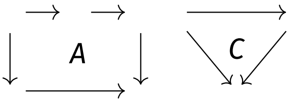

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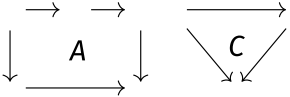

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Corollary (Coherence)
Any two structural morphisms in a monoidal category are equal.

Comparing algebras

A $\mathbf{2}$-functor

$$F : \mathcal{C} \to \mathcal{D}$$

between $\mathbf{2}$-categories is

- **essentially surjective** when for every $d \in \mathcal{D}$ there is $c \in \mathcal{C}$ such that $F(c) \simeq d$

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- **a local equivalence** when for every $c, c' \in \mathcal{C}$ the 1-functor

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- a **biequivalence** when there is an adjoint 2-functor $G : \mathcal{D} \to \mathcal{C}$ such that the components of unit and the counit are equivalences

$$c \simeq G(Fc) \qquad\qquad F(Gd) \simeq d$$

Algebras

Given a 2-LW $\mathcal{T}$, we write $\mathsf{Alg}(\mathcal{T})$ for the 2-category of algebras of $\mathcal{T}$.

Theorem (Yanofsky'00)
A morphism $\mathsf{F} : \mathcal{T} \to \mathcal{T}'$ of theories is a biequivalence if and only if the functor $\mathsf{Alg}(F) : \mathsf{Alg}(\mathcal{T}') \to \mathsf{Alg}(\mathcal{T})$ induced by precomposition is a biequivalence.

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Given $\mathcal{W}$ 2-rigid if we could show that the functor

$$\mathcal{T} \to \mathcal{T}/\mathcal{W}$$

is a biequivalence, we would deduce that

$$\mathsf{Alg}(\mathcal{T}/\mathcal{W}) \to \mathsf{Alg}(\mathcal{T})$$

is a biequivalence... but this is not the case!

Local equivalences vs biequivalences

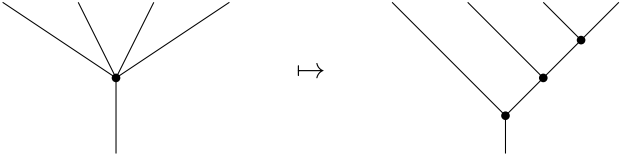
With $\mathcal{W}$ = all 2-cells, the functor

$$\mathsf{Mon} \to \mathsf{Mon}/\mathcal{W}$$

is an essentially surjective local equivalence (an equivalence on homs), there is a natural operation

$$\mathsf{Mon}/\mathcal{W} \to \mathsf{Mon}$$

but this is only a pseudofunctor:



A conjecture

Conjecture
When $\mathcal{W}$ is $\mathbf{2}$-rigid, the canonical $\mathbf{2}$-functor

$$\mathsf{Alg}(\mathcal{T}/\mathcal{W}) \to \mathsf{Alg}(\mathcal{T})$$

has a left adjoint such that the components of the unit are equivalences.

The case of symmetric monoidal categories

The theory of **commutative monoids** is

$$
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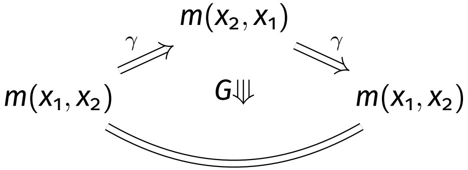

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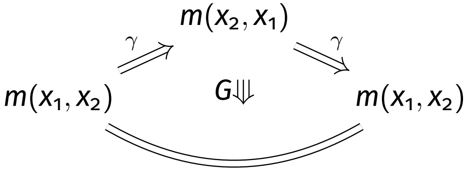

Its algebras are **symmetric monoidal categories**.

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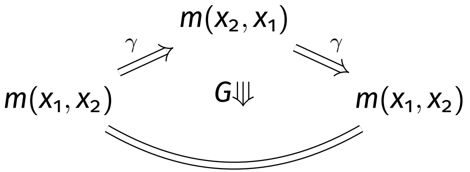

- if we take $\mathcal{W}$ generated by α, λ, ρ and add 3-cells as before, we are W-convergent: every symmetric monoidal category is equivalent to a strict one
- but we can do more!

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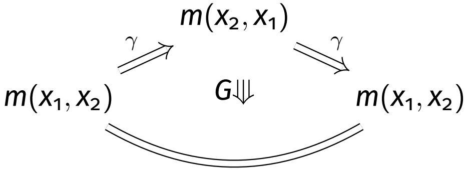

- it can be completed as a locally confluent presentation by adding a generator δ and a bunch of coherence relations

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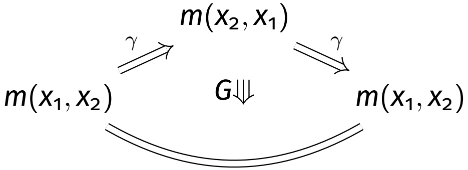
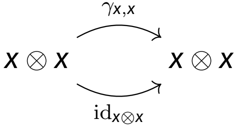


- it is not terminating otherwise we could show "full coherence" including



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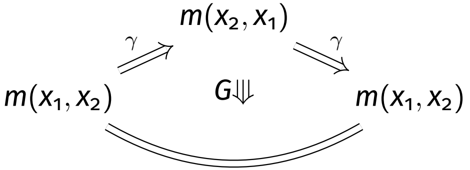

- restricting to affine terms (without repeated variables is not enough):

$$
m(x_1, x_2) \xRightarrow{\gamma(x_1, x_2)} m(x_2, x_1) \xRightarrow{\gamma(x_2, x_1)} m(x_1, x_2)
$$

- but we don't need both $m(x_1, x_2) \Rightarrow m(x_2, x_1)$ and $m(x_2, x_1) \Rightarrow m(x_1, x_2)$!

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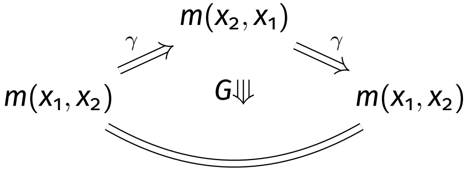
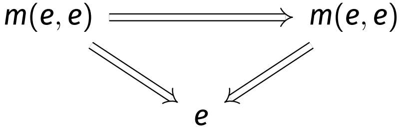


- if we only keep morphisms "sorting variables", we are almost terminating excepting for situations such as $m(e, e) \Rightarrow m(e, e)$ which can be removed:



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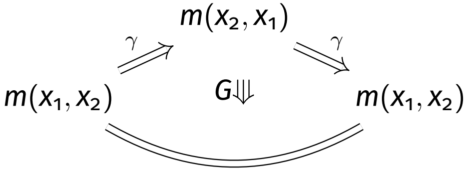

Theorem
In a symmetric monoidal category, every diagram whose source is a tensor product of distinct objects commutes.

Part III

Conclusion

Rigidity!

A quotient of (2-)category by a subgroupoid $\mathcal{W}$ is **coherent** when $\mathcal{W}$ is **rigid**.

This is the case when $\mathcal{W}$ is generated by a **convergent rewriting system**.

This also explains situations such as coherence for rig categories:

$$\delta_{x,y,z} : x \otimes (y \oplus z) \to (x \otimes y) \oplus (x \otimes z)$$
$$\delta'_{x,y,z} : (x \oplus y) \otimes z \to (x \otimes z) \oplus (y \otimes z)$$

$$\begin{array}{ccccc} & & (a+b)(c+d) & & \\ & \swarrow & & \searrow & \\ a(c+d)+b(c+d) & & & & (a+b)c+(a+b)d \\ \downarrow & & & & \downarrow \\ ac+ad+bc+bd & & \xrightarrow[\sim]{\qquad\qquad} & & ac+bc+ad+bd \end{array}$$

Thanks!

Questions?