COHERENCE IN CARTESIAN THEORIES USING REWRITING

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Abstract. The celebrated Squier theorem allows to prove coherence properties of algebraic structures, such as MacLane's coherence theorem for monoidal categories, based on rewriting techniques. We are interested here in extending the theory and associated tools simultaneously in two directions. Firstly, we want to take in account situations where coherence is partial, in the sense that it only applies for a subset of structural morphisms (for instance, in the case of the coherence theorem for symmetric monoidal categories, we do not want to strictify the symmetry). Secondly, we are interested in structures where variables can be duplicated or erased. We develop theorems and rewriting techniques in order to achieve this, first in the setting of abstract rewriting systems, and then extend them to term rewriting systems, suitably generalized in order to take coherence in account. As an illustration of our results, we explain how to recover the coherence theorems for monoidal and symmetric monoidal categories.

1. Introduction

1.1. **Coherence results.** Coherence results are fundamental in category theory. They can be seen both as a way of formally simplifying computations, by ensuring that we can consider strict algebraic structures without loss of generality, and as a guide for generalizing computations, by ensuring that we have correctly generalized algebraic structures in higher dimensions and taken higher-dimensional cells in account. Such results have been obtained for a wide variety of algebraic structures on categories such as monoidal categories [34], symmetric monoidal categories [34], braided monoidal categories [19], rig categories [29], compact closed categories [21], bicategories and pseudofunctors [32], etc. The coherence results are often quickly summarized as "all diagrams commute". However, this is quite misleading: firstly, we only want to consider diagrams made of structural morphisms, and secondly, we actually usually want to consider only a subset of those diagrams. Moreover, the commutation of diagrams is not the only way of formulating the coherence results. One of the aims of this article is to clarify the situation and the relationship between those various approaches.

Key words and phrases: coherence, rewriting system, Lawvere theory.

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1.2. **Coherence from rewriting theory.** A field which provides many computational techniques in order to show that diagrams commutes is rewriting [4, 47]. Namely, when a rewriting system is terminating and locally confluent (which can be verified algorithmically by computing its critical branchings), it is confluent and thus has the Church-Rosser property, which implies that any two zig-zags can be filled by local confluence diagrams. By properly extending the notion of rewriting system with higher-dimensional cells in order to take coherence in account (those cells specifying which confluence diagrams commute) one is then able to show coherence results of the form "all diagrams commute" up to the coherence laws. This idea of extending rewriting theory in order to take coherence in account dates back to pioneering work from people such as Power [41], Street [46] and Squier [44]. It has been generalized in higher dimensions in the context of polygraphs [45, 9, 3], as well as homotopy type theory [23], and used to recover various coherence theorem [27, 15], which can often be interpreted as computing (polygraphic) resolutions in suitable settings [3]. However, the relationship between rewriting and various formulations of coherence was, to our knowledge, largely unexplored.

1.3. **Coherence for monoidal monoidal categories.** One of the first and most important instance of a coherence theorem is the one for monoidal categories, originally due to Mac Lane. Since it will be used in the following as one of the main illustrations, we begin by recalling it here, and discuss its various possible formulations.

A *monoidal category* consists of a category C equipped with a tensor bifunctor and and unit element respectively noted

$$\otimes : C \times C \to C \qquad\qquad e : 1 \to C$$

together with natural isomorphisms

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \to x \otimes (y \otimes z) \qquad \lambda_x : e \otimes x \to x \qquad \rho_x : x \otimes e \to x$$

called *associator* and *left* and *right unitors*, satisfying two well-known axioms stating that the diagrams

$$\begin{array}{ccccc}
((x \otimes y) \otimes z) \otimes w & \xrightarrow{\alpha_{x,y,z}\otimes w} & (x \otimes (y \otimes z)) \otimes w & & \\
 & & & \searrow^{\alpha_{x,y\otimes z,w}} & \\
\downarrow{\scriptstyle \alpha_{x\otimes y,z,w}} & & & & x \otimes ((y \otimes z) \otimes w) \\
 & & & & \downarrow{\scriptstyle x\otimes\alpha_{y,z,w}} \\
(x \otimes y) \otimes (z \otimes w) & & \xrightarrow{\qquad\alpha_{x,y,z\otimes w}\qquad} & & x \otimes (y \otimes (z \otimes w))
\end{array}$$

and

$$\begin{array}{ccc}
(x \otimes e) \otimes y & \xrightarrow{\alpha_{x,e,y}} & x \otimes (e \otimes y) \\
{\scriptstyle \rho_x\otimes y}\searrow & & \swarrow{\scriptstyle x\otimes\rho_y} \\
 & x \otimes y &
\end{array}$$

commute for any objects x, y and z of C.

Thanks to these axioms, the way tensor expressions are bracketed does not really matter: we can always rebracket expressions using the structural morphisms (α, λ and ρ), and any two ways of rebracketing an expression into the other are equal. In fact, and this is an important point in this article, there are various ways to formalize this [1]:

(C1) Every diagram in a free monoidal category made up of α, λ and ρ commutes [19, Corollary 1.6], [34, Theorem VI.2.1].
(C2) Every diagram in a monoidal category made up of α, λ and ρ commutes [35, Theorem 3.1], [34, Theorem XI.3.2].
(C3) Every monoidal category is monoidally equivalent to a strict monoidal category [19, Corollary 1.4], [34, Theorem XI.3.1].
(C4) The forgetful 2-functor from strict monoidal categories to monoidal categories has a left adjoint and the components of the unit are equivalences.

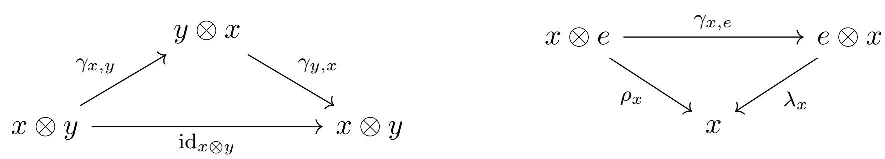
Condition (C2) implies (C1) as a particular case and the converse implication can also be shown, so that the two are easily seen to be equivalent. Condition (C4) implies (C3) as a particular case, and it can be shown that (C3) in turn implies (C2), see [34, Theorem XI.3.2].

1.4. **Coherence for symmetric monoidal categories.** Although fundamental, taking the previous example coherence theorem as a guiding example can be misleading as it hides the fact that the coherence results are in general more subtle: usually, we do not want all the diagrams made of structural morphisms to commute. In order to illustrate this, let use consider the following variant of monoidal categories.

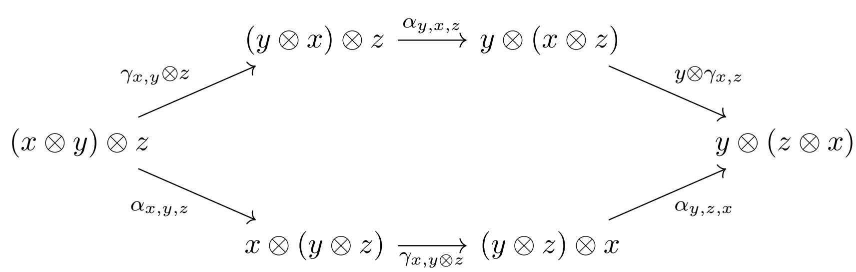
A *symmetric monoidal category* is a monoidal category equipped with a natural transformation

$$\gamma_{x,y} : x \otimes y \to y \otimes x$$

called *symmetry* such that the diagrams



and



commute for every objects x, y and z of C.

Analogous coherence theorems as above hold and can be formulated as follows:

(S1) Every diagram in a (free) symmetric monoidal category made up of α, λ ρ and γ commutes when the two sides have the same underlying symmetry [19, Corollary 26], [34, Theorem XI.1.1].
(S2) Every generic diagram in a (free) symmetric monoidal category made up of α, λ, ρ and γ commutes.
(S3) Every (free) symmetric monoidal category is symmetric monoidally equivalent to a strict symmetric monoidal category [37, Proposition 4.2], [19, Corollary 26].
(S4) The forgetful 2-functor from strict symmetric monoidal categories to symmetric monoidal categories has a left adjoint and the components of the unit are equivalences.

We can see above that the formulations do not anymore require that “all diagrams commute”. In order to illustrate why it has to be so, observe that the diagram

$$x \otimes x \underset{\mathrm{id}_{x\otimes x}}{\overset{\gamma_{x,x}}{\rightrightarrows}} x \otimes x \tag{1.1}$$

does *not* commute in general in monoidal categories, although its morphisms are structural ones. For a concrete example, consider the category of sets and functions equipped with cartesian product as tensor product and x to be any set with at least two distinct elements a and b. We namely have

$$\gamma_{x,x}(a,b) = (b,a) \neq (a,b) = \mathrm{id}_{x\otimes x}(a,b).$$

However, note that the two morphisms do not have the same “underlying symmetry” ($\gamma_{x,x}$ corresponds to a transposition whereas $\mathrm{id}_{x\otimes x}$ to an identity on a 2-element set). In fact, as stated in (S1), restricting diagrams where the two morphisms induce the same symmetry is enough to have them always commute. Another way to ensure that the diagrams should commute is to require them to be *generic* as in (S2), by which we roughly mean that all the objects occurring in the source (or target) object should be distinct: this is not the case in (1.1) since the source object is $x \otimes x$, in which x occurs twice. Intuitively, this condition ensures that the underlying symmetry of the morphisms is uniquely determined and thus that the diagram commutes as a particular instance of (S1). The same subtlety is implicitly present in the condition (S3): for a strict symmetric monoidal category, we do not require that we the symmetric should be strict (only the associator and unitors, such a category is sometimes also called a *permutative category* [37]).

1.5. **Other forms of coherence.** There are still other possible formulations of the coherence theorem involving what are called *unbiased* variants of the structures. In the case of monoidal categories, an *unbiased monoidal category* is a category equipped with n-ary tensor products for every natural number n, satisfying suitable axioms [31, Section 3.1]. The following variant of (C4) can then be shown:

(C4’) The forgetful 2-functor from strict monoidal categories to unbiased monoidal categories has a left adjoint and the components of the unit are equivalences.

This result is in fact a particular instance of a very general coherence theorem due to Power [40] (see also [24, 43]), which originates in the following observation: there is a 2-monad T on **Cat** whose strict algebras are strict monoidal categories and whose pseudo algebras are unbiased monoidal categories. Given a 2-monad T on a 2-category, under suitable assumptions (which are satisfied in the case of the monad of monoidal categories), it can be shown that the inclusion T-**StrAlg** $\to$ T-**PsAlg** of 2-categories, from the 2-category of strict T-algebras (and strict morphisms) to the 2-category of pseudo-2-algebras (and pseudo-morphisms) admits a left 2-adjoint (which can be interpreted as a strictification 2-functor) such that the components of the unit of the adjunction are internal equivalences in T-pseudo-algebras.

We do not insist much on this general route, as our main concern here is the relationship with rewriting, which provides ways of handling biased notions of algebras.

1.6. **Contents of the paper.** We first investigate (in section 2) an abstract version of this situation and formally compare the various coherence theorems: we show that quotienting a theory by a subtheory $\mathcal{W}$ gives rise to an equivalent theory if and only if $\mathcal{W}$ is coherent (or rigid), in the sense that all diagrams commute (theorem 14). Moreover, this is the case if and only if they give rise to equivalent categories of algebras (proposition 19), which can be thought of as a strengthened version of (C4). We also provide rewriting conditions which allow showing coherence in practice (proposition 40).

We then extend (in section 3) our results to the 2-dimensional cartesian theories, which are able to axiomatize (symmetric) monoidal categories. Our work is based on the notion of Lawvere 2-theory [14, 49, 50]. The rewriting counterpart is based on a coherent extension of term rewriting systems, following [11, 7, 36]. One of the main novelties here consists in allowing for coherence with respect to a sub-theory (which is required to handle coherence for symmetric monoidal categories), building on recent works in order to work in structures modulo substructures [10, 39, 12]. This article is an extended version of [38], and also corrects a few mistakes unfortunately present there.

2. Relative coherence and abstract rewriting systems

2.1. **Quotient of categories.** Suppose fixed a category $\mathcal{C}$ together with a set W of isomorphisms of $\mathcal{C}$. Although the situation is very generic, and the following explanation is only vague for now, it can be helpful to think of $\mathcal{C}$ as a theory describing a structure a category can possess and W as the morphisms we are interested in strictifying. For instance, if we are interested in the coherence theorem for symmetric monoidal categories, we can think of the objects of $\mathcal{C}$ as formal iterated tensor products, the morphisms of $\mathcal{C}$ as the structural morphisms (the composites of α, λ, ρ and γ), and we would typically take W as consisting of all instances of α, λ and ρ (but not γ). This will be made formal in section 3.

Definition 1. A functor $F : \mathcal{C} \to \mathcal{D}$ is *W-strict* when it sends every morphism of W to an identity.

Definition 2. The *quotient* $\mathcal{C}/W$ of $\mathcal{C}$ under W is the category equipped with a W-strict functor $\mathcal{C} \to \mathcal{C}/W$, such that every W-strict functor $F : \mathcal{C} \to \mathcal{D}$ extends uniquely as a functor $\tilde{F} : \mathcal{C}/W \to \mathcal{D}$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & \nearrow_{\tilde{F}} & \\ \mathcal{C}/W & & \end{array}$$

We write $\mathcal{W}$ for the subcategory of $\mathcal{C}$ generated by W. This subcategory will be assimilated to the smallest subset of morphisms of $\mathcal{C}$ which contains W is closed under composition and identities. The category $\mathcal{W}$ is a groupoid and it can shown that passing from W to $\mathcal{W}$ does not change the quotient.

Lemma 3. *The categories $\mathcal{C}/W$ and $\mathcal{C}/\mathcal{W}$ are isomorphic.*

Proof. By definition of quotient categories (definition 2), it is enough to show that the category $\mathcal{C}/W$ is a quotient of $\mathcal{C}$ by $\mathcal{W}$. It follows easily from the fact that a functor $\mathcal{C} \to \mathcal{D}$ is W-strict if and only if it is $\mathcal{W}$-strict. Namely, the left-to-right implication follows from functoriality and the right-to-left implication from the inclusion $W \subseteq \mathcal{W}$. □

Thanks to the above lemma, we will be able to assume, without loss of generality, that we always quotient categories by a subgroupoid which has the same objects as $\mathcal{C}$.

We will see that quotients are much better behaved when the groupoid we quotient by satisfies the following property.

Definition 4. A groupoid $\mathcal{W}$ is *rigid* when any two morphisms $f, g : x \to y$ which are parallel (i.e. have the same source, and have the same target) are necessarily equal.

Such a groupoid can be thought of as a "coherent" sub-theory of $\mathcal{C}$. It does not have non-trivial geometric structure in the sense of proposition 7 below.

We will need to use the following properties of categories.

Definition 5. A category is

– *discrete* when its only morphisms are identities,
– *contractible* when it is equivalent to the terminal category,
– *connected* when there is a morphism between any two objects,
– *propositional* when rigid and connected.

Lemma 6. *A propositional category with an object is contractible.*

Proof. Given a propositional category $\mathcal{C}$, the terminal functor $\mathcal{C} \to 1$ is full (because $\mathcal{C}$ is connected), faithful (because $\mathcal{C}$ is rigid) and surjective (because $\mathcal{C}$ has an object). □

Proposition 7. *Given a groupoid $\mathcal{W}$, the following are equivalent*

(i) $\mathcal{W}$ *is rigid,*
(ii) $\mathcal{W}$ *has identities as only automorphisms,*
(iii) $\mathcal{W}$ *is equivalent to a discrete category,*
(iv) $\mathcal{W}$ *is a coproduct of contractible categories.*

Proof. (i) implies (ii). Given a rigid category, any automorphism $f : x \to x$ is parallel with the identity and thus has to be equal to it.
(ii) implies (i). Given two parallel morphisms $f, g : x \to y$, we have $g^{-1} \circ f = \mathrm{id}_x$ and thus $f = g$.
(i) implies (iii). Write $\mathcal{D}$ for category of connected components of $\mathcal{W}$: this is the discrete category whose objects are the equivalence classes $[x]$ of objects x of $\mathcal{W}$ under the equivalence relation identifying x and y whenever there is a morphism $f : x \to y$ in $\mathcal{W}$. The quotient functor $Q : \mathcal{W} \to \mathcal{D}$ is full because $\mathcal{D}$ is discrete, faithful because $\mathcal{W}$ is rigid, and surjective on objects by construction of $\mathcal{D}$. It is thus an equivalence of categories.
(iii) implies (i). Suppose given an equivalence $F : \mathcal{W} \to \mathcal{D}$ to a discrete category $\mathcal{D}$. Given two parallel morphisms $f, g : x \to y$, they have the same image $Ff = Fg$ (which is an identity) because $\mathcal{D}$ is discrete, and are thus equal because F is faithful.
(iii) implies (iv). Consider a functor $F : \mathcal{W} \to \mathcal{D}$ which is an equivalence with $\mathcal{D}$ discrete. Given $x \in \mathcal{D}$, we write $F^{-1}x$ for the full subcategory of $\mathcal{W}$ whose objects are sent to x by F. Since $\mathcal{D}$ is discrete, for any morphism $f : x \to y$ in $\mathcal{D}$, we have $Fx = Fy$, from which follows that $\mathcal{W} \cong \sqcup_{x \in D} F^{-1}x$. Since $\mathcal{D}$ is discrete, each $F^{-1}x$ is non-empty, connected and rigid and thus contractible by lemma 6.
(iv) implies (iii). If $\mathcal{W} \cong \sqcup_{i \in I} \mathcal{W}_i$ with $\mathcal{W}_i$ contractible, i.e. $\mathcal{W}_i \simeq 1$, then $\mathcal{W} \simeq \sqcup_{i \in I} 1$ because equivalences are closed under coproducts and thus $\mathcal{W}$ is equivalent to a discrete category □

The fact that $\mathcal{W} \subseteq \mathcal{C}$ is rigid can be thought of here as the fact that coherence condition (C1) holds for $\mathcal{C}$, relatively to $\mathcal{W}$: any two parallel structural morphisms are equal. Condition (iii) and (iv) can also be interpreted as stating that $\mathcal{W}$ is a set, up to equivalence.

General notions of quotients (with respect to a subcategory, as opposed to a subgroupoid, or with respect to a general notion of congruence both on objects and morphisms) have been developed in [6], and a non-trivial to study and construct. However, when quotienting by a rigid subgroupoid we have the following simple description.

Proposition 8. *Suppose given a rigid subgroupoid $\mathcal{W}$ of a category $\mathcal{C}$. We define the following equivalence relations.*

– *We write $\sim$ for the equivalence relation on objects of $\mathcal{C}$ such that $x \sim y$ whenever there is a morphism $f : x \to y$ in $\mathcal{W}$. When it exists, such a morphism is unique by rigidity of $\mathcal{W}$ and noted $w_{x,y} : x \to y$.*
– *We also write $\sim$ for the equivalence relation on morphisms of $\mathcal{C}$ such that for $f : x \to y$ and $f' : x' \to y'$ we have $f \sim f'$ whenever there exists morphisms $v : x \to x'$ and $w : y \to y'$ in $\mathcal{W}$ making the following diagram commute:*

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ x' & \xrightarrow[f']{} & y' \end{array}$$

The quotient category $\mathcal{C}/\mathcal{W}$ is isomorphic to the category where

– *an object $[x]$ is an equivalence class of an object x of $\mathcal{C}$ under $\sim$,*
– *a morphism $[f] : [x] \to [y]$ is the equivalence class of a morphism $f : x \to y$ in $\mathcal{C}$ under $\sim$*
– *given $f : x \to y$ and $g : y' \to z$ with $[y] = [y']$, the composition is $[g] \circ [f] = [g \circ w_{y,y'} \circ f]$:*

$$x \xrightarrow{f} y \xrightarrow{w_{y,y'}} y' \xrightarrow{g} z$$

– *the identity on an object $[x]$ is $[\mathrm{id}_x]$.*

Proof. We first need to show that the definition makes sense.

– Composition is compatible with the equivalence relation. Given $f_1 : x_1 \to y_1$, $f_2 : x_2 \to y_2$, $g_1 : y'_1 \to z_1$, $g_2 : y'_2 \to z_2$ such that $f_1 \sim f_2$ and $g_1 \sim g_2$ (and thus $x_1 \sim x_2$, $y_1 \sim y_2$, $y'_1 \sim y'_2$ and $z_1 \sim z_2$) which are composable (i.e. $y_1 \sim y'_1$ and $y_2 \sim y'_2$), the following diagram shows that $[g_1] \circ [f_1] = [g_2] \circ [f_2]$:

$$\begin{array}{ccccccc} x_1 & \xrightarrow{f_1} & y_1 & \xrightarrow{w_{y_1,y'_1}} & y'_1 & \xrightarrow{g_1} & z_1 \\ {\scriptstyle w_{x_1,x_2}}\downarrow & & \downarrow{\scriptstyle w_{y_1,y_2}} & & \downarrow{\scriptstyle w_{y'_1,y'_2}} & & \downarrow{\scriptstyle w_{z_1,z_2}} \\ x_2 & \xrightarrow[f_2]{} & y_2 & \xrightarrow[w_{y_2,y'_2}]{} & y'_2 & \xrightarrow[g_2]{} & z_2 \end{array}$$

where the squares on the left and right respectively commute because $f_1 \sim f_2$ and $g_1 \sim g_2$ and the one in the middle does by rigidity of $\mathcal{W}$.
– Identities are compatible with the equivalence relations. Given objects x and y of $\mathcal{C}$ such that $x \sim y$, the diagram

$$\begin{array}{ccc} x & \xrightarrow{\mathrm{id}_x} & x \\ {\scriptstyle w_{x,y}}\downarrow & & \downarrow{\scriptstyle w_{x,y}} \\ y & \xrightarrow[\mathrm{id}_y]{} & y \end{array}$$

commutes showing that we have $\mathrm{id}_x \sim \mathrm{id}_y$.

Associativity of composition and the fact that identities are neutral element for composition follow immediately from the fact that those properties are satisfied in $\mathcal{C}$. Finally, we can show that the category has the universal property of definition 2. The quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{W}$ (sending objects and morphisms to their equivalence class) is $\mathcal{W}$-strict for any morphism $w : x \to y$, we have $[w] = [\mathrm{id}_y]$:

$$\begin{array}{ccc} x & \xrightarrow{w} & y \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_y} \\ y & \xrightarrow[\mathrm{id}_y]{} & y \end{array}$$

Moreover, a $\mathcal{W}$-strict functor $F : \mathcal{C} \to \mathcal{D}$ induces a unique functor $\tilde{F} : \mathcal{C}/\mathcal{W} \to \mathcal{D}$. Namely, by $\mathcal{W}$-strictness, two objects (resp. morphisms) which are equivalent have the same image by F. □

When $\mathcal{W} \subseteq \mathcal{C}$ is not rigid, we can have a similar description of the quotient, but the description is more complicated. Namely, if we are trying to compose two morphisms $[f]$ and $[g]$ in the quotient with $f : x \to y$ and $g : y' \to z$, we might have multiple morphisms $y \to y'$ in $\mathcal{W}$ (say v and w),

$$x \xrightarrow{f} y \underset{w}{\overset{v}{\rightrightarrows}} y' \xrightarrow{g} z$$

and, in such a situation, the compositions $g \circ v \circ f$ and $g \circ w \circ f$ should be identified in the quotient. This observation suggests that the construction of the quotient category $\mathcal{C}/\mathcal{W}$, when $\mathcal{W}$ is not rigid, is better described in two steps: we first formally make $\mathcal{W}$ rigid, and then apply proposition 8.

Definition 9. A functor $F : \mathcal{C} \to \mathcal{D}$ is *$\mathcal{W}$-rigid* when for any parallel morphisms $f, g : x \to y$ of $\mathcal{C}$ we have $Ff = Fg$.

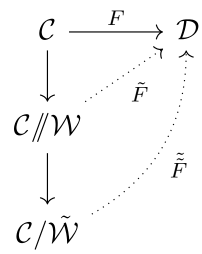
Definition 10. The *$\mathcal{W}$-rigidification* $\mathcal{C}/\!\!/\mathcal{W}$ of $\mathcal{C}$ is the category equipped with a $\mathcal{W}$-rigid functor $\mathcal{C} \to \mathcal{C}/\!\!/\mathcal{W}$, such that any $\mathcal{W}$-rigid functor $F : \mathcal{C} \to \mathcal{D}$ extends uniquely as a functor $\tilde{F} : \mathcal{C}/\!\!/\mathcal{W} \to \mathcal{D}$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & \nearrow{\scriptstyle \tilde{F}} & \\ \mathcal{C}/\!\!/\mathcal{W} & & \end{array}$$

Lemma 11. *The category $\mathcal{C}/\!\!/\mathcal{W}$ is the category obtained from $\mathcal{C}$ by quotienting morphisms under the smallest congruence (wrt composition) identifying any two parallel morphisms of $\mathcal{W}$.*

Proposition 12. *The quotient $\mathcal{C}/\mathcal{W}$ is isomorphic to $(\mathcal{C}/\!\!/\mathcal{W})/\tilde{\mathcal{W}}$ where $\tilde{\mathcal{W}}$ is the set of equivalence classes of morphisms in $\mathcal{W}$ under the equivalence relation of lemma 11.*

Proof. Since any $\mathcal{W}$-strict functors are $\mathcal{W}$-rigid, we have that any $\mathcal{W}$-strict functor extends as a unique functor $\tilde{F} : \mathcal{C}/\!\!/\mathcal{W} \to \mathcal{D}$ which is $\mathcal{W}$-rigid, and thus as a unique functor $\mathcal{C}/\tilde{\mathcal{W}} \to \mathcal{D}$:



□

A consequence of the preceding explicit description of the quotient is the following:

Lemma 13. *The quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{W}$ is surjective on objects and full.*

Proof. By proposition 12, the quotient functor is the composite of the quotient functors $\mathcal{C} \to \mathcal{C}/\!\!/\mathcal{W} \to \mathcal{C}/\mathcal{W}$. The first one is surjective on objects and full by lemma 11 and the second one is surjective on objects and full by proposition 8. □

This entails the following theorem, which is the main result of the section. Its meaning can be explained by taking the point of view given above: thinking of the category $\mathcal{C}$ as describing a structure, and of $\mathcal{W}$ as the part of the structure we want to strictify, the structure is equivalent to its strict variant if and only if the quotiented structure does not itself bear non-trivial geometry (in the sense of proposition 7).

Theorem 14. *Suppose that $\mathcal{W}$ is a subgroupoid of $\mathcal{C}$. The quotient functor $[-] : \mathcal{C} \to \mathcal{C}/\mathcal{W}$ is an equivalence of categories if and only if $\mathcal{W}$ is rigid.*

Proof. Since the quotient functor is always surjective and full by lemma 13, it remains to show that it is faithful if and only if $\mathcal{W}$ is rigid. Suppose that the quotient functor is faithful. Given $w, w' : x \to y$ in $\mathcal{W}$, by lemma 11 and proposition 12 we have $[w] = [w']$ and thus $w = w'$ by faithfulness. Suppose that $\mathcal{W}$ is rigid. The category $\mathcal{C}/\mathcal{W}$ then admits the description given in proposition 8. Given $f, g : x \to y$ in $\mathcal{C}$ such that $[f] = [g]$, there is $v : x \to x$ and $w : y \to y$ such that $w \circ f = g \circ v$. By rigidity, both v and w are identities and thus $f = g$. □

Example 15. As a simple example, consider the groupoid $\mathcal{C}$ freely generated by the graph

$$x \underset{g}{\overset{f}{\rightrightarrows}} y$$

The subgroupoid generated by $W = \{g\}$ is rigid, so that $\mathcal{C}$ is equivalent to the quotient category $\mathcal{C}/W$, which is the groupoid generated by the graph

$$x \circlearrowleft f$$

However, the groupoid generated by $W = \{f, g\}$ is not rigid (since we do not have $f = g$). And indeed, in this case, $\mathcal{C}$ is not equivalent to the quotient category $\mathcal{C}/W$, which is the terminal category.

Remark 16. By taking $\mathcal{C}$ to be $\mathcal{W}$ in theorem 14, we obtain that a category $\mathcal{W}$ is rigid if and only if it is equivalent to the discrete category of its connected components. This thus provides an alternative proof of condition (iii) of proposition 7.

2.2. **Coherence for algebras.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, an *algebra* of $\mathcal{C}$ in $\mathcal{D}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$. In the following, we will mostly be interested in the case where $\mathcal{D} = \mathbf{Cat}$: if we think of the category $\mathcal{C}$ as describing an algebraic structure (e.g. the one of monoidal categories), an algebra can be thought of as a category actually possessing this structure (an actual monoidal category). We write $\mathrm{Alg}(\mathcal{C}, \mathcal{D})$ for the category whose objects are algebras and morphisms are natural transformations, and $\mathrm{Alg}(\mathcal{C}) = \mathrm{Alg}(\mathcal{C}, \mathbf{Cat})$. Note that any functor

$$F : \mathcal{C} \to \mathcal{C}'$$

induces, by precomposition, a functor

$$\mathrm{Alg}(F, \mathcal{D}) : \mathrm{Alg}(\mathcal{C}', \mathcal{D}) \to \mathrm{Alg}(\mathcal{C}, \mathcal{D}).$$

We can characterize situations where two categories give rise to the same algebras as follows. A variant adapted to 2-representations of 2-categories can be found in [13, Lemma 5.3.1].

Proposition 17. *Suppose given a functor $F : \mathcal{C} \to \mathcal{C}'$ between categories. The functor F is an equivalence if and only if there is a family of equivalence of categories $\mathrm{Alg}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Alg}(\mathcal{C}', \mathcal{D})$ which is natural in $\mathcal{D}$.*

Proof. Given a 2-category $\mathcal{K}$, one can define a Yoneda functor

$$\begin{aligned} Y_{\mathcal{K}} : \mathcal{K}^{\mathrm{op}} &\to [\mathcal{K}, \mathbf{Cat}] \\ c &\mapsto \mathcal{K}(c, -) \end{aligned}$$

where $\mathbf{Cat}$ is the 2-category of categories, functors and natural transformations, and $[\mathcal{K}, \mathbf{Cat}]$ denotes the 2-category of 2-functors $\mathcal{K} \to \mathbf{Cat}$, transformations and modifications. In particular, given 0-cells $c \in \mathcal{K}^{\mathrm{op}}$ and $d \in \mathcal{K}$, we have $Y_{\mathcal{K}} c d = \mathcal{K}(c, d)$. The Yoneda lemma states that this functor is a local isomorphism (this is a particular case of the Yoneda lemma for bicategories [18, Corollary 8.3.13]): this means that, for every objects $c, d \in \mathcal{K}^{\mathrm{op}}$, we have an isomorphism of categories

$$[\mathcal{K}, \mathbf{Cat}](Y_{\mathcal{K}} c, Y_{\mathcal{K}} d) \cong \mathcal{K}(d, c).$$

In particular, taking $\mathcal{K} = \mathbf{Cat}$ (and ignoring size issues, see below for a way to properly handle this), the Yoneda functor sends a category $\mathcal{C} \in \mathbf{Cat}^{\mathrm{op}}$ to $Y_{\mathbf{Cat}} \mathcal{C}$, i.e. $\mathbf{Cat}(\mathcal{C}, -)$, i.e. $\mathrm{Alg}(\mathcal{C}, -)$. Given category $\mathcal{C}$ and $\mathcal{C}'$, by the Yoneda lemma, we thus have an isomorphism of categories

$$[\mathbf{Cat}, \mathbf{Cat}](\mathrm{Alg}(\mathcal{C}, -), \mathrm{Alg}(\mathcal{C}', -)) \cong \mathbf{Cat}(\mathcal{C}', \mathcal{C})$$

which is compatible with 0-composition in $\mathbf{Cat}$. The categories $\mathcal{C}$ and $\mathcal{C}'$ are thus equivalent, if and only if the categories $\mathrm{Alg}(\mathcal{C}, -)$ and $\mathrm{Alg}(\mathcal{C}', -)$ are equivalent, which is the case if and only if there is a family of equivalences of categories between $\mathrm{Alg}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Alg}(\mathcal{C}, \mathcal{D})$ natural in $\mathcal{D}$. □

Alternative proof of proposition 17. We provide here an alternative more pedestrian proof, which does not require ignoring size issues. Suppose that F is an equivalence of categories, with pseudo-inverse $G : \mathcal{C}' \to \mathcal{C}$, i.e. we have $G \circ F \cong \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{Id}_{\mathcal{C}'}$. We define functors

$$\begin{aligned} \mathrm{Alg}(F, \mathcal{D}) : \mathrm{Alg}(\mathcal{C}', \mathcal{D}) &\to \mathrm{Alg}(\mathcal{C}, \mathcal{D}) \\ A &\mapsto A \circ F \end{aligned} \qquad \begin{aligned} \mathrm{Alg}(G, \mathcal{D}) : \mathrm{Alg}(\mathcal{C}, \mathcal{D}) &\to \mathrm{Alg}(\mathcal{C}', \mathcal{D}) \\ A &\mapsto A \circ G \end{aligned}$$

Those induce an equivalence between $\mathrm{Alg}(\mathcal{C},\mathcal{D})$ and $\mathrm{Alg}(\mathcal{C}',\mathcal{D})$ since, for $A \in \mathrm{Alg}(\mathcal{C},\mathcal{D})$ and $A' \in \mathrm{Alg}(\mathcal{C}',\mathcal{D})$, we have

$$\begin{aligned}\mathrm{Alg}(G,\mathcal{D}) \circ \mathrm{Alg}(F,\mathcal{D}) &= A' \circ F \circ G \simeq A' \\ \mathrm{Alg}(F,\mathcal{D}) \circ \mathrm{Alg}(G,\mathcal{D}) &= A \circ G \circ F \simeq A\end{aligned}$$

Moreover, the family of equivalences $\mathrm{Alg}(F,\mathcal{D})$ is natural in $\mathcal{D}$, and similarly for $\mathrm{Alg}(G,\mathcal{D})$. Namely, given $H : \mathcal{D} \to \mathcal{D}'$ and considering the functor

$$\begin{aligned}\mathrm{Alg}(\mathcal{C},H) : \mathrm{Alg}(\mathcal{C},\mathcal{D}) &\to \mathrm{Alg}(\mathcal{C},\mathcal{D}') \\ A &\mapsto H \circ A\end{aligned}$$

as well as the variant with $\mathcal{C}'$ instead of $\mathcal{C}$, the diagram

$$\begin{array}{ccc} \mathrm{Alg}(\mathcal{C}',\mathcal{D}) & \xrightarrow{\mathrm{Alg}(\mathcal{C}',H)} & \mathrm{Alg}(\mathcal{C}',\mathcal{D}') \\ {\scriptstyle \mathrm{Alg}(F,\mathcal{D})}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Alg}(F,\mathcal{D}')} \\ \mathrm{Alg}(\mathcal{C},\mathcal{D}) & \xrightarrow[\mathrm{Alg}(\mathcal{C},H)]{} & \mathrm{Alg}(\mathcal{C},\mathcal{D}') \end{array}$$

commutes: an object $A \in \mathrm{Alg}(\mathcal{C}',\mathcal{D})$ is sent to $H \circ A \circ F$ by both sides.

Conversely, suppose given an equivalence of categories

$$\Phi_{\mathcal{D}} : \mathrm{Alg}(\mathcal{C}',\mathcal{D}) \leftrightarrow \mathrm{Alg}(\mathcal{C},\mathcal{D}) : \Psi_{\mathcal{D}}$$

which is natural in $\mathcal{D}$. We define

$$F = \Phi_{\mathcal{C}'}(\mathrm{Id}_{\mathcal{C}'}) : \mathcal{C} \to \mathcal{C}' \qquad\qquad G = \Psi_{\mathcal{C}}(\mathrm{Id}_{\mathcal{C}}) : \mathcal{C}' \to \mathcal{C}$$

and we have

$$G \circ F = G \circ \Phi_{\mathcal{C}'}(\mathrm{Id}_{\mathcal{C}'}) = \Phi_{\mathcal{C}'}(G \circ \mathrm{Id}_{\mathcal{C}'}) = \Phi_{\mathcal{C}'}(\Psi_{\mathcal{C}}(\mathrm{Id}_{\mathcal{C}})) \cong \mathrm{Id}_{\mathcal{C}'}$$

(the second equality is naturality), and similarly for $G \circ F \cong \mathrm{Id}_{\mathcal{C}}$. □

Remark 18. We would like to underline out a subtle point with respect to naturality in the above theorem. Given a functor $F : \mathcal{C} \to \mathcal{C}'$, it is always the case that the induced functors $\mathrm{Alg}(F,\mathcal{D})$ form a family which is natural in $\mathcal{D}$. Suppose moreover that all the functors $\mathrm{Alg}(F,\mathcal{D})$ are equivalences. We do not see any argument to show that the pseudo-inverse functors form a natural family, which is why we have to additionally impose this condition.

As a particular application, we have the following proposition, which can be interpreted as the equivalence of coherence conditions (C1) and a strengthened variant of (C4):

Proposition 19. *Fix a category $\mathcal{C}$ and a subgroupoid $\mathcal{W}$. Given a category $\mathcal{D}$, we have a functor*

$$\mathrm{Alg}(\mathcal{C}/\mathcal{W},\mathcal{D}) \to \mathrm{Alg}(\mathcal{C},\mathcal{D}) \tag{2.1}$$

induced by precomposition with the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{W}$. These functors form a family of equivalences of categories, natural in $\mathcal{D}$, if and only if $\mathcal{W}$ is rigid.

Proof. By theorem 14, $\mathcal{W}$ is rigid if and only if the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{W}$ is an equivalence, and we conclude by proposition 17. □

Remark 20. It can be wondered whether the case where $\mathcal{D} = \mathbf{Cat}$ is enough, i.e. whether the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{W}$ is an equivalence whenever the functor $\mathrm{Alg}(\mathcal{C}/W) \to \mathrm{Alg}(\mathcal{C})$ it induces is an equivalence. We leave it as an open question, but remark here that it cannot follow from general results: it is not the case that two categories $\mathcal{C}$ and $\mathcal{C}'$ are equivalent whenever $\mathrm{Alg}(\mathcal{C})$ and $\mathrm{Alg}(\mathcal{C}')$ are equivalent. It is namely known that two Cauchy equivalent categories give rise to the same algebras in **Cat**, see [25], so that the categories

$$\mathcal{C} = \; x \circlearrowleft e \qquad\qquad \mathcal{C}' = \; x \xrightarrow{f} y$$

where $e \circ e = e$ have the same algebras.

2.3. **Coherent abstract rewriting systems.** Previous sections illustrate the importance of the property of being rigid for a groupoid, and we now provide tools to show this in practice, based on tools originating from rewriting theory. In the same way the theory of rewriting can be studied "abstractly" [17, 4, 47], i.e. without taking in consideration the structure of the objects getting rewritten, we first develop the coherence theorems of interest in this article in an abstract setting. The categorical formalization of the notion of rewriting system given here is based on the notion of polygraph [45, 9, 3].

Definition 21. An *abstract rewriting system*, or ARS, or 1-*polygraph* is a diagram

$$\mathsf{P}_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \mathsf{P}_1$$

in the category **Set**.

An ARS is simply another name for a directed graph. It consists of a set P_0 whose elements are the *objects* of interest, a set P_1 of *rewriting rules* and two functions s_0 and t_0 respectively associating to a rewriting rule its *source* and *target*. We often write

$$a : x \to y$$

to denote a rewriting rule a with $s_0(a) = x$ and $t_0(a) = y$. We write P_1^* for the set of *rewriting paths* in the ARS: its elements are (possibly empty) finite sequences $a_1, \ldots, a_n$ of rewriting steps, which are composable in the sense that $t_0(a_i) = s_0(a_{i+1})$ for $1 \leq i < n$. Writing $x_i = t_0(a_i)$, such a path can thus be represented as

$$x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} x_n.$$

The source and target of such a rewriting path are respectively $s_0(a_1)$ and $t_0(a_n)$ (those are respectively x_0 and x_n in the above diagram), and n is called the *length* of the path. We sometimes write

$$p : x \overset{*}{\to} y$$

to indicate that p is a rewriting path with x as source and y as target. Given two composable paths $p : x \overset{*}{\to} y$ and $q : y \overset{*}{\to} z$, we write $p \cdot q : x \overset{*}{\to} z$ for their concatenation.

A *morphism* $f : \mathsf{P} \to \mathsf{Q}$ of ARS is a pair of functions $f_0 : \mathsf{P}_0 \to \mathsf{Q}_0$ and $f_1 : \mathsf{P}_1 \to \mathsf{Q}_1$ such that $s_0 \circ f_1 = f_0 \circ s_0$ and $t_0 \circ f_1 = f_0 \circ t_0$:

$$\begin{array}{ccc} \mathsf{P}_0 & \underset{t_0}{\overset{s_0}{\leftleftarrows}} & \mathsf{P}_1 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ \mathsf{Q}_0 & \underset{t_0}{\overset{s_0}{\leftleftarrows}} & \mathsf{Q}_1 \end{array}$$

We write $\mathbf{Pol}_1$ for the resulting category or ARS and their morphisms. There is a forgetful functor $\mathbf{Cat} \to \mathbf{Pol}_1$, sending a category C to the ARS whose objects are those of C and whose rewriting steps are the morphisms of C.

Lemma 22. *The forgetful functor* $\mathbf{Cat} \to \mathbf{Pol}_1$ *admits a left adjoint* $-^* : \mathbf{Pol}_1 \to \mathbf{Cat}$. *It sends an* ARS *to the category with* P_0 *as objects and* P_1^* *as morphisms, where composition is given by concatenation of paths and identities are the empty paths*

Proof. This fact is easily checked directly, but an abstract argument for the existence of the left adjoint in such situations is the following: the categories $\mathbf{Cat}$ and $\mathbf{Pol}_1$ are models of projectives sketches and the forgetful functor $\mathbf{Cat} \to \mathbf{Pol}_1$ is induced by a functor of sketches (the "inclusion" of the sketch of ARS into the sketch of categories) and, as such, it admits a left adjoint [5, Theorem 4.1]. □

As a variant of the preceding situation, we can consider the forgetful functor $\mathbf{Gpd} \to \mathbf{Pol}_1$, from the category of groupoids. It also admits a left adjoint $-^\sim : \mathbf{Pol}_1 \to \mathbf{Gpd}$, which can be described as follows. Given an ARS P, we write $\mathsf{P}^\pm$ for the ARS

$$\mathsf{P}_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \mathsf{P}_1 \sqcup \mathsf{P}_1$$

with the same objects as P and with $\mathsf{P}_1 \sqcup \mathsf{P}_1$ as rewriting rules: concretely, a rewriting rule a^ϵ in $\mathsf{P}_1^\pm$ is a pair consisting of a rewriting rule $a \in \mathsf{P}_1$ and $\epsilon \in \{-, +\}$. The source and target maps are given by

$$s_0(a^+) = s_0(a) \qquad t_0(a^+) = t_0(a) \qquad s_0(a^-) = t_0(a) \qquad t_0(a^-) = s_0(a)$$

We can think of a^+ as corresponding to a and a^- as corresponding to a taken "backward". A *rewriting zig-zag* is a path $a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}$ in $\mathsf{P}^\pm$. The intuition is that a zig-zag is a "non-directed" rewriting path, consisting of rewriting steps, some of which are taken backward. We write

$$p : x \overset{\sim}{\to} y$$

to indicate that p is a zig-zag from x to y. Two zig-zags are *congruent* when they are related by the smallest congruence $\sim$ such that, for every rewriting rule $a : x \to y$, we have

$$a^+a^- \sim \mathrm{id}_x \qquad\qquad a^-a^+ \sim \mathrm{id}_y \tag{2.2}$$

We write $\mathsf{P}_1^\sim$ for the set of zig-zags up to congruence.

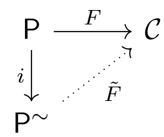
Lemma 23. *The category* $\mathsf{P}^\sim$ *with* P_0 *as objects,* $\mathsf{P}_1^\sim$ *as morphisms, where composition is given by concatenation of paths up to congruence, is the free groupoid on* P.

We have a canonical morphism $i_1 : \mathsf{P}_1 \to \mathsf{P}_1^\sim$, sending a rewriting step a to a^+. Writing $s_0^\sim, t_0^\sim : \mathsf{P}_1^\sim \to \mathsf{P}_0$ for the source and target maps, it induces a morphism of ARS by taking the identity on objects:

$$\begin{array}{ccc} \mathsf{P}_0 & \underset{t_0}{\overset{s_0}{\leftleftarrows}} & \mathsf{P}_1 \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle i_1} \\ \mathsf{P}_0 & \underset{t_0^\sim}{\overset{s_0^\sim}{\leftleftarrows}} & \mathsf{P}_1^\sim \end{array}$$

Writing $i : \mathsf{P} \to \mathsf{P}^\sim$ for this morphism of ARS, the universal property of P states that any morphism of ARS $F : \mathsf{P} \to \mathcal{C}$ where $\mathcal{C}$ is a groupoid extends uniquely as a functor $\tilde{F} : \mathsf{P}^\sim \to \mathcal{C}$

making the following diagram commute:



In the following, in order to avoid working with equivalence classes when working with elements of $\mathsf{P}_1^\sim$, we will instead only consider zig-zags $a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n}$ which are *reduced*, in the sense that they satisfy the following property: for every index i with $1 \leq i < n$, we have that $a_i = a_{i+1}$ implies $\epsilon_i = \epsilon_{i+1}$. This is justified by the following result.

Lemma 24. *The equivalence class under $\sim$ of a zig-zag contains a unique reduced zig-zag.*

Proof. Consider the string rewriting system on words over $\mathsf{P}_1^\pm$ with rules $a^+a^- \Rightarrow \mathrm{id}$ and $a^-a^+ \Rightarrow \mathrm{id}$ for $a \in \mathsf{P}_1$, corresponding to (2.2). It is length-reducing and thus terminating. Its critical branchings (whose sources are $a^-a^+a^-$ and $a^+a^-a^+$) are confluent, it is thus confluent. We deduce that any equivalence class contains a unique normal form, and those are precisely reduced zig-zags. □

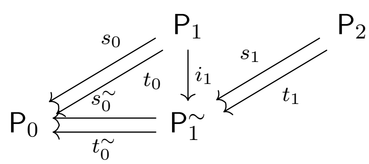
Given a path $p : x \overset{*}{\to} y$, we write $p^+ : x \overset{\sim}{\to} y$ (resp. $p^- : y \overset{\sim}{\to} x$) for the zig-zag obtained by adding a "+" (resp. "−") exponent to every step of a rewriting path. In particular, the first operation induces a canonical inclusion $i_1^* : \mathsf{P}_1^* \to \mathsf{P}_1^\sim$, defined by $i_1^*(p) = p^+$, witnessing for the fact that rewriting paths are particular zig-zags. We will sometimes leave its use implicit in the following. Note that lemma 24 implies that i_1^* is injective.

We think here as an ARS abstractly describing some algebraic structures. It is thus natural to extend this notion in order to take in account the coherence laws that these structures should posses. This can be done as follows.

Definition 25. An *extended abstract rewriting system*, or 2-ARS, or 2-*polygraph*, P consists of an ARS as above, together with a set P_2 and two functions $s_1, t_1 : \mathsf{P}_2 \to \mathsf{P}_1^\sim$, such that

$$s_0^\sim \circ s_1 = s_0^\sim \circ t_1 \qquad\qquad t_0^\sim \circ s_1 = t_0^\sim \circ t_1$$

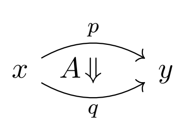
This data can be summarized as a diagram



In a 2-ARS, the elements of P_2 are *coherence relations* and the functions s_1 and t_1 respectively describe their source and target, which are rewriting zig-zags. We sometimes write

$$A : p \Rightarrow q$$

to indicate that $A \in \mathsf{P}_2$ is a coherence relation which admits p (resp. q) as source (resp. target), which can be thought of as a 2-cell



where x (resp. y) is the common source (resp. target) of p and q.

Definition 26. The *groupoid presented* by a 2-ARS P, denoted by $\overline{\mathsf{P}}$, is the groupoid obtained from the free groupoid generated by the underlying ARS by quotienting morphisms under the smallest congruence identifying the source and the target of any element of P_2.

More explicitly, the groupoid $\overline{\mathsf{P}}$ thus has P_0 as set of objects, the set $\mathsf{P}_1^{\sim}$ of rewriting zig-zags as morphisms, quotiented by the smallest equivalence relation $\overset{*}{\Leftrightarrow}$ such that

$$p \cdot q \cdot r \overset{*}{\Leftrightarrow} p \cdot q' \cdot r$$

for every rewriting zig-zags p and r and coherence relation $A : q \Rightarrow q'$, which are suitably composable:

$$x' \xrightarrow{\quad p \quad} x \underset{q'}{\overset{q}{\Downarrow A}} y \xrightarrow{\quad r \quad} y' \tag{2.3}$$

We write $\Leftrightarrow$ for the smallest symmetric relation identifying path $p \cdot q \cdot r$ and $p \cdot q' \cdot r$ when there is a coherence relation $A : q \Rightarrow q'$ as pictured above, so that $\overset{*}{\Leftrightarrow}$ is the reflexive transitive closure of $\Leftrightarrow$. Given a rewriting zig-zag $p \in \mathsf{P}_1^{\sim}$, we write $\overline{p}$ for the corresponding morphism in $\overline{\mathsf{P}}$, i.e. its equivalence class under $\overset{*}{\Leftrightarrow}$. Given a zig-zag $p : x \to y$ in $P^{\sim}$, we write $\overline{p} : x \to y$ for its equivalence class.

Remark 27. A more categorical approach to the equivalences between zig-zags can be developed as follows, see [3] for details. A 2-*groupoid* is a 2-category whose 1- and 2-cells are invertible. A 2-ARS freely generates a 2-groupoid, whose underlying 1-groupoid is the one freely generated by the underlying 1-ARS of P, and containing the coherence relations as 2-cells. Given zig-zags $p, q : x \to y$, we then have $p \overset{*}{\Leftrightarrow} q$ if and only if there is a 2-cell $p \Rightarrow q$ in the free 2-groupoid: the 2-cells can thus be thought of as witnesses for the equivalences of zig-zags. We do not further detail this approach here, since it is not required, but it would be for instance needed if we were interested in higher coherence laws.

There are many 2-ARS presenting a given groupoid. In particular, one can always perform the following transformations on 2-ARS, while preserving the presented groupoid. Those are analogous to the transformations that one can perform on group presentations (while preserving the presented group) first studied by Tietze [48, 33].

Definition 28. The *Tietze transformations* are the following possible transformations on a 2-ARS P:

(T1) given a zig-zag $p : x \overset{\sim}{\to} y$, add a new rewriting rule $a : x \to y$ in P_1 together with a new coherence relation $A : a \Rightarrow p$ in P_2,

(T2) given zig-zags $p, q : x \overset{\sim}{\to} y$ such that $p \overset{*}{\Leftrightarrow} q$, add a new coherence relation $A : p \Rightarrow q$ in P_2.

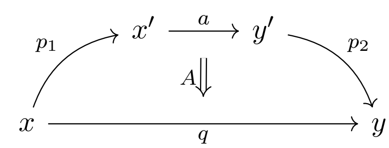
The *Tietze equivalence* is the smallest equivalence relation on 2-ARS identifying P and Q whenever Q can be obtained from P by a Tietze transformation (T1) or (T2).

It is easy to see that the Tietze transformations are "correct", in the sense that they preserve the presented groupoid. With more work [3, Chapter 5], it is even possible to show that those transformations are "complete", in the sense that any two 2-ARS presenting the same groupoid are Tietze equivalent. We only state the first direction here since this is the only one we will need here:

Proposition 29. *Any two Tietze equivalent 2-*ARS *present isomorphic groupoids.*

From the transformations of definition 28, it is possible to derive other transformations, which thus also preserve the presented groupoid.

Lemma 30. *Suppose given a 2-*ARS P *containing a rewriting rule* $a : x \to y$ *and a relation* $A : p \Rightarrow q$ *such that* a *occurs exactly once in the source* p*, i.e.* $p = p_1 \cdot a \cdot p_2$*, and does not occur in the target*



Then P *is Tietze equivalent to the 2-*ARS *where*

- *we have removed the rewriting rule* a*,*
- *we have removed the coherence relation* A*,*
- *we have replaced every occurrence of* a *in the source or target of a coherence relation by* $p_1^- \cdot q \cdot p_2^-$*.*

Rewriting properties. Suppose fixed a 2-ARS P. For simplicity, we suppose that for every coherence relation $A : p \Rightarrow q$ in P_2, we have that p and q are rewriting paths (as opposed to zig-zags). We also suppose fixed a set $W \subseteq \mathsf{P}_1$. We can think of W as inducing a rewriting subsystem W of P, with P_0 as objects, W as rewriting steps and

$$\mathsf{W}_2 = \{A \in \mathsf{P}_2 \mid s_1(A) \in W^* \text{ and } t_1(A) \in W^*\}$$

as coherence relations, and formulate the various traditional rewriting concepts with respect to it. In such a situation, consider the presented groupoid $\mathcal{C} = \overline{\mathsf{P}}$. The set W of rewriting rules, induces a set of morphisms of $\mathcal{C}$, namely $\{\overline{w} \mid w \in W\}$ that we still write W, which generates a subgroupoid $\mathcal{W}$ of $\mathcal{C}$. Our aim here is to provide rewriting tools in order to show that $\mathcal{W}$ is rigid, so that $\mathcal{C}$ is equivalent to the quotient $\mathcal{C}/\mathcal{W}$ by theorem 14, and moreover provide a concrete description of the quotient category.

Definition 31. The 2-ARS P is W*-terminating* if there is no infinite sequence $a_1, a_2, \ldots$ of elements of W such that every finite prefix is a rewriting path, i.e. belongs to W^*.

Definition 32. An element $x \in \mathsf{P}_0$ is a W*-normal form* when there is no rewriting step in W with x as source. We say that P is *weakly* W*-normalizing* when for every $x \in \mathsf{P}_0$ there exists a normal form $\hat{x}$ and a rewriting path $x \xrightarrow{*} \hat{x}$. In this case, we write $n_x : x \xrightarrow{*} \hat{x}$ for an arbitrary choice of such a path, which is however supposed to be the identity when x is a normal form.

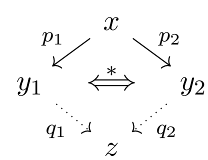
Lemma 33. *If* P *is* W*-terminating then it is weakly* W*-normalizing.*

Proof. Consider a maximal rewriting path $a_1, a_2, \ldots$ in W^* starting from x. Because P is W-terminating, this path is necessarily finite, and its target is a normal form by maximality. □

Definition 34. A W*-branching* is a pair of rewriting paths

$$p_1 : x \xrightarrow{*} y_1 \qquad\qquad q_2 : x \xrightarrow{*} y_2$$

in W^* which are coinitial, i.e. have the same source x which is called the *source* of the branching. A W-branching is *local* when both p_1 and p_2 are rewriting steps. A W-branching as above is *confluent* when there is a pair of cofinal (with the same target) rewriting paths $q_1 : y_1 \overset{*}{\to} z$ and $q_2 : y_2 \overset{*}{\to} z$ in W^* such that $p_1 \cdot q_1 \overset{*}{\Leftrightarrow} p_2 \cdot q_2$:



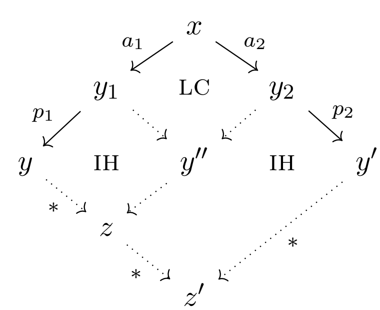
Note that, in the above definitions, not only we require that we can close a span of rewriting steps by a cospan of rewriting paths (as in the traditional definition of confluence), but also that the confluence square can be filled coherence relations.

Definition 35. The ARS P is *locally W-confluent* when W-branching is confluent. It is *W-confluent* when for every $p_1 : x \overset{*}{\to} y_1$ and $p_2 : x \overset{*}{\to} y_2$ in W^*, there exist $q_1 : y_1 \overset{*}{\to} z$ and $q_2 : y_2 \overset{*}{\to} z$ in W^* such that $p_1 \cdot q_1 \overset{*}{\Leftrightarrow} p_2 \cdot q_2$. We say that P is *W-convergent* when it is both W-terminating and W-confluent.

The celebrated Newman's lemma (also sometimes called the diamond lemma) along with its traditional proof easily generalizes to our setting:

Proposition 36. *If P is W-terminating and locally W-confluent then it is W-confluent.*

Proof. The relation on objects defined by $x \geq y$ whenever there exists a rewriting path $p : x \overset{*}{\to} y$ in W^* is a well-founded partial order because P is W-terminating. We say that P is *W-confluent at x* when every W-branching with source x confluent. We are going to show that P is locally W-confluent at x for every object x, by well-founded induction on x. In the base case, x is a W-normal form and the result is immediate. Otherwise, consider a W-branching consisting of paths $a_1 \cdot p_1$ and $a_2 \cdot p_2$ for some rewriting steps a_1 and a_2 and rewriting paths p_1 and p_2 (we suppose that the paths are non-empty, otherwise the result is immediate). The following diagram shows the W-confluence at x:



Above, the diagram LC is W-confluent by local confluence, and the two diagrams IH are by induction hypothesis. ☐

Definition 37. The 2-ARS P is *W-coherent* if for any parallel zig-zags $p, q : x \overset{\sim}{\to} y$ in $W^\sim$, we have $p \overset{*}{\Leftrightarrow} q$.

The following is immediate:

Lemma 38. *A 2-ARS P is W-coherent precisely when $\overline{\mathsf{W}}$ is a rigid subgroupoid of $\overline{\mathsf{P}}$.*

The traditional Church-Rosser property [47, Theorem 1.2.2] generalizes as follows in our setting:

Proposition 39. *If* P *is weakly* W*-normalizing and* W*-confluent then for any zig-zag* $p : x \overset{\sim}{\rightarrow} y$ *in* $W^{\sim}$*, we have* $\hat{x} = \hat{y}$ *and* $p \cdot n_y \overset{*}{\Leftrightarrow} n_x$*, i.e. the diagram*

$$\begin{array}{ccc} x & \xrightarrow{p} & y \\ {\scriptstyle n_x}\downarrow & & \downarrow{\scriptstyle n_y} \\ \hat{x} & = & \hat{y} \end{array}$$

commutes in $\overline{\mathsf{P}}$.

Proof. By confluence, given a rewriting path $p : x \overset{*}{\rightarrow} y$ in W^*, we have $\hat{x} = \hat{y}$ and $p \cdot n_y \overset{*}{\Leftrightarrow} n_x$, and thus $p^+ \cdot n_y \overset{*}{\Leftrightarrow} n_x$ and $n_x \cdot p^- \overset{*}{\Leftrightarrow} n_y$, i.e. the following diagrams commute in $\overline{\mathsf{P}}$:

$$\begin{array}{ccc} x & \xrightarrow{p^+} & y \\ {\scriptstyle n_x}\downarrow & & \downarrow{\scriptstyle n_y} \\ \hat{x} & = & \hat{y} \end{array} \qquad\qquad \begin{array}{ccc} x & \xleftarrow{p^-} & y \\ {\scriptstyle n_x}\downarrow & & \downarrow{\scriptstyle n_y} \\ \hat{x} & = & \hat{y} \end{array}$$

Any zig-zag $p : x \overset{\sim}{\rightarrow} y$ in $W^{\sim}$ decomposes as $p = p_1^- q_1^+ p_2^- p_2^+ \ldots p_n^- p_n^+$ for some $n \in \mathbb{N}$ and paths p_i and q_i in W^*. We thus have $p \cdot n_y \overset{*}{\Leftrightarrow} n_x$, since all the squares of the following diagram commute in $\overline{W}$ by the preceding remark:

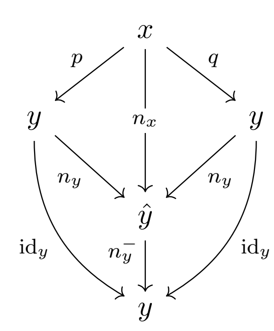
$$\begin{array}{ccccccccccccc} x & \xrightarrow{p_1^-} & y_1 & \xrightarrow{q_1^+} & x_2 & \longrightarrow & \cdots & \longrightarrow & x_n & \xrightarrow{p_n^-} & y_n & \xrightarrow{q_n^-} & y \\ {\scriptstyle n_x}\downarrow & & \downarrow{\scriptstyle n_{y_1}} & & \downarrow{\scriptstyle n_{x_2}} & & & & \downarrow{\scriptstyle n_{x_n}} & & \downarrow{\scriptstyle n_{y_n}} & & \downarrow{\scriptstyle n_y} \\ \hat{x} & = & \hat{x} & = & \hat{x} & = & \cdots & = & \hat{x} & = & \hat{x} & = & \hat{x} \end{array}$$

which allows us to conclude. □

This implies the following "abstract" variant of Squier's homotopical theorem [44, 26, 16]:

Proposition 40. *If* P *is weakly* W*-normalizing and is* W*-confluent then it is* W*-coherent.*

Proof. Given two parallel zig-zags $p, q : x \overset{\sim}{\rightarrow} y$ in $W^{\sim}$, we have $p \overset{*}{\Leftrightarrow} q$, since the following diagram commutes in $\overline{\mathsf{P}}$:



Namely, we have $\hat{x} = \hat{y}$ by confluence, the two triangles above commute by proposition 39, and the two triangles below do because n_y^- is an inverse for n_y. □

Example 41. As a variant of example 15, consider the 2-ARS P with $\mathsf{P}_0 = \{x, y\}$, $\mathsf{P}_1 = \{a, b : x \to y\}$ and $\mathsf{P}_2 = \emptyset$, i.e. $x \underset{b}{\overset{a}{\rightrightarrows}} y$. With $W = \{a\}$, we have that P is W-terminating and locally W-confluent, thus W-confluent by proposition 36, and thus W-coherent by lemma 33 and proposition 40. With $W = \{a, b\}$, we have seen in example 15 that the groupoid $\overline{W}$ is not rigid and, indeed, P is not W-confluent because $\overline{a} \neq \overline{b}$ (because $\mathsf{P}_2 = \emptyset$).

Definition 42. We write $\mathrm{N}(\overline{\mathsf{P}})$ for the *category of normal forms* of $\overline{\mathsf{P}}$, defined as the full subcategory of $\overline{\mathsf{P}}$ whose objects are those in W-normal form.

Lemma 43. *If* P *is weakly* W*-normalizing, the inclusion functor* $\mathrm{N}(\overline{\mathsf{P}}) \to \overline{\mathsf{P}}$ *is an equivalence of categories.*

Proof. An object x of P admits a normal form $\hat{x}$, by lemma 33. Writing $n_x : x \xrightarrow{*} \hat{x}$ for a normalization path, we have an isomorphism $\overline{n_x} : x \to \hat{x}$ in $\overline{\mathsf{P}}$. The inclusion functor is thus full and faithful (by definition), and every object of $\overline{\mathsf{P}}$ is isomorphic to an object in the image, it is thus an equivalence of categories. □

When P is W-convergent, the equivalence given in the above lemma is precisely the one with the quotient category:

Proposition 44. *If* P *is* W*-convergent, the quotient category is isomorphic to the category of normal forms:* $\overline{\mathsf{P}}/W \cong \mathrm{N}(\overline{\mathsf{P}})$.

Proof. Since P is W-convergent, by proposition 40 and lemma 38, the groupoid generated by W is rigid and we thus have the description of the quotient $\overline{\mathsf{P}}/W$ given by proposition 8. We have a canonical functor $\mathrm{N}(\overline{\mathsf{P}}) \to \overline{\mathsf{P}}/W$, obtained as the composite of the inclusion functor $\mathrm{N}(\overline{\mathsf{P}}) \to \overline{\mathsf{P}}$ with the quotient functor $\overline{\mathsf{P}} \to \overline{\mathsf{P}}/W$. An object of $\overline{\mathsf{P}}/W$ is an equivalence class $[x]$ of objects which, by convergence, contains a unique normal form, namely $\hat{x}$. The functor is bijective on objects. By weak normalization (lemma 33), any morphism $f : x \to y$ is equivalent to one with both normal source and target, namely $\hat{f} = n_y \circ f \circ n_x^- : \hat{x} \to \hat{y}$, hence the functor is full. Suppose given two morphisms $f, g : \hat{x} \to \hat{y}$ in $\mathrm{N}(\mathsf{P})$ with the same image $[f] = [g]$: by definition of the equivalence on morphisms, there exist morphisms $v : \hat{x} \to \hat{x}$ and $w : \hat{y} \to \hat{y}$ in $W^\sim$ making the diagram

$$
\begin{array}{ccc}
\hat{x} & \xrightarrow{f} & \hat{y} \\
{\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\
\hat{x} & \xrightarrow[g]{} & \hat{y}
\end{array}
$$

commute. By the Church-Rosser property (proposition 40), we have $v = n_{\hat{x}} \circ n_{\hat{x}}^-$ and thus $v = \mathrm{id}_x$ (since $n_{\hat{x}} = \mathrm{id}_{\hat{x}}$ by hypothesis), and similarly $w = \mathrm{id}_y$. Hence $f = g$ and the functor is faithful. The functor is thus an isomorphism as being full, faithful and bijective on objects. □

We would now like to provide an explicit description of $\mathrm{N}(\overline{\mathsf{P}})$, by a 2-ARS. A good candidate is the following 2-ARS $\mathsf{P} \setminus W$ obtained by "restricting P to normal forms". More precisely,

- $(\mathsf{P} \setminus W)_0$: the objects of $\mathsf{P} \setminus W$ are the those of P in W-normal form,
- $(\mathsf{P} \setminus W)_1$: the rewriting rules of $\mathsf{P} \setminus W$ are those of P whose source and target are both in $(\mathsf{P} \setminus W)_0$ (in particular, it does not contain any element of W, thus the notation),
- $(P \setminus W)_2$: the coherence relations are those of P_2 whose source and target both belong to $(\mathsf{P} \setminus W)_1^\sim$.

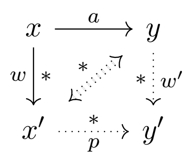
It is not the case in general that this 2-ARS presents $\mathrm{N}(\overline{\mathsf{P}})$, but we provide here conditions which ensure that it holds, see also [10, 39] for alternative conditions. For simplicity, we suppose here that the source and target of every rewriting step in P_2 is a path (as opposed to a zig-zag).

Proposition 45. *Suppose that*

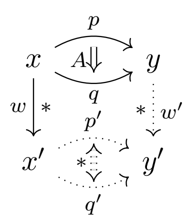
(1) P *is* W*-convergent,*

(2) *every rule* $a : x \to y$ *in* P_1 *whose source* x *is* W*-normal also has a* W*-normal target* y*,*

(3) *for every coinitial rule* $a : x \to y$ *in* P_1 *and path* $w : x \overset{*}{\to} x'$ *in* W^**, there are paths* $p : x' \overset{*}{\to} y'$ *in* P_1^* *and* $w' : y \overset{*}{\to} y' \in W^*$ *such that* $a \cdot w' \overset{*}{\Leftrightarrow} w \cdot p$*:*



(4) *for every coherence relation* $A : p \Rightarrow q : x \overset{*}{\to} y$*, and for every path* $w : x \overset{*}{\to} x'$*, the paths* $p', q' : x' \overset{*}{\to} y'$ *in* P_1^* *and* $w' : y \overset{*}{\to} y'$ *in* W^* *such that* $p \cdot w' \overset{*}{\Leftrightarrow} w \cdot p'$ *and* $q \cdot w' \overset{*}{\Leftrightarrow} w \cdot q'$ *induced by (3) satisfy* $p' \overset{*}{\Leftrightarrow} q'$*:*



Then $\mathrm{N}(\overline{\mathsf{P}})$ *is isomorphic to* $\overline{\mathsf{P} \setminus W}$*.*

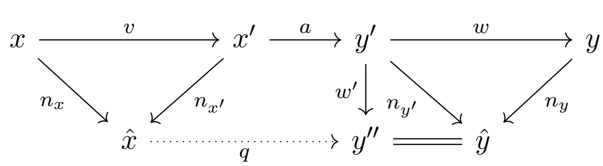
Proof. We write $\mathsf{Q} = \mathsf{P} \setminus W$. Since Q is, by definition, a sub-2-ARS of P there is a canonical functor $\overline{\mathsf{Q}} \to \overline{\mathsf{P}}$. Moreover, since the objects of P are, by definition, in W-normal form, this functor corestricts as a functor $F : \overline{\mathsf{Q}} \to \mathrm{N}(\overline{\mathsf{P}})$ which is the identity on objects.

First, note that condition (2) implies that for any path p of the form

$$x_0 \xrightarrow{a_1} x_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} x_n$$

with x_0 in W-normal form, we have that all the x_i are in W-normal form and thus p belongs to Q_1^*. Similarly, every coherence relation $A : p \Rightarrow q : x \overset{*}{\to} y$ with x in W-normal form belongs to Q_2.

We claim that for every zig-zag $p : x \overset{\sim}{\to} y$ in $\mathsf{P}_1^\sim$ there is zig-zag $q \in \mathsf{Q}_1^\sim$ such that $p \overset{*}{\Leftrightarrow} n_x \cdot q \cdot n_y^-$. We have that p is of the form $p = w_0 \cdot a_1 \cdot w_1 \cdot a_2 \cdot w_2 \cdot \ldots \cdot a_n \cdot w_n$ where the a_i are rules in P_1 which are not in W (possibly taken backward) and the w_i are in $W^\sim$. For instance, consider the case $n = 1$ and a path p of the form $p = v \cdot a \cdot w$ with $a \in \mathsf{Q}_1$ and $v, w \in W^\sim$ (the case where a is reversed is similar, and the general case follows by induction):



By hypothesis (1) and proposition 39, we have $v \overset{*}{\Leftrightarrow} n_x \cdot n_{x'}^-$ and $w \overset{*}{\Leftrightarrow} n_{y'} \cdot n_y^-$. By hypothesis (3), there exist paths $q : \hat{x} \overset{*}{\to} y''$ in P_1^* and $w' : y' \overset{*}{\to} y''$ in W^* such that $a \cdot w' \overset{*}{\Leftrightarrow} n_{x'} \cdot q$. By hypothesis (2) and the remark at the beginning of this proof, we have that $q \in \mathsf{Q}_1^*$ and y'' is a normal form. By (1), we thus have $y'' = \hat{y}$ and $w' \overset{*}{\Leftrightarrow} n_{y'}$, and we conclude. As a particular case of the property we have just shown, for any zig-zag $p : x \overset{\sim}{\to} y$ whose source and target are in W-normal form, we have that that p is equivalent to a zig-zag $q : x \overset{\sim}{\to} y$ (since in this case both n_x and n_y are identities). The functor $F : \overline{\mathsf{Q}} \to \mathrm{N}(\overline{\mathsf{P}})$ is thus full.

In the following, given a path $p : x \to y$ in P_1^*, we write $\underline{p} : \hat{x} \to \hat{y}$ in Q_1^* for a path such that $p = n_x \cdot \underline{p} \cdot n_y^-$. Such a path always exists by the previous reasoning and can be constructed in a functorial way (i.e. $\underline{p_1 \cdot p_2} = \underline{p}_1 \cdot \underline{p}_2$). Now, suppose given two zig-zags $p, p' : x \overset{\sim}{\to} y$ in $\mathsf{P}_1^\sim$ such that $p \overset{*}{\Leftrightarrow} p'$. The relation $p \overset{*}{\Leftrightarrow} p'$ means that there is a sequence $p_1, \ldots, p_n$ of zig-zags in $\mathsf{P}_1^\sim$ such that $p_1 = p$, $p_n = p'$ and each p_i is related to p_{i+1} by taking a relation in context, as in (2.3):

$$p = p_1 \Leftrightarrow p_2 \Leftrightarrow \ldots \Leftrightarrow p_n = p'$$

More formally, for $1 \leq i < n$, there is a decomposition

$$p_i = q_i \cdot r_i \cdot s_i \qquad\qquad p_{i+1} = q_i \cdot r'_i \cdot s_i$$

such that there is a relation $A : r_i \Rightarrow r'_i$ or $A : r'_i \Rightarrow r_i$ (another approach would consist in reasoning by induction on the 2-cells of the freely generated 2-groupoid of remark 27). By hypothesis (4), there is a relation $\underline{r}_i \overset{*}{\Leftrightarrow} \underline{r}'_i$, and thus $\underline{p}_i \overset{*}{\Leftrightarrow} \underline{p}_{i+1}$ by functoriality. By recurrence on n, we thus have $\underline{p} \overset{*}{\Leftrightarrow} \underline{p}'$. From this, we deduce that that the functor F is also faithful. □

In practice, condition (1) can be shown using traditional rewriting techniques (e.g. proposition 36) and condition (2) is easily checked by direct inspection of the rewriting rules. We provide below sufficient conditions in order to show the two remaining conditions:

Proposition 46. *We have the following.*

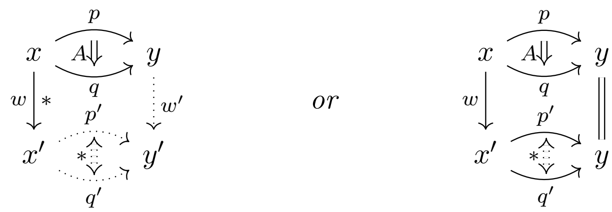
(3) *Suppose that for every coinitial rules $a : x \to y$ in P_1 and $w : x \to x'$ in W, there are paths $p : x' \overset{*}{\to} y'$ in P_1^* and $w' : y \overset{*}{\to} y'$ in W^* such that w is of length at most one and $a \cdot w' \overset{*}{\Leftrightarrow} w \cdot p$:*

$$\begin{array}{ccc} x & \xrightarrow{a} & y \\ {\scriptstyle w}\downarrow & & \vdots\, {\scriptstyle w'} \\ x' & \underset{p}{\overset{*}{\dashrightarrow}} & y' \end{array} \qquad \text{or} \qquad \begin{array}{ccc} x & \xrightarrow{a} & y \\ {\scriptstyle w}\downarrow & & \Vert \\ x' & \underset{p}{\overset{*}{\dashrightarrow}} & y' \end{array} \tag{2.4}$$

The condition (3) of proposition 45 is satisfied.

(4) *Suppose that condition (3) is satisfied and for every coherence relation $A : p \Rightarrow q : x \overset{*}{\to} y$ in P_2 and rule $w : x \to x'$ in W, the paths $p', q' : x' \overset{*}{\to} y'$ in P_1^* and $w' : y \overset{*}{\to} y'$ in W^* of length at most one such that $p \cdot w' \overset{*}{\Leftrightarrow} w \cdot p'$ and $q \cdot w' \overset{*}{\Leftrightarrow} w \cdot q'$ induced by (3) are such*

that $p' \overset{*}{\Leftrightarrow} q'$*:*



(2.5)

Then condition (4) of proposition 45 is satisfied.

Proof. Both properties are easily shown by recurrence on the length of w. □

We can finally summarize the results obtained in this section as follows. Given a 2-ARS P and a set $W \subseteq \mathsf{P}_1$, we have the following possible reasonable definitions of the fact that P is *coherent* wrt W:

(1) Every parallel zig-zags with edges in W are equal
(i.e. the subgroupoid of $\overline{P}$ generated by W is rigid).
(2) The quotient map $\overline{\mathsf{P}} \to \overline{\mathsf{P}}/W$ is an equivalence of categories.
(3) The canonical morphism $\mathrm{N}(\mathsf{P}) \to \overline{\mathsf{P}}$ is an equivalence.
(4) The inclusion $\mathrm{Alg}(\overline{\mathsf{P}}/W, \mathcal{D}) \to \mathrm{Alg}(\overline{\mathsf{P}}, \mathcal{D})$ is a natural equivalence of categories.

Theorem 47. *If* P *is* W*-convergent then all the above coherence properties hold.*

Proof. (1) is given by proposition 40, (2) is given by (1) and theorem 14, (3) is given by proposition 44, and (4) is given by (1) and proposition 19. □

3. Relative coherence and term rewriting systems

In order to use the previous developments in concrete situations, such as (symmetric) monoidal categories, we need to consider a more structured notion of theory. For this reason, we consider here Lawvere 2-theories, as well as the adapted notion of rewriting, which is a coherent extension of the traditional notion of term rewriting systems.

3.1. Lawvere 2-theories.

We begin by recalling the traditional notion due to Lawvere [30]:

Definition 48. A *Lawvere theory* $\mathcal{T}$ is a cartesian category, with $\mathbb{N}$ as set of objects, and cartesian product given on objects by addition. A morphism between Lawvere theories is a product-preserving functor and we write $\mathbf{Law}_1$ for the category of Lawvere theories.

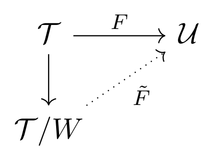
For simplicity, we restrict here to unsorted theories, but the developments performed here could easily adapted to the multi-sorted case. In such a theory, we usually restrict our attention to morphisms with 1 as codomain, since $\mathcal{T}(n, m) \cong \mathcal{T}(n, 1)^m$ by cartesianness.

A $(2,1)$*-category* is a 2-category in which every 2-cell is invertible (i.e. a category enriched in groupoids). The following generalization of Lawvere theory was introduced in various places, see [14, 49, 50] (as well as [42] for the enriched point of view):

Definition 49. A *Lawvere 2-theory* $\mathcal{T}$ is a cartesian $(2,1)$-category with $\mathbb{N}$ as objects, and cartesian product given on objects by addition. A morphism $F : \mathcal{T} \to \mathcal{U}$ between 2-theories is a 2-functor which preserves products. We write $\mathbf{Law}_2$ for the resulting category (which can be extended to a 3-category by respectively taking natural transformations and modifications as 2- and 3-cells).

We can reuse the properties developed in section 2 by working “hom-wise” as follows. Suppose fixed a 2-theory $\mathcal{T}$ together with a subset W of the 2-cells. We write $\mathcal{W}$ for the sub-2-theory of $\mathcal{T}$, with the same 0- and 1-cells, and whose 2-cells contain W (we often assimilate this 2-theory to its set of 2-cells). A morphism $F : \mathcal{T} \to \mathcal{U}$ of Lawvere 2-theories is *W-strict* when it sends every 2-cell in W to an identity.

Definition 50. The *quotient* 2*-theory* $\mathcal{T}/W$ is the theory equipped with a W-strict morphism $\mathcal{T} \to \mathcal{T}/W$ such that every W-strict morphism $F : \mathcal{T} \to \mathcal{U}$ extends uniquely as a morphism $\mathcal{T}/W \to \mathcal{U}$:



We have $\mathcal{T}/W \cong \mathcal{T}/\mathcal{W}$, so that we can always assume that we are quotienting by a sub-2-theory. On hom-categories, the quotient corresponds to the one introduced in section 2.1:

Lemma 51. *For every $m, n \in \mathbb{N}$, we have*

$$(\mathcal{T}/\mathcal{W})(m,n) = \mathcal{T}(m,n)/\mathcal{W}(m,n).$$

We say that a morphism

$$F : \mathcal{T} \to \mathcal{U}$$

is a *local equivalence* when for every objects $m, n \in \mathcal{T}$, the induced functor

$$F_{m,n} : \mathcal{T}(m,n) \to \mathcal{U}(m,n)$$

between hom-categories is an equivalence.

Definition 52. A theory $\mathcal{W}$ is 2*-rigid* when any two parallel 2-cells are equal.

Lemma 53. *A theory $\mathcal{W}$ is 2-rigid if and only if the category $\mathcal{W}(m,n)$ is rigid for every 0-cells m and n.*

By direct application of theorem 14, we have

Theorem 54. *The quotient 2-functor $\mathcal{T} \to \mathcal{T}/\mathcal{W}$ is a local equivalence iff $\mathcal{W}$ is 2-rigid.*

3.2. **Extended rewriting systems.** We briefly recall here the categorical setting for term rewriting systems. A more detailed presentation can be found in [11, 7, 36, 3].

Definition 55. A *signature* consists of a set S_1 of *symbols* together with a function $s_0 : \mathsf{S}_1 \to \mathbb{N}$ associating to each symbol an *arity* and we write $a : n \to 1$ for a symbol a of arity n. A morphism of signatures is a function between the corresponding sets of symbols which preserves arity, and we write $\mathbf{Pol}_1^\times$ for the corresponding category.

There is a forgetful functor $\mathbf{Law}_1 \to \mathbf{Pol}_1^\times$, sending a theory $\mathcal{T}$ on the set $\bigsqcup_{n\in\mathbb{N}} \mathcal{T}(n,1)$ with first projection as arity. This functor admits a left adjoint $-^* : \mathbf{Pol}_1^\times \to \mathbf{Law}_1$, which we now describe. Given a signature S_1, and $n \in \mathbb{N}$, $\mathsf{S}_1^*(n,1)$ is the set of *terms* of arity n: those are formed using operations, with variables in $\{x_1^n, x_2^n, \ldots, x_n^n\}$. Note that the superscript for variables is necessary to unambiguously recover the type of a variable, i.e. $x_i^n : n \to 1$, but for simplicity we will often omit it in the following. More explicitly, the family of sets $\mathsf{S}_1^*(n,1)$ is the smallest one such that

– for $1 \leq i \leq n$, we have

$$x_i^n \in \mathsf{S}_1^*(n, 1)$$

– given $m, n \in \mathbb{N}$, a symbol $a : n \to 1$ and terms $t_1, \ldots, t_n \in \mathsf{S}_1^*(m, 1)$, we have

$$a(t_1, \ldots, t_m) \in \mathsf{S}_1^*(m, 1)$$

More generally, a morphism f in $\mathsf{S}_1^*(n, m)$ is an m-uple

$$f = \langle t_1, \ldots, t_m \rangle$$

of terms t_i with variables in $\{x_1^n, \ldots, x_n^n\}$, which can be thought of as a formal *substitution*. Given such a substitution f and a term t, we write

$$t[f] \qquad \text{or} \qquad t[t_1/x_1, \ldots, t_n/x_n]$$

for the term obtained from t by formally replacing each variable x_i^n by t_i. This operation is thus defined inductively by

$$x_i^n[f] = t_i \qquad\qquad a(u_1, \ldots, u_k)[f] = a(u_1[f], \ldots, u_k[f])$$

The composition of two morphisms $\langle t_1, \ldots, t_m \rangle : \mathsf{S}_1^*(n, m)$ and $\langle u_1, \ldots, u_k \rangle : \mathsf{S}_1(m, k)$ is given by parallel substitution:

$$\langle u_1, \ldots, u_k \rangle \circ \langle t_1, \ldots, t_m \rangle = \langle u_1[t_1/x_1, \ldots, t_n/x_n], \ldots, u_k[t_1/x_1, \ldots, t_m/x_m] \rangle$$

and the identity in $\mathsf{S}_1^*(n, n)$ is $\langle x_1^n, \ldots, x_n^n \rangle$. The resulting category S_1^* is easily checked to be a Lawvere theory, which satisfies the following universal property:

Lemma 56. *The Lawvere theory S_1^* is the free Lawvere theory on the signature S_1.*

By abuse of notation, we sometimes write

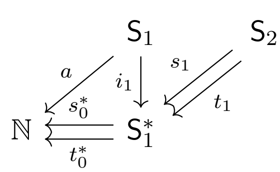
$$\mathsf{S}_1^* = \bigsqcup_{m,n \in \mathbb{N}} \mathsf{S}_1^*(m, n)$$

for the set of all substitutions and $s_0^*, t_0^* : \mathsf{S}_1^* \to \mathbb{N}$ for the source and target maps, and $i_1 : \mathsf{S}_1 \to \mathsf{S}_1^*$ for the map sending an operation $a : n \to 1$ to the substitution consisting of one term $\langle a(x_1^n, \ldots, x_n^n) \rangle$, so that we have $s_0^* \circ i_1 = s_0$ and $t_0^* \circ i_1 = 1$.

Definition 57. A *term rewriting system*, or TRS, S consists of a signature S_1 together with a set S_2 of *rewriting rules* and functions $s_1, t_1 : \mathsf{S}_2 \to \mathsf{S}_1^*$ which indicate the source and target of each rewriting rule, and are supposed to satisfy

$$s_0^* \circ s_1 = s_0^* \circ t_1 \qquad\qquad t_0^* \circ s_1 = t_0^* \circ t_1 = 1$$

This data can be summarized in the following diagram:



We sometimes write

$$\rho : t \Rightarrow u$$

for a rule ρ with t as source and u as target. The relations satisfied by TRS ensure that both t and u have the same arity.

We now need to introduce some notions in order to be able to define rewriting in such a setting. A *context* C of arity n is a term with variables in $\{x_1, \ldots, x_n, \square\}$ where the variable

$\square$ is a particular variable, the *hole*, occurring exactly once. Here, we define the number $|t|_i$ of occurrences of a variable x_i (and similarly for $\square$) in a term t by induction by

$$|x_i|_i = 1 \qquad\qquad |a(t_1, \ldots, t_n)|_i = \sum_{k=1}^{n} |t_k|_i$$

We write $\mathsf{S}_n^\square$ for the set of contexts of arity n. Given a context C and a term t, both of same arity n, we write $C[t]$ for the term obtained from C by replacing $\square$ by t. The composition of contexts C and D is given by substitution

$$D \circ C = D[C]$$

This composition is associative and admits the identity context $\square$ as neutral element. A *bicontext* from n to k, is a pair (C, f) consisting of a context C of arity n and a substitution $f \in \mathsf{S}_1^*(n, k)$. This data can be thought of as the specification of a function on terms

$$\begin{aligned}\mathsf{S}_1^*(n, 1) &\to \mathsf{S}_1^*(k, 1)\\ \langle t\rangle &\mapsto C[\langle t\rangle \circ f]\end{aligned}$$

In the following, for simplicity, we will omit the brackets and simply write t instead of $\langle t\rangle$ in such an expression, so that the image of the function can also be denoted $C[t[f]]$. This function will be referred as the *action* of a bicontext on terms. The composition of bicontexts (C, f) and (D, g) of suitable types is given by $(D \circ C, f \circ g)$. The action is compatible with this composition, in the sense that we have

$$D[C[- \circ f] \circ g] = (D \circ C[f])[- \circ (f \circ g)]$$

A *rewriting step* of arity n

$$C[\rho \circ f] : C[t \circ f] \Rightarrow C[u \circ f]$$

is a triple consisting of

– a rewriting rule $\rho : t \Rightarrow u$, with t and u of arity k,
– a context C of arity n,
– a substitution $f : n \to k$ in S_1^*.

A rewriting step can thus be thought of as a rewriting rule in a bicontext. Its source is the term $C[t \circ f]$ and its target is the term $C[u \circ f]$. We write $\mathsf{S}_2^{[]}$ for the set of rewriting steps. A *rewriting path* π is a composable sequence

$$C_1[t_1 \circ f_1] \xRightarrow{C_1[\rho_1 \circ f_1]} C_1[u_1 \circ f_1] = C_2[t_2 \circ f_2] \xRightarrow{C_2[\rho_2 \circ f_2]} \cdots \xRightarrow{C_n[\rho_n \circ f_n]} C_n[u_n \circ f_n]$$

of rewriting steps. We write S_2^* for the set of rewriting paths and adopt the previous notation, e.g. we write $\pi \cdot \pi'$ for the concatenation of two composable rewriting paths π and π'. As in section 2.3, we can also define a notion of *rewriting zig-zag* which is similar to rewriting paths excepting that some rewriting steps may be taken backwards, and write $\mathsf{S}_1^\sim$ for the corresponding set.

Given a signature S_1, there is a forgetful functor from the category of Lawvere 2-theories with S_1^* as underlying Lawvere theory to the category rewriting systems with S_1 as signature (with the expected notion of morphism).

Lemma 58. *Given a* TRS S*, the Lawvere* 2*-theory* $\mathsf{S}^\sim$ *with* S_1^* *as* 1*-cells and* $\mathsf{S}_2^\sim$ *as* 2*-cells is free on* S*.*

The action of bicontexts on terms extend to rewriting steps as follows. Given a rewriting step

$$C[\rho \circ f] : C[t \circ f] \Rightarrow C[u \circ f]$$

a context D and a substitution g of suitable types, we define $D[C[\rho \circ f] \circ g]$ to be the rewriting step

$$(D \circ C[g])[\rho \circ (f \circ g)] : (D \circ C[g])[t \circ (f \circ g)] \Rightarrow (D \circ C[g])[u \circ (f \circ g)]$$

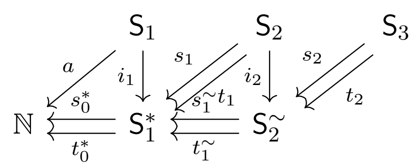
Moreover, we extend this action to rewriting paths and zig-zags by functoriality, i.e.

$$C[(p \cdot q) \circ f] = C[p \circ f] \cdot C[q \circ f]$$

Definition 59. An *extended term rewriting system*, or 2-TRS, consists of a term rewriting system as above, together with a set S_3 of *coherence relations* and functions $s_2, t_2 : \mathsf{S}_3 \to \mathsf{S}_2^{\sim}$, indicating their source and target, satisfying

$$s_1^{\sim} \circ s_2 = s_1^{\sim} \circ t_2 \qquad\qquad t_1^{\sim} \circ s_2 = t_1^{\sim} \circ t_2$$

Diagrammatically,



Given a 2-TRS as above, we sometimes write

$$A : \pi \Rrightarrow \pi'$$

to indicate that A is a coherence relation in S_3 with π as source and π' as target. Given two rewriting paths π and π', we write $\pi \overset{*}{\Leftrightarrow} \pi'$ when they are related by the smallest congruence identifying the source and target of any coherence relation.

Definition 60. The Lawvere 2-theory *presented* by a 2-TRS S is the $(2,1)$-category noted $\overline{\mathsf{S}}$, with $\mathbb{N}$ as 0-cells, S_1^* as 1-cells and, as 2-cells the quotient of $\mathsf{S}_2^{\sim}$ under the congruence $\overset{*}{\Leftrightarrow}$.

Example 61. The extended rewriting system Mon for monoids has symbols and rules

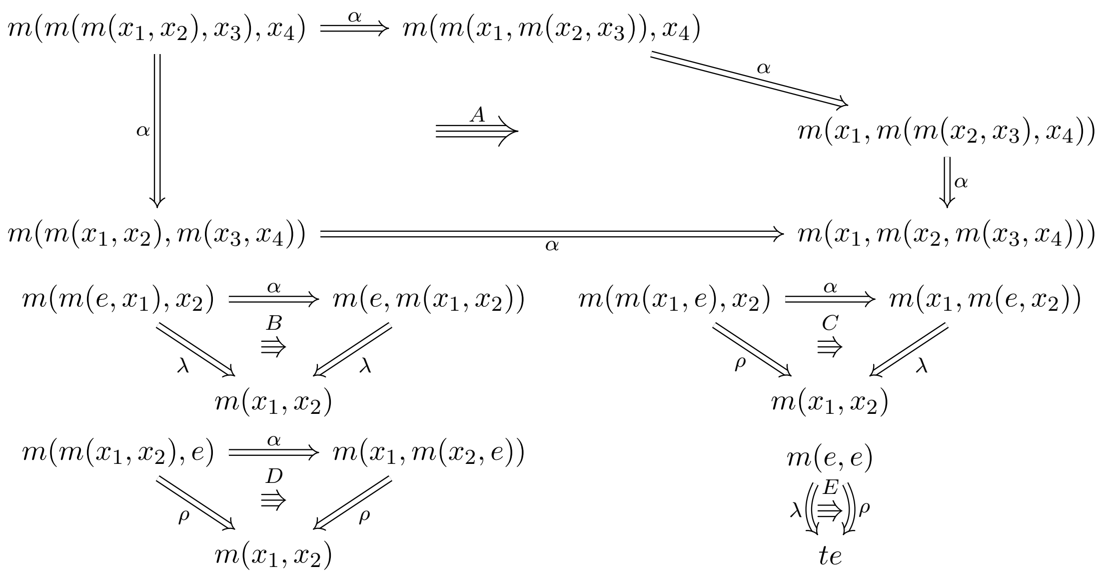
$$\mathsf{Mon}_1 = \{m : 2 \to 1, e : 0 \to 1\}$$

$$\mathsf{Mon}_2 = \left\{ \begin{array}{ll} \alpha : & m(m(x_1,x_2),x_3) \Rightarrow m(x_1,m(x_2,x_3)) \\ \lambda : & \qquad m(e,x_1) \Rightarrow x_1 \\ \rho : & \qquad m(x_1,e) \Rightarrow x_1 \end{array} \right\}$$

There are coherence relations A, B, C, D and E, respectively corresponding to a confluence for the five critical branchings of the rewriting system (as defined in section 3.2), whose 0-sources are

$$m(m(m(x_1,x_2),x_3),x_4) \quad m(m(e,x_1),x_2) \quad m(m(x_1,e),x_2) \quad m(m(x_1,x_2),e) \quad m(e,e)$$

Those coherence relations can be pictured as follows:



For concision, for each arrow, we did not indicate the proper rewriting step, but only the rewriting rule of the rewriting step (hopefully, the reader will easily be able to reconstruct it). For instance, the coherence relation C has type

$$C : m(\rho(x_1), x_2) \Rrightarrow \alpha(x_1, e, x_2) \cdot m(x_1, \lambda(x_2))$$

We mention here that the notion of Tietze transformation can be defined for 2-TRS in a similar way as for 2-ARS (definition 28):

Definition 62. The *Tietze transformations* are the following possible transformations on a 2-ARS P:

(T1) given a zig-zag $\pi : t \overset{\sim}{\to} u$, add a new rewriting rule $\alpha : t \to u$ in P_2 together with a new coherence relation $A : \alpha \Rrightarrow \pi$ in P_3,

(T2) given zig-zags $\pi, \rho : t \overset{\sim}{\to} u$ such that $\pi \overset{*}{\Leftrightarrow} \rho$, add a new coherence relation $A : \pi \Rrightarrow \rho$ in P_3.

The *Tietze equivalence* is the smallest equivalence relation on 2-ARS identifying P and Q whenever Q can be obtained from P by a Tietze transformation (T1) or (T2).

Proposition 63. *Any two Tietze equivalent 2-*TRS *present isomorphic groupoids.*

Suppose fixed a 2-TRS S together with $W \subseteq \mathsf{S}_2$. The 2-TRS S induces an 2-ARS in each hom-set: this point of view will allow reusing the work done on 2-ARS on section 2.

Definition 64. Given a 2-TRS S and $n \in \mathbb{N}$, we write $\mathsf{S}(n, 1)$ for the 2-ARS whose

– objects are the n-ary terms:

$$\mathsf{S}(n, 1)_0 = \mathsf{S}_1^*(n, 1)$$

– morphisms are the n-ary rewriting steps:

$$\mathsf{S}(n, 1)_1 = \mathsf{S}_2^{[]}(n, 1)$$

where $\mathsf{S}_2^{[]}(n, 1)$ is the set of rewriting steps

$$C[\rho \circ f] : C[t \circ f] \Rightarrow C[u \circ f]$$

with both $C[t \circ f]$ and $C[u \circ f]$ of arity n,
- coherence relations are triples (C, A, f), written $C[A \circ f]$, for some context C, coherence relation $A \in \mathsf{S}_3$ and substitution f, of suitable type, of the form

$$C[A \circ f] : C[\pi \circ f] \Rrightarrow C[\pi' \circ f] : C[t \circ f] \Rightarrow C[u \circ f]$$

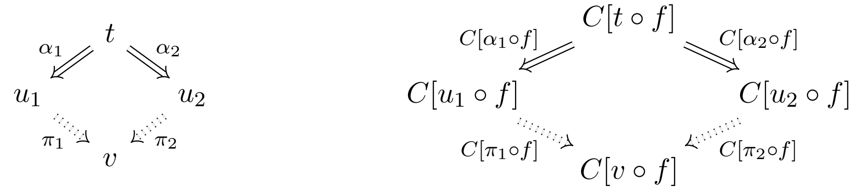
such that both $C[t \circ f]$ and $C[u \circ f]$ of arity n.

Similarly, a set W induces a set $W(m, 1) \subseteq \mathsf{S}(m, 1)_1$, where $W(m, 1)$ is the set of *W-rewriting steps*, i.e. rewriting steps of the form $C[\alpha \circ f]$ with $\alpha \in W$. We say that a 2-TRS S is *W-terminating* / *locally W-confluent* / *W-confluent* / *W-coherent* when each $\mathsf{S}(m, n)$ is with respect to $W(m, n)$. We say that S is *confluent* when it is W-confluent for $W = \mathsf{S}_2$ (and similarly for other properties). More explicitly,

Definition 65. A *W-branching* (α_1, α_2) is a pair of rewriting steps $\alpha_1 : t \Rightarrow u_1$ and $\alpha_2 : t \Rightarrow u_2$ in $\mathcal{W}^{\square}$ with the same source:

$$u_1 \xLeftarrow{\alpha_1} t \xRightarrow{\alpha_2} u_2$$

Such a W-branching is *W-confluent* when there are cofinal rewriting paths $\pi_1 : u_1 \Rightarrow v$ and $\pi_2 : u_2 \Rightarrow v$ in W^* such that $\overline{\alpha_1 \cdot \pi_1} = \overline{\alpha_2 \cdot \pi_2}$, which is depicted on the left



By extension of proposition 36, we have

Proposition 66. *If S is W-terminating and locally W-confluent then it is W-confluent.*

In practice, termination can be shown as follows [4, Section 5.2].

Definition 67. A *reduction order* $\geq$ is a well-founded preorder on terms in S_1^* which is compatible with context extension: given terms $t, u \in \mathsf{S}_1^*$, $t > u$ implies $C[t \circ f] > C[u \circ f]$ for every context C and substitution $f \in \mathsf{S}_1^*$ (whose types are such that the expressions make sense).

Proposition 68. *A 2-*TRS *S equipped with a reduction order such that $t > u$ for any rule $\alpha : t \Rightarrow u$ in W is W-terminating.*

Proof. For any rewriting step $C[\rho \circ f] : C[t \circ f] \Rightarrow C[u \circ f]$ we have $C[t \circ f] > C[u \circ f]$ and we conclude by well-foundedness. □

Moreover, in order to construct a reduction order one can use the following "interpretation method" [4, Section 5.3].

Proposition 69. *Suppose given a well-founded poset $(X, \leq)$ and an interpretation*

$$[\![a]\!] : X^n \to X$$

of each symbol $a \in \mathsf{S}_1$ of arity n as a function which is strictly decreasing in each argument. This induces an interpretation $[\![t]\!] : X^n \to X$ of every term t of arbitrary arity n defined by induction by

$$[\![x_i^n]\!] = \pi_i^n \qquad\qquad [\![a(t_1, \ldots, t_n)]\!] = [\![a]\!] \circ \langle [\![t_1]\!], \ldots, [\![t_n]\!] \rangle$$

where $\pi_i^n : X^n \to X$ is the projection on the i-th coordinate. We define an order on functions $f, g : X^n \to X$ by

$$f \succ g \qquad \text{iff} \qquad f(x_1, \ldots, x_n) \succ g(x_1, \ldots, x_n) \text{ for every } x_1, \ldots, x_n \in X$$

and we still write $\succeq$ for the order on terms such that $t \succeq u$ whenever $[\![t]\!] \succeq [\![u]\!]$. This order is always a reduction order.

Note that given a reduction order $\succeq$ defined as above, by proposition 68, if we have $t \succ u$ for every rule $\alpha : t \Rightarrow u$ the 2-TRS is W-terminating.

Example 70. Consider the 2-TRS Mon of example 61. We consider the set $X = \mathbb{N} \setminus 0$ and interpret the symbols as

$$[\![m(x_1, x_2)]\!] = 2x_1 + x_2 \qquad\qquad [\![e]\!] = 1$$

All the rules are decreasing since we have

$$[\![m(m(x_1, x_2), x_3)]\!] = 4x_1 + 2x_2 + x_3 > 2x_1 + 2x_2 + x_3 = [\![m(x_1, m(x_2, x_3))]\!]$$
$$[\![m(e, x_1)]\!] = 2 + x_1 > x_1 = [\![x_1]\!]$$
$$[\![m(x_1, e)]\!] = 2x_1 + 1 > x_1 = [\![x_1]\!]$$

and the rewriting system is terminating.

We now briefly recall the notion of *critical branching*, see [36] for a more detailed presentation. We say that a branching (α_1, α_2) is *smaller* than a branching (β_1, β_2) when the second can be obtained from the first by "extending the context", i.e. when there exists a context C and a morphism f of suitable types such that $\beta_i = C[\alpha_i \circ f]$ for $i = 1, 2$. In this case, the confluence of the first branching implies the confluence of the second one (see the diagram on the right above). The notion of context can be generalized to define the notion of a binary context C, with two holes, each of which occurs exactly once: we write $C[t, u]$ for the context where the holes have respectively been substituted with terms t and u. A branching is *orthogonal* when it consists of two rewriting steps at disjoint positions, i.e. when it is of the form

$$C[u_1 \circ f_1, t_2 \circ f_2] \xLeftarrow{C[\alpha_1 \circ f_1, t_2 \circ f_2]} C[t_1 \circ f_1, t_2 \circ f_2] \xRightarrow{C[t_1 \circ f_1, \alpha_2 \circ f_2]} C[t_1 \circ f_1, u_2 \circ f_2]$$

for some binary context C, rewriting rules $\alpha_i : t_i \Rightarrow u_i$ in S_2 and morphisms f_i in S_1^* of suitable types. A branching is *critical* when it is not orthogonal and minimal (wrt the above order). A TRS with a finite number of rewriting rules always have a finite number of critical branchings and those can be computed efficiently [4].

Lemma 71. *A 2-TRS S is locally W-confluent when all its critical W-branchings are W-confluent.*

Proof. Suppose that all critical W-branchings are confluent. A non-overlapping W-branching is easily shown to be W-confluent. A non-minimal W-branching is greater than a minimal one, which is W-confluent by hypothesis, and is thus itself also W-confluent. □

We write $W_3 \subseteq \mathsf{S}_3$ for the set of coherence relations $A : \pi \Rightarrow \rho$ such that both π and ρ belong to $W^\sim$. As a useful particular case, we have the following variant of the Squier theorem:

Lemma 72. *If 2-TRS S has a coherence relation in W_3 corresponding to a choice of confluence for every critical W-branching then it is locally W-confluent.*

Example 73. The 2-TRS Mon of example 61. By definition, every critical branching is confluent and Mon is thus locally confluent. From example 70 and proposition 66, we deduce that it is confluent.

As a direct consequence of proposition 40, we have

Lemma 74. *If* S *is* W*-terminating and locally* W*-confluent then it is* W*-coherent.*

From examples 70 and 73, we deduce that the 2-TRS Mon is coherent, thus showing the coherence property (C1) for monoidal categories.

Suppose given a W-convergent 2-TRS S. By lemma 74, $\overline{\mathsf{S}}$ is W-coherent, by theorem 54, the quotient functor $\overline{\mathsf{S}} \to \overline{\mathsf{S}}/W$ is a local equivalence, and by proposition 44, $\overline{\mathsf{S}}/W$ is obtained from $\overline{\mathsf{P}}$ by restricting to 1-cells in normal form. Moreover, in good situations, we can provide a description of the quotient category $\overline{\mathsf{S}}/W$ by applying proposition 45 hom-wise.

3.3. **Algebras for Lawvere 2-theories.** The notion of algebra for 2-theories was extensively studied by Yanofsky [49, 50], we refer to his work for details.

Definition 75. An *algebra* for a Lawvere 2-theory $\mathcal{T}$ is a 2-functor $C : \mathcal{T} \to \mathbf{Cat}$ which preserves products. By abuse of notation, we often write C instead of $C1$ and suppose that products are strictly preserved, so that $Cn = C^n$.

A *pseudo-natural transformation* $F : C \Rightarrow D$ between algebras C and D consists in a functor $F : C \to D$ together with a family $\phi_f : Df \circ F^n \Rightarrow F \circ Cf$ of natural transformations indexed by 1-cells $f : n \to 1$ in $\mathcal{T}$,

$$\begin{array}{ccc} C^n & \xrightarrow{Cf} & C \\ {\scriptstyle F^n}\big\downarrow & \overset{\phi_f}{\nearrow} & \big\downarrow{\scriptstyle F} \\ D^n & \xrightarrow[Df]{} & D \end{array}$$

which is compatible with products, composition and 2-cells of $\mathcal{T}$.

A *modification* $\mu : F \Rrightarrow G : C \Rightarrow D$ between two pseudo-natural transformations is a natural transformation $\mu : F \Rrightarrow G$ which is compatible with 2-cells of $\mathcal{T}$. We write $\mathrm{Alg}(\mathcal{T})$ for the 2-category of algebras of a 2-theory $\mathcal{T}$, pseudo-natural transformations and modifications.

Example 76. Consider the 2-TRS Mon of example 61. The 2-category $\mathrm{Alg}(\overline{\mathsf{Mon}})$ of algebras of the presented 2-theory is isomorphic to the category **MonCat** of monoidal categories, strong monoidal functors and monoidal natural transformations. It might be surprising that Mon has five coherence relations whereas the traditional definition of monoidal categories only features two axioms (which correspond to the coherence relations A and C). There is no contradiction here: the commutation of the two axioms can be shown to imply the one of the three other [22, 15].

We conjecture that one can generalize the classical proof that any monoidal category is monoidally equivalent to a strict one [34, Theorem XI.3.1] to show the following general (C3) coherence theorem, as well as its (C4) generalization:

Conjecture 77. *When* $\mathcal{W}$ *is* 2*-rigid, every* $\mathcal{T}$*-algebra is equivalent to a* $\mathcal{T}/\mathcal{W}$ *algebra.*

Conjecture 78. *When $\mathcal{W}$ is 2-rigid, the 2-functor $\mathrm{Alg}(\mathcal{T}/\mathcal{W}) \to \mathrm{Alg}(\mathcal{T})$ induced by pre-composition with the quotient functor $\mathcal{T} \to \mathcal{T}/W$ has a left adjoint such that the components of the unit are equivalences.*

This is left for future works, since it would require introducing some more categorical material, and our aim in this article is to focus on the rewriting techniques. Note that, apart from informal explanations, we could not find a proof of Conjectures 77 and 78 for symmetric or braided monoidal categories in the literature, e.g. in [35, 19, 34] (in [19, Theorem 2.5] the result is only shown for free braided monoidal categories).

3.4. **Symmetric monoidal categories.** A *symmetric monoidal category* is a monoidal category equipped with a natural isomorphism $\gamma_{x,y} : x \otimes y \to y \otimes x$, called *symmetry*, satisfying three classical axioms recalled in section 1.4. A symmetric monoidal category is *strict* when the structural isomorphisms α, λ and ρ are identities (but we do not require γ to be an identity). We write **SMonCat** (resp. $\mathbf{SMonCat}_{\mathrm{str}}$) for the category of symmetric monoidal categories (resp. strict ones). Using the same method as above, we can show the coherence theorems for symmetric monoidal categories [19]. This example illustrates the interest of the previous developments since we are quotienting by a $(2,1)$-category $\mathcal{W}$ which is not the whole category (contrarily to the case of monoidal categories presented above). Similar results using rewriting in polygraphs where obtained earlier [27, 2, 15]. They require heavier computations since manipulations of variables (duplication, erasure and commutation) need to be implemented as explicit rules in this context.

We write SMon for the 2-TRS obtained from Mon (see example 61) by adding a rewriting rule

$$\gamma : m(x_1, x_2) \Rightarrow m(x_2, x_1)$$

corresponding to symmetry, together with a coherence relation

$$F : \gamma(x_1, x_2) \cdot \gamma(x_2, x_1) \Rrightarrow \mathrm{id}_{m(x_1,x_2)}$$

which can be pictured as

$$\begin{array}{ccc} m(x_1,x_2) & \overset{\gamma}{\Longrightarrow} & m(x_2,x_1) \\ \Vert & \overset{F}{\Rrightarrow} & \Downarrow{\scriptstyle\gamma} \\ m(x_1,x_2) & = & m(x_1,x_2) \end{array}$$

as well as the relations

$$\begin{array}{ccccc} m(m(x_1,x_2),x_3) & \overset{\gamma}{\Rightarrow} & m(m(x_2,x_1),x_3) & \overset{\alpha}{\Rightarrow} & m(x_2,m(x_1,x_3)) \\ {\scriptstyle\alpha}\Downarrow & & \overset{G}{\Rrightarrow} & & \Downarrow{\scriptstyle\gamma} \\ m(x_1,m(x_2,x_3)) & \underset{\gamma}{\Rightarrow} & m(m(x_2,x_3),x_1) & \underset{\alpha}{\Rightarrow} & m(x_2,m(x_3,x_1)) \end{array} \qquad \begin{array}{ccc} m(e,x_1) & \overset{\gamma}{\Longrightarrow} & m(x_1,e) \\ & {\scriptstyle\lambda}\searrow \overset{I}{\Rrightarrow} \swarrow{\scriptstyle\rho} & \\ & x_1 & \end{array}$$

It is immediate to see that the algebras of SMon are precisely symmetric monoidal categories:

Proposition 79. *The category $\mathrm{Alg}(\overline{\mathsf{SMon}})$ is isomorphic to the category* **SMonCat**.

The 2-TRS SMon is not locally confluent. We now introduce a variant of it which has this property. We write SMon' for the 2-TRS obtained from SMon by adding a rewriting rule

$$\delta : m(x_1, m(x_2, x_3)) \to m(x_2, m(x_1, x_3))$$

removing the coherence relation G and adding coherence relations

$$\begin{array}{ccc} m(x_1,m(x_2,x_3)) & \xRightarrow{\delta} & m(x_2,m(x_1,x_3)) \\ \Vert & \overset{F'}{\Rrightarrow} & \Downarrow{\scriptstyle\delta} \\ m(x_1,m(x_2,x_3)) & = & m(x_1,m(x_2,x_3)) \end{array} \qquad \begin{array}{ccc} m(m(x_1,x_2),x_3) & \xRightarrow{\gamma} & m(m(x_2,x_1),x_3) \\ {\scriptstyle\alpha}\Downarrow & \overset{G'}{\Rrightarrow} & \Downarrow{\scriptstyle\alpha} \\ m(x_1,m(x_2,x_3)) & \xRightarrow[\delta]{} & m(x_2,m(x_1,x_3)) \end{array}$$

$$\begin{array}{ccccc} m(m(x_1,x_2),x_3) & & \xRightarrow{\gamma} & & m(x_3,m(x_1,x_2)) \\ {\scriptstyle\alpha}\Downarrow & & \overset{H}{\Rrightarrow} & & \Vert \\ m(x_1,m(x_2,x_3)) & \xRightarrow[\gamma]{} & m(x_1,m(x_3,x_2)) & \xRightarrow[\delta]{} & m(x_3,m(x_1,x_2)) \end{array} \qquad \begin{array}{ccc} m(x_1,e) & \xRightarrow{\gamma} & m(e,x_1) \\ {\scriptstyle\rho}\searrow & \overset{J}{\Rrightarrow} & \swarrow{\scriptstyle\lambda} \\ & x_1 & \end{array}$$

$$\begin{array}{ccccc} m(m(x_1,x_2),m(x_3,x_4)) & & \xRightarrow{\delta} & & m(x_3,m(m(x_1,x_2),x_4)) \\ {\scriptstyle\alpha}\Downarrow & & \overset{K}{\Rrightarrow} & & \Downarrow{\scriptstyle\alpha} \\ m(x_1,m(x_2,m(x_3,x_4))) & \xRightarrow[\delta]{} & m(x_1,m(x_3,m(x_2,x_4))) & \xRightarrow[\delta]{} & m(x_3,m(x_1,m(x_2,x_4))) \end{array}$$

$$\begin{array}{ccccc} m(x_1,m(m(x_2,x_3),x_4)) & & \xRightarrow{\delta} & & m(m(x_2,x_3),(x_1,x_4)) \\ {\scriptstyle\alpha}\Downarrow & & \overset{L}{\Rrightarrow} & & \Downarrow{\scriptstyle\alpha} \\ m(x_1,m(x_2,m(x_3,x_4))) & \xRightarrow[\delta]{} & m(x_2,m(x_1,m(x_3,x_4))) & \xRightarrow[\delta]{} & m(x_2,m(x_3,m(x_1,x_4))) \end{array}$$

$$\begin{array}{ccc} m(x_1,m(x_2,x_3)) & \xRightarrow{\delta} & m(x_2,m(x_1,x_3)) \\ {\scriptstyle\gamma}\Downarrow & \overset{M}{\Rrightarrow} & \Downarrow{\scriptstyle\gamma} \\ m(m(x_2,x_3),x_1) & \xRightarrow[\alpha]{} & m(x_2,m(x_3,x_1)) \end{array}$$

$$\begin{array}{ccccc} m(x_1,m(x_2,x_3)) & \xRightarrow{\delta} & m(x_2,m(x_1,x_3)) & \xRightarrow{\delta} & m(x_2,m(x_3,x_1)) \\ {\scriptstyle\gamma}\Downarrow & & \overset{N}{\Rrightarrow} & & \Downarrow{\scriptstyle\delta} \\ m(x_1,m(x_3,x_2)) & \xRightarrow[\delta]{} & m(x_3,m(x_1,x_2)) & \xRightarrow[\gamma]{} & m(x_3,m(x_2,x_1)) \end{array}$$

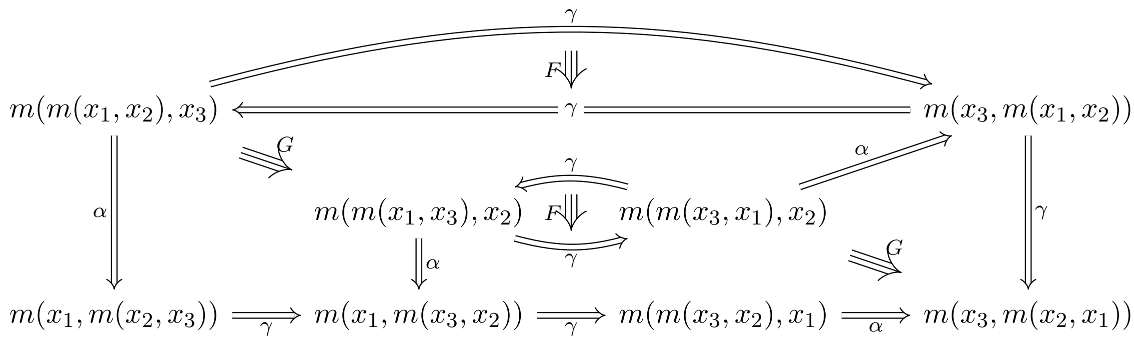
$$\begin{array}{ccccc} m(x_1,m(x_2,m(x_3,x_4))) & \xRightarrow{\delta} & m(x_2,m(x_1,m(x_3,x_4))) & \xRightarrow{\delta} & m(x_2,m(x_3,m(x_1,x_4))) \\ {\scriptstyle\delta}\Downarrow & & \overset{O}{\Rrightarrow} & & \Downarrow{\scriptstyle\delta} \\ m(x_1,m(x_3,m(x_2,x_4))) & \xRightarrow[\delta]{} & m(x_3,m(x_1,m(x_2,x_4))) & \xRightarrow[\delta]{} & m(x_3,m(x_2,m(x_1,x_4))) \end{array}$$

$$\begin{array}{ccc} m(e,m(x_1,x_2)) & \xRightarrow{\delta} & m(x_1,m(e,x_2)) \\ {\scriptstyle\lambda}\searrow & \overset{P}{\Rrightarrow} & \swarrow{\scriptstyle\lambda} \\ & m(x_1,x_2) & \end{array} \qquad \begin{array}{ccc} m(x_1,m(e,x_2)) & \xRightarrow{\delta} & m(e,m(x_1,x_2)) \\ {\scriptstyle\lambda}\searrow & \overset{Q}{\Rrightarrow} & \swarrow{\scriptstyle\lambda} \\ & m(x_1,x_2) & \end{array}$$

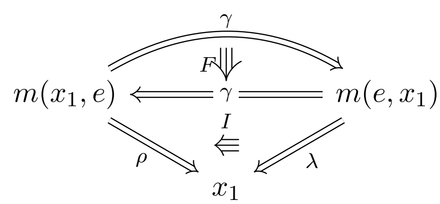
$$\begin{array}{ccc} m(x_1,m(x_2,e)) & \xRightarrow{\delta} & m(x_2,m(x_1,e)) \\ {\scriptstyle\rho}\searrow & \overset{R}{\Rrightarrow} & \swarrow{\scriptstyle\rho} \\ & m(x_1,x_2) & \end{array}$$

Proposition 80. *The 2-TRS* SMon *and* SMon′ *present isomorphic categories.*

Proof. By proposition 63, it is enough to show that both 2-TRS are Tietze equivalent. The commutation of H and I is immediate in presence of the other axioms, we can thus add them using Tietze transformations of type (T2). Namely, the commutation of H by using F and G twice:



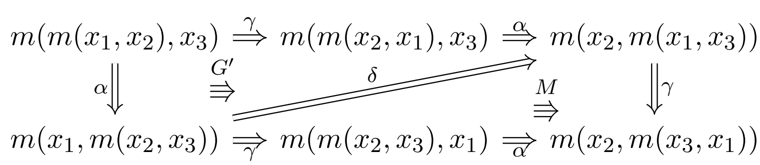
and the commutation of J can be obtained from F and I:



Next, by a Tietze transformation of type (T1), we can add the rule δ together with its definition

$$\delta(x_1,x_2,x_3) = \alpha(x_1,x_2,x_3) \circ m(\gamma(x_1,x_2),x_3) \circ \alpha(x_1,x_2,x_3)^{-1}$$

which is formally given by the relation G'. From this definition, one easily shows that the coherence relations K to R are derivable and can thus be added by Tietze transformation of type (T2). Finally, the coherence relation G is then superfluous, since it can be derived as



and can thus be removed by a Tietze transformation of type (T2). □

Lemma 81. *The 2-TRS* SMon' *is locally confluent.*

Proof. By lemma 71, it is enough to show that all the critical branchings are confluent, which holds by definition of SMon': the critical branchings involving α, λ and ρ are handled in example 61, and those involving γ or δ and another rewriting rule in the above definition of SMon'. □

Since our aim is to study the relationship between symmetric monoidal categories and their strict version, it is natural to consider the set of rewriting rules

$$W = \{\alpha, \lambda, \rho\}$$

i.e. all the rules excepting γ. Namely,

Lemma 82. *The category* $\mathrm{Alg}(\mathsf{SMon}/W)$ *is isomorphic to the category* $\mathbf{SMonCat}_{\mathrm{str}}$ *of strict monoidal categories.*

Lemma 83. *The 2-*TRS SMon *is* W*-coherent.*

Proof. Since W consists in α, λ and ρ only, this can be deduced as in the case of monoids: the 2-TRS is W-terminating by example 70 and W-locally confluent by definition (example 61), it is thus W-coherent by lemma 74. □

Provided that conjecture 77 holds, we could deduce that any symmetric monoidal category is monoidally equivalent to a strict one. Note that the above reasoning only depends on the convergence of the subsystem induced by W, i.e. on the fact that every diagram made of α, λ and ρ commutes, but it does not require anything on diagrams containing γ's. In particular, if we removed the compatibility relations G, H, I and J, the strictification theorem would still hold. The resulting notion of strict symmetric monoidal category would however be worrying since, for instance, in absence of I, the morphism

$$\gamma_{e,x_1} : m(e, x_1) \to m(x_1, e)$$

would induce, in the quotient, a non-trivial automorphism

$$\gamma_{e,x_1} : x_1 \to x_1$$

of each object x_1. We prove below (theorem 91) a variant of the coherence theorem is "stronger" in the sense that it requires these axioms to hold and implies that the identity is the only automorphism of x_1.

Every affine diagram commutes. We have seen that for the theory of monoidal categories "every diagram commutes", in the sense that $\overline{\mathsf{Mon}}$ is a 2-rigid $(2,1)$-category. For symmetric monoidal categories, we do not expect this to hold since we have two rewriting paths

$$\gamma_{x_1,x_1} : m(x_1, x_1) \Rightarrow m(x_1, x_1) \qquad\qquad \mathrm{id}_{m(x_1,x_1)} : m(x_1, x_1) \Rightarrow m(x_1, x_1)$$

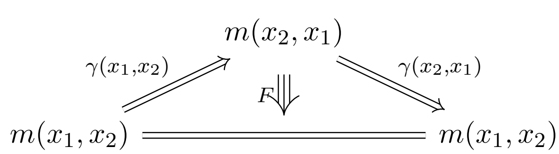
which are both from $m(x_1, x_1)$ to itself, and are not equal in general as explained in the introduction. It can however be shown that it holds for a subclass of 2-cells whose source and target are affine terms:

Definition 84. A term t is *affine* if no variable occurs twice, i.e. $|t|_i \leq 1$ for every index i.

We now explain this, thus recovering a well-known property [35, Theorem 4.1] using rewriting techniques. In order to use those, it will be convenient to work with the 2-TRS SMon' which is locally confluent instead of SMon. The 2-TRS SMon' is not terminating (even when restricted to affine terms) because of the rules γ and δ which witnesses for the commutativity of the operation m: for instance, we have the loop

$$m(x_1, x_2) \xRightarrow{\gamma(x_1,x_2)} m(x_2, x_1) \xRightarrow{\gamma(x_2,x_1)} m(x_1, x_2) \tag{3.1}$$

In order to circumvent this problem, we are going to formally "remove" the second morphism above and only keep instances of γ (resp. δ) which tend to make variables in decreasing order. Namely, by the coherence relation F, i.e.



we have $\gamma(x_2, x_1) = \gamma(x_1, x_2)^-$ so that $\gamma(x_2, x_1)$ is superfluous and we can remove it, by using Tietze transformations, without changing the presented $(2,1)$-category. Note that this operation is clearly not stable under substitution (for instance consider the substitution $[x_2, x_1]$ which exchanges the two variable names), so that this cannot actually be performed at the level of $(2,1)$-categories, but it can if we work within the hom-groupoids, which will be enough for our purposes. If we remove all the rewriting steps involving γ which tend to put variables in increasing order as explained above, we still have some loops such as

$$m(e, x_1) \xrightarrow{\gamma(e,x_1)} m(x_1, e) \xrightarrow{\gamma(x_1,e)} m(e, x_1)$$

Intuitively, this is because the above rewriting path involves terms containing a unit e, whereas our previous criterion relies on the order of variables. Fortunately, we can first remove all units by applying the rules λ and ρ, and then apply the above argument.

Fix an arity $n \in \mathbb{N}$, consider the 2-ARS $\mathsf{P} = \mathsf{SMon}'(n, 1)$ as defined in definition 64. We write P' for the 2-ARS obtained from P by

- removing from P_1 the terms where the unit e occurs excepting e itself,
- removing from P_2 the rewriting steps whose source or target terms contain e (in particular, we remove all rewriting steps involving λ or ρ),
- removing from P_3 the coherence relations where a removed step occurs in the source or the target.

Lemma 85. *The groupoid* $\overline{\mathsf{P}}'$ *it presents is equivalent to* $\overline{\mathsf{P}}$.

Proof. We write $W \subseteq \mathsf{P}_1$ for the set of rewriting steps involving λ or ρ. By lemma 81, the 2-ARS P' is locally W-confluent, and thus, by lemma 43, $\overline{\mathsf{P}}$ is equivalent to $N(\overline{\mathsf{P}})$, the full subcategory on W-normal forms. In turn, by proposition 45 (see below for details), the category $N(\overline{\mathsf{P}})$ is isomorphic to the groupoid presented $\mathsf{P} \setminus W$ and we conclude by observing that the 2-ARS P' is precisely $\mathsf{P} \setminus W$ (terms in normal form are precisely those where e does not occur, with the exception of e itself).

Let us explain why the conditions of proposition 45 are satisfied.

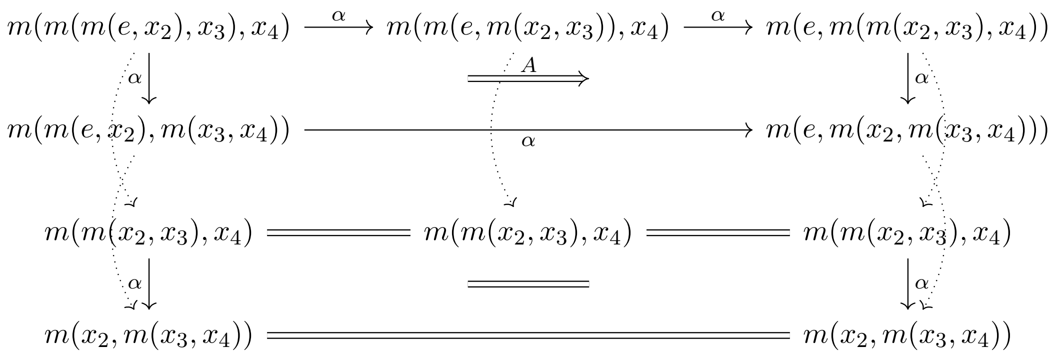
(1) We have seen above that P is W-convergent.
(2) It is immediate to check that no rewriting rule can produce a term containing e from a term which does not have this property.
(3) From proposition 46, in order to show this condition, we have to check that every diagram of the form

$$\begin{array}{ccc} t & \xrightarrow{\alpha} & u \\ {\scriptstyle\omega}\downarrow & & \\ t' & & \end{array}$$

can be closed as in (2.4) for arbitrary rewriting steps $\alpha \in \mathsf{P}_1$ and $\omega \in W$. It is enough to show this when they form a critical branching. There are five of them, which correspond to the coherence relations B, C, D, I and J, from which we conclude.
(4) From proposition 46, in order to show this condition, we have to check that every diagram of the form

$$\begin{array}{ccc} t & \overset{\alpha}{\underset{\beta}{\rightrightarrows}}\,{\scriptstyle X\Downarrow} & u \\ {\scriptstyle\omega}\downarrow & & \\ t' & & \end{array}$$

can be closed as in (2.5) for ω in W. Again, it is enough to show this in situations which are not orthogonal and minimal, in a similar sense as for critical branchings, see section 3.2, which we call a "critical branching between a rewriting rule and a coherence relation". For instance, one such critical branching between λ and A can be closed as follows:



where the vertical dotted arrows are rewriting steps involving λ (this is the only critical branching between λ and A and there are three critical branchings between ρ and A). □

Lemma 86. *The 2-ARS P' is locally confluent.*

Proof. We can deduce local confluence of P' from the one of P: given a local branching with t as source, it is confluent in P by lemma 81 and thus in P'. Namely, since t lies in P', the whole diagram does by property (2) shown in the proof of lemma 85 above. □

Given a term t, we write $\|t\|$ the list of variables occurring in it, from left to right, e.g. $\|m(m(x_2,e),x_1)\| = x_2x_1$. We order variables by $x_i \succeq x_j$ whenever $i \le j$ and extend it to lists of variables by lexicographic ordering. Given terms t and u, we write $t \succeq u$ when $\|t\|$ is greater than u according to the preceding order.

Lemma 87. *The preorder $\succeq$ is well-founded on affine terms with fixed arity.*

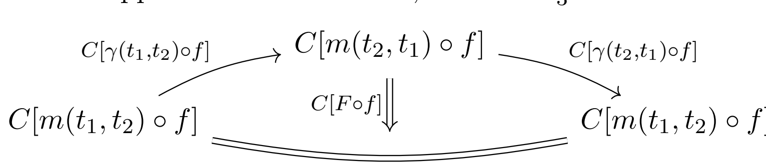
Proof. Any infinite decreasing sequence $t_1 \succ t_2 \succ \ldots$ of terms, would induce an infinite decreasing sequence $\|t\|_1 \succ \|t\|_2 \succ \ldots$ of lists of variables, but there is only a finite number of those since we consider affine terms (so that there are no repetitions of variables) of fixed arity (so that there is a finite number of variables). □

A rewriting step $\rho : t \Rightarrow u$ in P'_1 is *decreasing* when $t \succ u$. We write P'' for the 2-ARS obtained from P by

- removing from P'_1 all the rewriting steps of the form
$$C[\gamma(t_1,t_2)] : C[m(t_1,t_2)] \Rightarrow C[m(t_2,t_1)]$$
which are not decreasing,
- replacing in the source or target of a relation in P'_2 all the non-decreasing steps $C[\gamma(t_1,t_2)]$ by $C[\gamma(t_2,t_1)^-]$.

Lemma 88. *The 2-ARS P' and P'' present isomorphic groupoids.*

Proof. This is a direct application of lemma 30, because P'_3 contains the coherence rules



which allow us to conclude. □

Lemma 89. *The 2-ARS P'' is terminating on affine terms and locally confluent.*

Proof. By "terminating on affine terms", we mean here that there is no infinite sequence of rewriting steps $t_0 \to t_1 \to \ldots$ where the t_0 is affine. Note that the property of being affine, as well as the variables occurring in terms, are preserved by rewriting steps, so that all the t_i are also necessarily affine in this situation. In order to show termination, we can take the lexicographic product of the orders $\succeq$ of lemma 87 and the one of example 70. This order is well-founded as a lexicographic product of well-founded orders. The rewriting steps involving γ are strictly decreasing wrt $\succeq$ (by definition of P''). The rewriting steps involving α are left invariant by $\succeq$ but are strictly decreasing wrt the second order. We deduce that P'' is terminating.

We have seen in lemma 86 that P' is locally confluent. We thus have that P'' is also locally confluent because the confluence diagrams involving γ (namely G, H, I and J) only require decreasing instances of rewriting rules involving γ. □

As a direct consequence, we have:

Lemma 90. *Given two rewriting paths $p, q : t \xrightarrow{*} u$ in P'' such that t is affine, we have that u is also affine an $t \overset{*}{\Leftrightarrow} u$.*

Proof. By lemma 89, the restriction of P'' to affine terms is terminating and locally confluent, thus confluent by proposition 36 and thus coherent by proposition 40. □

From the properties shown in section 2.3, we deduce that P'' is coherent and we can thus conclude to the following coherence theorem:

Theorem 91. *In a symmetric monoidal category, every diagram whose 0-source is a tensor product of distinct objects, and whose morphisms are composites and tensor products of structural morphisms, commutes.*

Proof. Fix a symmetric monoidal category $\mathcal{C}$. By proposition 79, it can be seen as a product preserving functor $\overline{\mathsf{SMon}} \to \mathbf{Cat}$. A coherence diagram in $\mathcal{C}$ thus corresponds to a pair of rewriting paths $p, q : t \overset{*}{\Rightarrow} u$ in SMon_2^* for some terms t and u of a given arity n. Thus of paths $p, q : t \xrightarrow{*} u$ in S_1^* with $\mathsf{S} = \mathsf{SMon}(n, 1)$. Writing $\mathsf{P} = \mathsf{SMon}'(n, 1)$, we have

$$
\begin{aligned}
\overline{\mathsf{S}}(p, q) &\cong \overline{\mathsf{P}}(p, q) && \text{by proposition 80} \\
&\cong \overline{\mathsf{P}}'(p, q) && \text{by lemma 85,} \\
&\cong \overline{\mathsf{P}}''(p, q) && \text{by lemma 88,} \\
&\cong 1 && \text{by lemma 90,}
\end{aligned}
$$

from which we conclude. □

4. Future works

We believe that the developed framework applies to a wide variety of algebraic structures, which will be explored in subsequent work. In fact, the full generality of the framework was not needed for (symmetric) monoidal categories, since the rules of the corresponding theory never need to duplicate or erase variables (and, in fact, those can be handled by traditional polygraphs [27, 15]). This is however, needed for the case of rig categories [29], which feature two monoidal structures $\oplus$ and $\otimes$, and natural isomorphisms such as $\delta_{x,y,z} : x \otimes (y \oplus z) \to (x \otimes y) \oplus (x \otimes z)$ (note that x occurs twice in the target), generalizing the laws for rings. Those were a motivating example for this work, and we will develop elsewhere a proof of coherence of those structures based on our rewriting framework, as well as related approaches on the subject [8, Appendix G].

Also, the importance of the notion of polygraph can be explained by the fact that they are the cofibrant objects in a model structure on ω-categories [28, 3]. It would be interesting to develop a similar point of view for higher term rewriting systems: a first step in this direction is the model structure developed in [50].

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