Notes on toposes

Samuel Mimram

August 27, 2018

Some notes for the reading group on topos theory. These notes are (mainly) based on [5, 3]. They are unpolished, use at you own risk.

Contents

1 Logic **2**

2 Models **5**

3 Grothendieck topos **14**

4 The syntactic category **21**

5 The classifying topos **24**

6 Higher toposes **39**

1 Logic

Signature. A *signature* Σ consists in

- a set of *sorts*,
- a set of *function symbols*

$$f \quad : \quad (A_1, \ldots, A_n) \quad \to \quad A$$

  together with a typing in sorts,
- a set of *relation symbols*

$$R \quad : \quad (A_1, \ldots, A_n)$$

  together with a typing.

Well-formedness. A *context* Γ in a given signature is a list of pairs

$$\Gamma \quad = \quad x_1 : A_1, \ldots, x_n : A_n$$

of variables together with a sort.

A term t of sort A is valid in context Γ, what we write

$$\vdash_\Gamma t : A$$

when this can be derived using the rules

- given $(x : A) \in \Gamma$,

$$\vdash_\Gamma x : A$$

- given $f : (A_1, \ldots, A_n) \to A$ in Σ,

$$\frac{\vdash_\Gamma t_1 : A_1 \qquad \ldots \qquad \vdash_\Gamma t_n : A_n}{\vdash_\Gamma f(t_1, \ldots, t_n) : A}$$

Similarly, a formula ϕ is valid in context Γ, written $\vdash_\Gamma \phi$ when it can be derived using

- given $R : (A_1, \ldots, A_n)$ in Σ,

$$\frac{\vdash_\Gamma t_1 : A_1 \qquad \ldots \qquad \vdash_\Gamma t_n : A_n}{\vdash_\Gamma R(t_1, \ldots, t_n)}$$

- equality:

$$\frac{\vdash_\Gamma s : A \qquad \vdash_\Gamma t : A}{\vdash_\Gamma s = t}$$

- conjunction:

$$\frac{\vdash_\Gamma \phi \qquad \vdash_\Gamma \psi}{\vdash_\Gamma \phi \wedge \psi} \qquad\qquad \frac{}{\vdash_\Gamma \top}$$

– disjunction:

$$\frac{\vdash_\Gamma \phi \qquad \vdash_\Gamma \psi}{\vdash_\Gamma \phi \vee \psi} \qquad\qquad \frac{}{\vdash_\Gamma \bot}$$

– implication:

$$\frac{\vdash_\Gamma \phi \qquad \vdash_\Gamma \psi}{\vdash_\Gamma \phi \Rightarrow \psi}$$

– negation:

$$\frac{\vdash_\Gamma \phi}{\vdash_\Gamma \neg\phi}$$

– first-order quantifications:

$$\frac{\vdash_{\Gamma,x:A} \phi}{\vdash_\Gamma (\exists x)\phi} \qquad\qquad \frac{\vdash_{\Gamma,x:A} \phi}{\vdash_\Gamma (\exists x)\phi}$$

– infinitary disjunction and conjunction:

$$\frac{\vdash_\Gamma \phi_i \qquad i \in I}{\vdash_\Gamma \bigvee_i \phi_i} \qquad\qquad \frac{\vdash_\Gamma \phi_i \qquad i \in I}{\vdash_\Gamma \bigwedge_i \phi_i}$$

(note that even though there can be infinitely many formulas, those necessarily share finitely many variables).

Sequent calculus. Sequents are of the form

$$\phi \vdash_\Gamma \psi$$

meaning that ϕ entails ψ. We always implicitly suppose that the sequents we manipulate are well-formed, meaning that both

$$\vdash_\Gamma \phi \qquad \text{and} \qquad \vdash_\Gamma \psi$$

hold. The rules for deriving sequents are

no rule for equality?

– conjunction:

$$\frac{}{\phi \vdash_\Gamma \top} \qquad \frac{}{\phi \wedge \psi \vdash_\Gamma \phi} \qquad \frac{}{\phi \wedge \psi \vdash_\Gamma \psi} \qquad \frac{\phi \vdash_\Gamma \psi \qquad \phi \vdash_\Gamma \chi}{\phi \vdash_\Gamma \psi \wedge \chi}$$

(and similarly for infinitary version)

– disjunction:

$$\frac{}{\bot \vdash_\Gamma \phi} \qquad \frac{}{\phi \vdash_\Gamma \phi \vee \psi} \qquad \frac{}{\psi \vdash_\Gamma \phi \vee \psi} \qquad \frac{\phi \vdash_\Gamma \chi \qquad \psi \vdash_\Gamma \chi}{\phi \vee \psi \vdash_\Gamma \chi}$$

(and similarly for infinitary version)

– implication:

$$\frac{\phi \wedge \psi \vdash_\Gamma \chi}{\phi \vdash_\Gamma \psi \Rightarrow \chi} \qquad\qquad \frac{\phi \vdash_\Gamma \psi \Rightarrow \chi}{\phi \wedge \psi \vdash_\Gamma \chi}$$

– existential quantification: with $x \notin \mathrm{FV}(\psi)$,

$$\frac{\phi \vdash_{\Gamma,x:A} \psi}{(\exists x)\phi \vdash_\Gamma \psi} \qquad\qquad \frac{(\exists x)\phi \vdash_\Gamma \psi}{\phi \vdash_{\Gamma,x:A} \psi}$$

– universal quantification: with $x \notin \mathrm{FV}(\phi)$,

$$\frac{\phi \vdash_{\Gamma,x:A} \psi}{\phi \vdash_\Gamma (\forall x)\psi} \qquad\qquad \frac{\phi \vdash_\Gamma (\forall x)\psi}{\phi \vdash_{\Gamma,x:A} \psi}$$

– axioms:

  – distributivity:

$$\overline{\phi \wedge (\psi \vee \chi) \vdash_\Gamma (\phi \wedge \psi) \vee (\phi \wedge \chi)}$$

  there is also an infinitary variant:

$$\overline{\phi \wedge \bigvee_{i \in I} \psi_i \vdash_\Gamma \bigvee_{i \in I} (\phi \wedge \psi_i)}$$

  – Frobenius: with $x \notin \mathrm{FV}(\phi)$,

$$\overline{\phi \wedge (\exists x)\psi \vdash_\Gamma (\exists x)(\phi \wedge \psi)}$$

  – excluded middle:

$$\overline{\top \vdash_\Gamma \phi \vee \neg\phi}$$

We identify the following fragments of logic:

Logic	$=$	$\wedge/\top$	$\vee/\bot$	$\exists$	$\forall$	$\bigvee$	$\bigwedge$	$\neg/\Rightarrow$
Algebraic	✓							
Horn	✓	✓						
Regular	✓	✓		✓				
Coherent	✓	✓	✓	✓				
First-order	✓	✓	✓	✓	✓			✓
Geometric	✓	✓	✓	✓		✓		
Infinitary f-o	✓	✓	✓	✓	✓	✓	✓	✓

where the axioms are supposed to hold when they make sense (e.g. coherent logic has the distributivity axiom, etc.). Intuitionistic vs classical (first-order logic) refers to whether we don't or do accept the excluded middle. For algebraic theories, we restrict to axioms of the form $\top \vdash_\Gamma \psi$.

Theory. A *theory* is a collection of sequents, called *axioms.*

Barr's theorem. The excluded middle is admissible in geometric logic [2]: if a sequent is provable using excluded middle, it can also be proved without (if we assume that the ambient set theory satisfies the axiom of choice). The proof of this statement is however "highly non-constructive".

2 Models

Suppose that $\mathcal{C}$ is a cartesian category.

Set-theoretic models. In a set-theoretic model, we interpret

- sorts as sets: the elements of this sort, e.g. Nat is interpreted as $\mathbb{N}$,
- function symbols as functions, e.g. $m : (\mathsf{Nat}, \mathsf{Nat}) \to \mathsf{Nat}$ as the multiplication $\times : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$,
- relations as subsets of the type, e.g. $\mathsf{even} : \mathsf{Nat}$ as $2\mathbb{N} \subseteq \mathbb{N}$,
- logical operations as "expected" set-theoretic operations, e.g. $\vee$ is union, $\wedge$ is intersection, etc.
- implication $\phi \vdash \psi$ as inclusion of subsets.

We will generalize this to arbitrary categories. In particular, we will replace subsets by subobjects.

Structure. Given a signature Σ, a Σ-*structure* in $\mathcal{C}$ consists of

- an object $[\![A]\!]$ of $\mathcal{C}$ for every sort A,
- a function
$$[\![f]\!] \quad : \quad [\![A_1]\!] \times \ldots \times [\![A_n]\!] \quad \to \quad [\![A]\!]$$
for every function symbol $f : (A_1, \ldots, A_n) \to A$,
- a subobject
$$[\![R]\!] \quad \hookrightarrow \quad [\![A_1]\!] \times \ldots \times [\![A_n]\!]$$
for every relation symbol $R : (A_1, \ldots, A_n)$.

A morphism of structures consists of morphisms

$$h_A : [\![A]\!] \to [\![A]\!]'$$

indexed by sorts A, such that for every function symbol f we have

$$\begin{tikzcd}
[\![A_1]\!] \times \ldots \times [\![A_n]\!] \arrow[r, "[\![f]\!]"] \arrow[d, "h_{A_1} \times \ldots \times h_{A_n}"'] & [\![A]\!] \arrow[d, "h_A"] \\
[\![A_1]\!]' \times \ldots \times [\![A_n]\!]' \arrow[r, "[\![f]\!]'"'] & [\![A]\!]'
\end{tikzcd}$$

and for relation symbol R there exists a commuting diagram

$$\begin{tikzcd}
\llbracket R \rrbracket \arrow[r, hook] \arrow[d] & \llbracket A_1 \rrbracket \times \ldots \times \llbracket A_n \rrbracket \arrow[d, "h_{A_1} \times \ldots \times h_{A_n}"] \\
\llbracket R \rrbracket' \arrow[r, hook] & \llbracket A_1 \rrbracket' \times \ldots \times \llbracket A_n \rrbracket'
\end{tikzcd}$$

(note that the morphism $\llbracket R \rrbracket \to \llbracket R \rrbracket'$ can be an arbitrary morphism).

Terms. We can extend it as an interpretation of

– contexts: as objects

$$\llbracket x_1 : A_1, \ldots, x_n : A_n \rrbracket \quad = \quad \llbracket A_1 \rrbracket \times \ldots \times \llbracket A_n \rrbracket$$

– well-formed terms: as morphisms

$$\llbracket \vdash_\Gamma x : A \rrbracket \quad = \quad \llbracket \Gamma \rrbracket \xrightarrow{\pi} \llbracket A \rrbracket$$

the canonical projection,

$$\llbracket \vdash_\Gamma f(t_1, \ldots, t_n) : A \rrbracket \quad = \quad \llbracket \Gamma \rrbracket \xrightarrow{\langle \llbracket \vdash_\Gamma t_1 \rrbracket, \ldots, \llbracket \vdash_\Gamma t_n \rrbracket \rangle} \llbracket A_1 \rrbracket \times \ldots \times \llbracket A_n \rrbracket \xrightarrow{\llbracket f \rrbracket} \llbracket A \rrbracket$$

for a function symbol $f : (A_1, \ldots, A_n) \to A$,

Subobjects. A morphism $m : a \hookrightarrow c$ *factors through* a morphism $n : b \hookrightarrow c$ if there exists h such that $m = n \circ h$:

$$\begin{tikzcd}
a \arrow[rr, dotted, "h"] \arrow[dr, "m"'] & & b \arrow[dl, "n"] \\
& c &
\end{tikzcd}$$

Note that when n is mono, there exists at most one such h, so that the resulting category is a preorder: the monomorphisms into c can be preordered by the "factors through" relation (i.e., $m \leq n$ in the above situation) and, we obtain a poset by quotienting by the relation identifying m and n when $m \leq n$ and $n \leq m$. This poset is written $\mathrm{Sub}(c)$ and its elements are called subobjects of c.

Note that the pullback of a mono along any morphism is a mono. A morphism

$$f \quad : \quad a \quad \to \quad b$$

thus induces a *pullback functor* (pullback increasing function, really)

$$f^* \quad : \quad \mathrm{Sub}(b) \quad \to \quad \mathrm{Sub}(a)$$

Relations. In order to interpret relations, we need to suppose that our category has finite limits.

The interpretation of

$$\vdash_\Gamma R(t_1, \ldots, t_n)$$

is the pullback

$$\begin{tikzcd}
\llbracket \vdash_\Gamma R(t_1, \ldots, t_n) \rrbracket \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket R \rrbracket \arrow[d, hook] \\
\llbracket \Gamma \rrbracket \arrow[r, "{\langle \llbracket \vdash_\Gamma t_1 \rrbracket, \ldots, \llbracket \vdash_\Gamma t_n \rrbracket \rangle}"'] & \llbracket A_1 \rrbracket \times \ldots \times \llbracket A_n \rrbracket
\end{tikzcd}$$

for a relation symbol $R : (A_1, \ldots, A_n)$.

The interpretation of

$$\vdash_\Gamma s = t$$

with s and t of type A is the equalizer

$$\begin{tikzcd}
\llbracket \vdash_\Gamma s = t \rrbracket \arrow[r, hook, dotted] & \llbracket \Gamma \rrbracket \arrow[r, shift left, "\llbracket \vdash_\Gamma s \rrbracket"] \arrow[r, shift right, "\llbracket \vdash_\Gamma t \rrbracket"'] & \llbracket A \rrbracket
\end{tikzcd}$$

(note that this is necessarily a mono).

We say that a sequent

$$\phi \vdash_\Gamma \psi$$

is *satisfied* when

$$\llbracket \vdash_\Gamma \phi \rrbracket \leq \llbracket \vdash_\Gamma \psi \rrbracket$$

in $\mathrm{Sub}(\llbracket \Gamma \rrbracket)$. A *model* of a theory is a structure in which all the axioms are satisfied.

Disjunction. Since our category $\mathcal{C}$ has finite limits, the poset $\mathrm{Sub}(c)$ is an inf-semi-lattice where $m \vee n$ is given by pullback:

$$\begin{tikzcd}
& m \vee n \arrow[dl, hook, dotted] \arrow[dr, hook, dotted] & \\
a \arrow[dr, hook] & & b \arrow[dl, hook] \\
& c &
\end{tikzcd}$$

and maximal element being the identity $\mathrm{id}_c : c \hookrightarrow c$. We thus interpret a conjunction

$$\vdash_\Gamma \phi \wedge \psi$$

as the pullback

$$\begin{tikzcd}
\llbracket \vdash_\Gamma \phi \wedge \psi \rrbracket \arrow[r, hook, dotted] \arrow[d, hook, dotted] & \llbracket \vdash_\Gamma \phi \rrbracket \arrow[d, hook] \\
\llbracket \vdash_\Gamma \psi \rrbracket \arrow[r, hook] & \llbracket \Gamma \rrbracket
\end{tikzcd}$$

and top element

$$\vdash_\Gamma \top$$

as the top element of $\mathrm{Sub}(\llbracket\Gamma\rrbracket)$.

Note that the rules for conjunction

$$\frac{}{\phi \vdash_\Gamma \top} \qquad\qquad \frac{}{\phi \wedge \psi \vdash_\Gamma \phi} \qquad\qquad \frac{}{\phi \wedge \psi \vdash_\Gamma \psi} \qquad\qquad \frac{\phi \vdash_\Gamma \psi \qquad \phi \vdash_\Gamma \chi}{\phi \vdash_\Gamma \psi \wedge \chi}$$

are validated in our model (meaning that if the premises are satisfied then the conclusion is also satisfied). In fact, they "say precisely" that each $\mathrm{Sub}(c)$ is a bounded inf-semilattice.

Quantifications: Set-theoretic interpretation. A function $f : X \to Y$ induces a function $f^* : \wp(Y) \to \wp(X)$ sending a subset $P \subseteq X$ to the set $f^{-1}(P) \subseteq Y$ of preimages under f. In particular, consider the projection

$$p \quad : \quad X \times Y \quad \to \quad X$$

the induced function

$$p^* \quad : \quad \wp(X) \quad \to \quad \wp(X \times Y)$$

admits a left and a right adjoint

$$\exists_p, \forall_p \quad : \quad \wp(X \times Y) \quad \to \quad \wp(X)$$

defined by

$$\begin{array}{rcl} \exists_p(P) & = & \{x \in X \mid \exists y \in Y, (x, y) \in P\} \\ \forall_p(P) & = & \{x \in X \mid \forall y \in Y, (x, y) \in P\} \end{array}$$

which provide the expected interpretation of the first-order quantifiers.

Existential quantification: regular categories. In order to interpret existential quantification, we have to have more structure. The *image* of a morphism $f : a \to b$ is a factorization through a subobject $\mathrm{Im}(f)$ of b, which is minimal with this property:

$$\begin{tikzcd} & {} \arrow[ddr, hook, dashed, bend left] & \\ & \mathrm{Im}(f) \arrow[u, dotted] \arrow[dr, hook] & \\ a \arrow[ur, dotted] \arrow[uur, dotted, bend left] \arrow[rr, "f"'] & & b \end{tikzcd}$$

A *regular category* is a finitely complete category which has image factorizations which are stable under pullback. In such a category, we write $c(f) : a \to \mathrm{Im}(f)$ for the left part of the factorization (the *cover map*), which is necessarily a regular epi (it is the coequalizer of the kernel pair of f). In fact, (regular epi, mono) forms a factorization system.

In such a category, the interpretation of

$$\vdash_\Gamma (\exists x)\phi$$

is the image of the composite

$$\begin{tikzcd}
\llbracket \vdash_{\Gamma,x:A} \phi \rrbracket \arrow[r, dotted] \arrow[d, hook] & \llbracket (\exists x)\phi \rrbracket \arrow[d, hook, dotted] \\
\llbracket \Gamma, x : A \rrbracket \arrow[r, "\pi"] & \llbracket \Gamma \rrbracket
\end{tikzcd}$$

It can be also noted $\exists_\pi \llbracket \vdash_{\Gamma,x:A} \phi \rrbracket$: given a morphism

$$f \quad : \quad a \quad \to \quad b$$

the pullback functor

$$f^* \quad : \quad \mathrm{Sub}(b) \quad \to \quad \mathrm{Sub}(a)$$

admits a left adjoint

$$\exists_f \quad : \quad \mathrm{Sub}(a) \to \mathrm{Sub}(b)$$

which to a mono $m : c \hookrightarrow a$ assigns $\mathrm{Im}(f \circ m)$:

$$\begin{tikzcd}
c \arrow[r, "c(f \circ m)"] \arrow[d, hook, "m"'] & \mathrm{Im}(f \circ m) \arrow[d, hook] \\
a \arrow[r, "f"'] & b
\end{tikzcd}$$

More on regular categories. Equivalently, a category is *regular* (= has "well-behaved relations") when

1. it is finitely complete,
2. the kernel pair
$$\begin{tikzcd}
a \times_b a \arrow[r, "p_2"] \arrow[d, "p_1"'] & a \arrow[d, "f"] \\
a \arrow[r, "f"'] & b
\end{tikzcd}$$
of any morphism $f : a \to b$ has a coequalizer
$$\begin{tikzcd}
a \times_b a \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & b \arrow[r, dotted] & b/f
\end{tikzcd}$$
3. regular epimorphisms are stable under pullback.

A *regular epimorphism* $f : a \to b$ is such that "b is an union of the parts of a": formally, f can be obtained as the coequalizer

$$\begin{tikzcd}
a' \arrow[r, shift left, dotted] \arrow[r, shift right, dotted] & a \arrow[r, "f"] & b
\end{tikzcd}$$

of some diagram. Note:

split epi implies regular epi implies epi but the converse implications are not true in general. An *effective epimorphism* is an epimorphism which has a kernel pair and is its coequalizer.

The categories **Cat**, **Pos** and **Top** are not regular (and thus not toposes): consider the posets

- A being $\{a, b\} \times (0 \to 1)$
- B being $(0 \to 1 \to 2)$
- C being $(0 \to 2)$

and the morphisms

- $p : A \to B$ which identifies $(a, 1)$ with $(b, 0)$:

$$\begin{tikzcd}
 & & 2 \\
a1 \arrow[rr, dotted, bend left] & b1 \arrow[ur, dotted] & 1 \arrow[u] \\
a0 \arrow[u] \arrow[rr, dotted, bend right] & b0 \arrow[u] \arrow[ur, dotted] & 0 \arrow[u]
\end{tikzcd}$$

- the inclusion $i : C \to B$

The morphism p is a regular epi, but it is not stable under pullback:

$$\begin{tikzcd}
\{0, 2\} \arrow[r, dotted] \arrow[d, dotted, "i^*(p)"'] & A \arrow[d, "p"] \\
C \arrow[r, "i"'] & B
\end{tikzcd}$$

the morphism $i^*(p) : \{0, 2\} \to C$ is the inclusion, which is an epi but not a regular one.

Disjunction: coherent categories. A *coherent category* $\mathcal{C}$ is a regular category in which $\mathrm{Sub}(c)$ has unions for every object c and those are preserved by $f^* : \mathrm{Sub}(b) \to \mathrm{Sub}(a)$. In those, we can interpret $\vee$ and $\perp$, and thus coherent logic.

Infinitary disjunction: geometric categories. A *geometric category* is a regular category which is *well-powered* (every $\mathrm{Sub}(c)$ is small) and $\mathrm{Sub}(c)$ have arbitrary unions which are stable under pullback. In those, we can interpret geometric logic.

Given a morphism h of structures and a geometric formula ϕ, by induction, there exists a commutative diagram of the form

$$\begin{tikzcd}
\llbracket \vdash_\Gamma \phi \rrbracket \arrow[r, hook] \arrow[d] & \llbracket \Gamma \rrbracket \arrow[d, "h_{A_1} \times \ldots \times h_{A_n}"] \\
\llbracket \vdash_\Gamma \phi \rrbracket' \arrow[r, hook] & \llbracket \Gamma \rrbracket'
\end{tikzcd}$$

In other words, morphisms of structures preserve the validity of geometric formulas. This is not true in general for non-geometric formulas. For instance, consider the inclusion $\{0\} \to \{0,1\}$. The first satisfies $(\forall x)(\forall y)(x = y)$ but not the second. As a more realistic example, consider the theory of preorders: a morphism of models is an increasing function, and such a function does not preserve for instance the fact of having a greatest element. A morphism which preserves the validity of every formula is called *elementary*.

Universal quantification: Heyting categories. A *Heyting category* is a coherent category $\mathcal{C}$ in which $f^* : \mathrm{Sub}(b) \to \mathrm{Sub}(a)$ has a right adjoint $\forall_f : \mathrm{Sub}(a) \to \mathrm{Sub}(b)$. In such a category, the posets $\mathrm{Sub}(c)$ are Heyting algebras (see below) and $\Rightarrow$ is stable under pullback. We recall that a Heyting algebra is an inf-semilattice which is cartesian closed, i.e., there exists $b \Rightarrow c$ such that

$$\frac{a \wedge b \leq c}{a \leq b \Rightarrow c}$$

In other words, $b \Rightarrow c$ is the lub of a such that $a \wedge b \leq c$:

$$\begin{tikzcd}
 & & & b \Rightarrow c \\
b & & c \arrow[ur] & \\
 & b \wedge c \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

It is above c (because $c \wedge b \leq c$), but goes not too far so that $(b \Rightarrow c) \wedge b \leq c$ (and in fact $\leq b \wedge c$). Given $a \hookrightarrow c$ and $b \hookrightarrow c$ in $\mathrm{Sub}(c)$, we can construct $(a \Rightarrow b) \hookrightarrow c$ as $\forall_a(a \wedge b)$:

$$\begin{tikzcd}
a \wedge b \arrow[r, hook, dotted] \arrow[d, hook, dotted] & b \arrow[d, hook] & a \Rightarrow b \arrow[dl, hook, dotted] \\
a \arrow[r, hook] & c &
\end{tikzcd}$$

We can interpret full first-order logic: omplication is interpreted as $\Rightarrow$ and negation as pseudo complement, i.e., $\neg\phi$ is the same as $\phi \Rightarrow \bot$.

As a general matter of fact, if $\mathcal{C}$ is category with pullbacks, a morphism

$$f \quad : \quad a \to b$$

induces a *pullback* or *change-of-base* functor

$$f^* \quad : \quad \mathcal{C}/b \quad \to \quad \mathcal{C}/a$$

which always admits a left adjoint

$$\exists_f \quad : \quad \mathcal{C}/a \quad \to \quad \mathcal{C}/b$$

defined by post-composition by f (for subobjects it is more subtle since we have to formally make the result a mono again). If $\mathcal{C}/b$ is cartesian closed then it also has a right adjoint

$$\forall_f \quad : \quad \mathcal{C}/a \quad \to \quad \mathcal{C}/b$$

Excluded middle: Boolean categories. A coherent category *Boolean* when every subobject $m : a \hookrightarrow c$ is *complemented*, i.e., there exists a unique $n : b \hookrightarrow c$ such that $m \vee n = 1_c$ and $m \wedge n = 0_c$. A Boolean category is a Heyting category, a geometric category is a Heyting category. Models in Boolean categories satisfy the excluded middle.

In presence of a subobject classifier. We can more generally interpret logic in an elementary topos (every Grothendieck topos is an elementary one), i.e., a category $\mathcal{C}$ which is

- cartesian closed,
- cocomplete,
- has a subobject classifier.

A *subobject classifier* is a mono $\top : 1 \to \Omega$ such that for every mono $m : b \to c$ there exists a unique morphism $\chi_m : c \to \Omega$, the *character* of m, forming a pullback diagram

$$\begin{tikzcd} b \arrow[r, dotted] \arrow[d, hook, dotted, "m"'] & 1 \arrow[d, hook, "\top"] \\ c \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

When $\mathcal{C}$ has finite limits and is locally small, this is the same as saying that the subobject functor

$$\mathrm{Sub} \quad : \quad \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

is representable, i.e.,

$$\mathrm{Sub}(c) \quad \simeq \quad \mathcal{C}(c, \Omega)$$

which shows that a locally small category with a subobject classifier is necessarily well-powered: we have a small set $\mathrm{Sub}(c)$ for every object c. Note that in a topos, the image of $f : a \to b$ can be obtained by computing the pushout

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "f"'] & b \arrow[d, "p"] \\ b \arrow[r, "q"'] & r \end{tikzcd}$$

and compute the coequalizer $\mathrm{Im}(f)$ of p and q, which yields a factorization of f by universal property

$$\begin{tikzcd} \mathrm{Im}(f) \arrow[r, hook] & b \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & r \\ a \arrow[u, dotted, "c(f)"] \arrow[ur, dashed, "f"'] & & \end{tikzcd}$$

In this situation,connectives can be implemented on Ω. For instance, the character of

$$\langle \top, \top \rangle \quad : \quad 1 \quad \hookrightarrow \quad \Omega \times \Omega$$

is a morphism

$$\wedge \quad : \quad \Omega \times \Omega \quad \to \quad \Omega$$

Two subobjects of c, $m : a \hookrightarrow c$ and $n : b \hookrightarrow c$ can be seen as characters $\chi_m, \chi_n : c \to \Omega$, and we declare that $m \wedge n$ is the mono corresponding to the character

$$c \xrightarrow{\langle \chi_m, \chi_n \rangle} \Omega \times \Omega \xrightarrow{\wedge} \Omega$$

Similarly,

- $\perp : 1 \to \Omega$ is the character of the initial/terminal morphism $0 \to 1$
- $\vee : \Omega \times \Omega \to \Omega$ is the character of the image of

$$[(\top, \mathrm{id}_\Omega), (\mathrm{id}_\Omega, \top)] \quad : \quad \Omega + \Omega \quad \to \quad \Omega \times \Omega$$

- $\Rightarrow : \Omega \times \Omega \to \Omega$ is the character of the equalizer

$$\begin{tikzcd} \leq \arrow[r, hook, dotted, "e"] & \Omega \times \Omega \arrow[r, shift left, "\wedge"] \arrow[r, shift right, "\pi_1"'] & \Omega \end{tikzcd}$$

- $\forall_a : \Omega^a \to \Omega$ is the unique morphism making

$$\begin{tikzcd} 1 \arrow[r, "\ulcorner \top_a \urcorner"] \arrow[d] & \Omega^a \arrow[d, "\forall_a"] \\ 1 \arrow[r, "\top"'] & \Omega \end{tikzcd}$$

  a pullback, where $\ulcorner \top_a \urcorner$ is the exponential adjoint of

$$1 \times a \xrightarrow{\pi_a} a \longrightarrow 1 \xrightarrow{\top} \Omega$$

- etc.

Syntactically, this can be implemented by adding a distinguished kind Ω with appropriate rules. Note that relations are simply terms of kind Ω, not a distinguished syntactic class, and logical operations can be typed on Ω as above, e.g.

$$\wedge \quad : \quad \Omega \times \Omega \quad \to \quad \Omega$$

Powerset objects. Instead of requiring that a topos is cartesian closed, it is enough to require that the object Ω is closed. We generally write Pb instead of Ω^b in this case: we require to every object b is assigned an object Pb such that

$$\frac{a \times b \to \Omega}{a \to Pb}$$

the "counit" being generally denoted

$$\in_a \quad : \quad a \times Pa \quad \to \quad \Omega$$

Equivalently, we ask that for every object b there is an object Pb and a morphism $\in_b : b \times Pb \to \Omega$ such that for every $f : b \times a \to \Omega$ there is a unique $g : a \to Pb$ such that

$$\begin{tikzcd} a \arrow[d, dotted, "g"'] & b \times a \arrow[r, "f"] \arrow[d, dotted, "1 \times g"'] & \Omega \arrow[d] \\ Pb & b \times Pb \arrow[r, "\in_b"'] & \Omega \end{tikzcd}$$

This point of view corresponds to adding term constructors

$$\frac{\vdash_\Gamma t : A \qquad \vdash_\Gamma \tau : PA}{\vdash_\Gamma t \in \tau : \Omega} \qquad\qquad \frac{\vdash_{\Gamma, x:A} R : \Omega}{\vdash_\Gamma \{x \in A \mid R(x)\} : PA}$$

together with appropriate rules, see [4].

Soundness. If a sequent is derivable in a theory T then it is satisfied in any model of the theory.

Completeness. If a sequents is satisfied in all models of a theory in a category with enough structure then it is derivable.

3 Grothendieck topos

Sieve. Given a small category $\mathcal{C}$ and $c \in \mathcal{C}$, a *sieve* C on c is a collection of morphisms of codomain c such that given $f \in S$ and g composable with f, $f \circ g \in S$. A sieve S is *generated* by a collection F of morphisms whenever every morphism of S factors through a morphism in F.

Topology. A *topology* on a small category $\mathcal{C}$ consists of a function J which to every object $c \in \mathcal{C}$ associates a set $J(c)$ of sieves on c such that

1. maximality: the maximal sieve on c belongs to $J(c)$,
2. pullback stability: for every morphism $f : b \to c$ and $S \in J(c)$, $f^*(S)$ belongs to $J(b)$ where
$$f^*(S) \quad = \quad \{g : a \to b \mid f \circ g \in S\}$$
3. transitivity: a sieve S on a such that there exists $T \in J(a)$ such that for every $f : b \to a$ in T $f^*(S) \in J(b)$ then $S \in J(a)$.

Site. A *site* is a pair $(\mathcal{C}, J)$ consisting of a category $\mathcal{C}$ and a Grothendieck topology J on $\mathcal{C}$.

Examples.

- the trivial topology: the only covering sieves are the maximal ones
- the chaotic topology: $J(c)$ has only one sieve consisting of endomorphisms of c (multiplicative $\mathbb{N}$ seen as a category with one object, we get the arithmetic site)
- the discrete topology: $J(c)$ consists of all sieves
- the atomic topology: $J(c)$ consists of all non-empty sieves (we have to suppose that $\mathcal{C}$ satisfies the right Ore condition: cospans can be completed in a commutative diagram)

- Given a topological space X, $\mathcal{O}(X)$ is the category of opens in X and morphisms are inclusions. The natural topology on $\mathcal{O}(X)$: covering sieves are those generated by covering families, i.e., families $(U_i \to U)_{i\in I}$ such that $\bigcup_{i\in I} U_i = U$.

- Given a scheme X, $Et(X)$ the category of étale coverings on X equipped with the étale topology.

Locales. A *lattice* is a poset such that every pair of element has an inf and a sup. A *frame* is a complete distributive lattice, i.e., a lattice with arbitrary infs and sups and $a \wedge \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \wedge b_i)$. Morphisms are those preserving finite infs and arbitrary sups and **Frm** is the resulting category. Locales: $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$.

Note that for any topological space X, $\mathcal{O}(X)$ is a locale. There is an adjunction between **Top** and **Loc**. Note that it generalized the lattice of opens of a space in that opens are not necessarily subsets of X. Topology is secretly an algebraic geometry:

topology	algebraic geometry
frame	commutative ring
local	affine scheme
without points	$\mathrm{Spec}(\mathbb{Q}[X]/(X^2+1))$

A locale L can be seen as a category whose objects are the elements of L and there is a morphism $a \to b$ whenever $a \leq b$. The *canonical topology* on L: covering sieves are generated by families $(L_i \leq L)_{i\in I}$ such that $\bigvee_{i\in I} L_i = L$.

Note, wrt to Stone-type dualities, given a preorder category $\mathcal{C}$ and a topology J, we define the set $\mathrm{Id}_J(\mathcal{C})$ of J-ideals of $\mathcal{C}$, which form a locale: subsets I of objects of $\mathcal{C}$ such that

- for every $f : a \to b$ is $b \in I$ then $a \in I$,

- for every covering sieve $S \in J(c)$ if for every $f \in S$, $\mathrm{dom}(f) \in I$ then $c \in I$

Theorem: $\mathbf{Sh}(\mathcal{C}, J) \cong \mathbf{Sh}(\mathrm{Id}_J(\mathcal{C}), J_{\mathrm{can}})$:

$$\begin{tikzcd}
 & \mathbf{Sh}(\mathcal{C}, J) \arrow[r, "\sim"] \arrow[ddl, dotted, no head] & \mathbf{Sh}(\mathrm{Id}_J(\mathcal{C}), J_{\mathrm{can}}) \arrow[ddr, dotted, no head] & \\
 & \mathbf{Sh}(\mathcal{C}', J') \arrow[r, "\sim"] \arrow[ddl, dotted, no head] & \mathbf{Sh}(\mathrm{Id}_{J'}(\mathcal{C}'), J_{\mathrm{can}}) \arrow[ddr, dotted, no head] & \\
\mathcal{C} \arrow[d] & & & \mathrm{Id}_J(\mathcal{C}) \\
\mathcal{C} & & & \mathrm{Id}_{J'}(\mathcal{C}') \arrow[u]
\end{tikzcd}$$

Topos. A *presheaf* on $\mathcal{C}$ is a functor $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. Given a site $(\mathcal{C}, J)$, a *sheaf* is a presheaf $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ such that for every $S \in J(c)$ and family $(x_f \in P(\mathrm{dom}\, f))_{f\in S}$ such that for every morphism g, $P(g)(x_f) = x_{f\circ g}$ there exists a unique $x \in P(c)$ such that $x_f = P(f)(x)$. We write $\mathbf{Sh}(\mathcal{C}, J)$ for the category of sheaves and natural transformations. A category is a *Grothendieck topos* if it is equivalent to a category of sheaves (on a *site of definition*).

Some examples:

- $\mathbf{Set} = \mathbf{Sh}(\{*\}, J_{\text{triv}})$
- $[\mathcal{C}^{\text{op}}, \mathbf{Set}] = \mathbf{Sh}(\mathcal{C}, J_{\text{triv}})$
- Connes-Consani 2014, $N^\times$ seen as a category with one object, the *arithmetic site* is $\hat{\mathbb{N}}^\times = [\mathbb{N}^{\text{op}}, \mathbf{Set}]$ (this is the same as $\mathbb{N}^\times$-equivariant sets). $(\hat{\mathbb{N}}^\times, (\mathbb{Z}, \max, +))$ its "points" with values in $\mathbb{R}^+$ are related to zeros of the Riemann ζ function.
- Given a topological space X, $\mathbf{Sh}(\mathcal{O}_X, J_{\mathcal{O}_X}) \simeq \mathbf{Sh}(X)$.
- Given a locale L, $\mathbf{Sh}(L, J_L)$ is a Grothendieck topos such that $L \simeq \text{Sub}(1)$.
- The site of frequencies $[0, +\infty] \rtimes \mathbb{N}^\times$ whose objects are the open intervals of $[0, \infty]$ (with the topology induced by $\mathbb{R}$) and morphisms are indexed by $n \in \mathbb{N}$: $n : \Omega \to \Omega$ if $n\Omega \subseteq \Omega'$. The topology J on $\mathcal{C}$ is induced by families $\Omega_i \xrightarrow{1} \Omega$ such that $\bigcup_{i\in I} \Omega_i = \Omega$. The points of $\mathbf{Sh}(\mathcal{C}, J)$ are related to zeros of ζ.
- Given a topological group G, $\text{Cont}(G)$ whose objects are continuous left actions of G and morphisms are G-equivariant functions. If we take the full subcategory $\mathcal{C}$ of transitive actions of G, $\text{Cont}(G) \cong \mathbf{Sh}(\mathcal{C}, J_{\text{atomic}})$. Theorem (Buntz-Moerdijk): every Grothendieck topos with enough points can be represented as $\text{Cont}(G)$ with G topological groupoid.

Suppose given $\mathcal{D}$ full subcategory of $\mathcal{C}$ equipped with a topology J. $\mathcal{D}$ is *J-dense* if for every object $c \in \mathcal{C}$, the sieve generated by families of arrows to c of objects in $\mathcal{D}$ is J-covering. In this case, we can define a topology $J|_{\mathcal{D}}$ on $\mathcal{D}$ such that $S \in J|_{\mathcal{D}}(d)$ iff $\overline{S} \in J(d)$ (where $\overline{S}$ is the sieve generated by S in $\mathcal{D}$). Theorem (SGA4 III 4.1): Given $(\mathcal{C}, J)$ a small category and $\mathcal{D}$ a full subcategory of $\mathcal{C}$ then $\mathbf{Sh}(\mathcal{C}, J) \equiv \mathbf{Sh}(\mathcal{D}, J|_{\mathcal{D}})$.

Subobject classifier. A subobject is an equivalence class of monomorphisms under the relation identifying two monomorphisms which mutually factor each other. A *subobject classifier* is a monomorphism $\top : 1 \to \Omega$ such that every mono $m : a \to b$ fits into a pullback diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d, hook, "m"'] & 1 \arrow[d] \\ b \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

for a unique $\chi_m : b \to \Omega$.

Given a site $(\mathcal{C}, J)$, $\mathbf{Sh}(\mathcal{C}, J)$ has a subobject classifier. Given S a sieve in $\mathcal{C}$ on an object c, we say that S is J-closed if $\forall f : d \to c, f^*(S) \in J(d)$ then $f \in S$. We define $\Omega_J : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ by

$$\Omega_J(c) \quad = \quad \{R \mid R \text{ is a closed sieve of } c\}$$

and on morphisms by $\Omega_J(f) = f^*(-)$. We also define $\top : 1_{\mathbf{Sh}(\mathcal{C},J)} \to \Omega_J$ by $\top(-)(c) = M_c$, the maximal sieve of c.

Given a Grothendieck topos $\mathcal{E}$ and $a \in \mathcal{E}$, $\mathrm{Sub}_{\mathcal{E}}(a)$ is a Heyting algebra. For every morphism $f : a \to b$, the pullback $f^* : \mathrm{Sub}(b) \to \mathrm{Sub}(a)$ has left and right adjoints $\exists_f, \forall_f : \mathrm{Sub}(a) \to \mathrm{Sub}(b)$.

Geometric morphism. Given topological spaces X and Y, a function induces functors

- *direct image*: $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$
- *inverse image*: $f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$

such that

$$f^* \quad \dashv \quad f_*$$

where f_* is obtained by composition with f^{-1}, i.e., given on $F : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ by

$$f_*(F)(U) \quad = \quad F(f^{-1}(U))$$

More generally, given $f : C \to D$, $f^* : \hat{D} \to \hat{C}$ is defined by $f^*(P) = P \circ f$, and this operation restricts to a functor between sheaves. The inverse image is obtained by sheafification of the left adjoint of f^* (which can be computed as the left Kan extension of P along f).

A *geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ between topoi consists of a pair of adjoint functors

$$f^* : \mathcal{E} \to \mathcal{F} \qquad\qquad f_* : \mathcal{F} \to \mathcal{E}$$

such that

- f^* is left adjoint to f_*, and
- f^* is left exact (preserves finite limits).

A *geometric transformation* $\alpha : f \to g$ is a natural transformation $f^* \to g^*$. We can thus define a 2-category $\mathbf{Geom}(\mathcal{F}, \mathcal{E})$.

Examples:

- A functor $f : \mathcal{C} \to \mathcal{D}$ induces a Geometric functor $\hat{\mathcal{D}} \to \hat{\mathcal{C}}$ where $\hat{f}_*$ is given by precomposition.
- There is a geometric morphism $\mathbf{Sh}(\mathcal{C}, J) \to \hat{\mathcal{C}}$ where direct image is inclusion and inverse image is sheafification.

A *point* in a topos is a geometric morphism $\mathbf{Set} \to \mathcal{E}$. For instance, a point in $\mathbf{Sh}(\mathcal{O}_X, J_X)$ for a topological space is a geometric morphism such that the inverse image is the fiber functor at the point. Similarly, a point of $\hat{\mathcal{C}}$ has inverse image is evaluation at a point. Note that there are toposes without points. The set of points of $\mathbf{Sh}(\mathcal{O}_X, J_X)$ is the set of points (in the usual sense) of X iff X sober.

If $\mathcal{E}$ classifies a theory T then the set of points of $\mathcal{E}$ is the category of set-theoretic models of T.

Diaconescu's theorem. In algebraic topology, a fundamental functor is

$$\Delta \quad : \quad \triangle \quad \to \quad \mathbf{Top}$$

which to an object $n \in \triangle$ of the simplicial category associate the space Δ^n, the n-dimensional simplex. It induces the nerve functor

$$N \quad : \quad \mathbf{Top} \quad \to \quad \hat{\triangle}$$

defined by

$$N(X)(n) \quad = \quad \mathrm{Hom}(\Delta^n, X)$$

This functor admits a left adjoint, the *geometric realization*

$$R \quad : \quad \hat{\triangle} \quad \to \quad \mathbf{Top}$$

defined on $P \in \hat{\triangle}$ by

$$R(P) \quad = \quad \left(\coprod_{n \in \mathbb{N}} \coprod_{a \in Pn} \Delta^n \right) / \sim$$

where $\sim$ is an appropriate equivalence relation which glues the simplices along their faces.

More generally, given a functor $A : \mathcal{C} \to \mathcal{E}$, where $\mathcal{C}$ is small and $\mathcal{E}$ locally small, we have an induced functor

$$N_A \quad : \quad \mathcal{E} \to \hat{\mathcal{C}}$$

defined by

$$N_A(X) \quad = \quad \mathrm{Hom}(A-, X)$$

which could be called the *nerve* along F (thus the notation here, the traditional one being R_A for right adjoint). This functor admits a left adjoint

$$- \otimes_{\mathcal{C}} A \quad : \quad \hat{\mathcal{C}} \quad \to \quad \mathcal{E}$$

the *geometric realization* along A, which can constructed in various ways:

- as the colimit of

  $$\mathrm{El}(P) \xrightarrow{p} \mathcal{C} \xrightarrow{A} \mathcal{E}$$

  where the functor on the left is the canonical projection of the category of elements of P on the base category $\mathcal{C}$,

- as the left Kan extension

  $$\begin{tikzcd} & \mathcal{E} & \\ \mathcal{C} \arrow[ur, "A"] \arrow[rr, "Y"'] & & \hat{\mathcal{C}} \arrow[ul, dotted] \end{tikzcd}$$

- as the coend

  $$P \otimes_{\mathcal{C}} A \quad = \quad \int^{c} Pc \cdot Ac$$

  where the "$\cdot$" is the copower, i.e., we take the coproduct of $|Ac|$ copies of Pc.

A functor $A : \mathcal{C} \to \mathcal{E}$ is *flat* when $- \otimes_{\mathcal{C}} A$ preserves finite limits and write $\mathbf{Flat}(\mathcal{C}, \mathcal{E})$ for the category of flat functors and natural transformations. We have an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \hat{\mathcal{C}}) \quad \cong \quad \mathbf{Flat}(\mathcal{C}, \mathcal{E})$$

which sends

- a geometric morphism $f : \mathcal{E} \to \hat{\mathcal{C}}$ to the flat functor $Y \circ f^*$:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "f"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathcal{C} \arrow[l, bend right, "f^*"'] \arrow[r, "Y"] & \hat{\mathcal{C}} \end{tikzcd}$$

- a flat functor $A : \mathcal{C} \to E$ to the geometric morphism given by N_A and $- \otimes_{\mathcal{C}} A$

Proof. Given a flat functor A, the pair $(N_A, - \otimes_{\mathcal{C}} A)$ is a geometric morphism by definition of A being flat. Conversely, given a geometric morphism $f : \mathcal{C} \to \mathcal{E}$, we have $f^* = - \otimes_{\mathcal{C}} (f^* \circ Y)$ since the two agree on representable functors and preserve colimits. This can be shown to establish an equivalence of categories. □

When C is small, $F : C \to \mathbf{Set}$ is flat precisely when its Yoneda extension (left Kan extension along Yoneda) preserves finite limits:

$$\begin{tikzcd} & \mathbf{Set} & \\ C \arrow[ur, "F"] \arrow[rr, "Y"'] & & \hat{C} \arrow[ul, dotted] \end{tikzcd}$$

Therefore F preserves finite limits which exist in C and conversely if C has finite limits and F preserves them then F is flat: when C has finite limits, flat functors coincide with left exact functors.

A category $\mathcal{C}$ is *filtered* if

- it is non-empty,
- for every pair of objects a and b there exists cofinal morphisms

$$\begin{tikzcd} & c & \\ a \arrow[ur, dotted, "f"] & & b \arrow[ul, dotted, "g"'] \end{tikzcd}$$

- for every pair of morphisms $f, g : a \to b$, there exists $h : b \to c$ such that $h \circ f = h \circ g$:

$$\begin{tikzcd} c \\ b \arrow[u, dotted, "h"] \\ a \arrow[u, shift left, "g"'] \arrow[u, shift right, "f"] \end{tikzcd}$$

or equivalently, every finite diagram has a cocone. A functor $A : \mathcal{C} \to \mathbf{Set}$ is flat if and only if

– it is a filtered colimit of representables,

– its category of elements is filtered.

More generally, when $\mathcal{E}$ is a topos, $A : \mathcal{C} \to \mathcal{E}$ is flat if and only if it is *filtering*, i.e., for every object $E \in \mathcal{E}$ there exists an epimorphic family $(e_i : E_i \to E)_{i \in E}$ and for every index $i \in I$ and $c_i \in C$ there is a generalized element $E_i \to Fc_i$ in $\mathcal{E}$. When the category $\mathcal{C}$ has finite limits, $A : \mathcal{C} \to \mathcal{E}$ is flat if and only if it preserves finite limits (note that $\mathcal{C} = \triangle$ is not cartesian, so no contradiction here).

Given a site $(\mathcal{C}, J)$ a functor $F : \mathcal{C} \to \mathcal{E}$ is J-*continuous* if and only if it sends J-covering sieves to epimorphic families. The above equivalence restricts to an equivalence of cateogories

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \quad \cong \quad \mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$$

where on the right, we consider J-continuous flat functors.

In the case of $\mathbb{N}^\times$, we have $\mathbf{Flat}(\mathcal{C}, \mathbf{Set})$ is the category of ordered subgroups of $(\mathbb{Q}, \mathbb{Q}^+)$, and thus the points are this.

Internal language. Given a category $\mathcal{C}$ (with finite products), we can canonically define a signature on Σ whose

– sorts are the objects of $\mathcal{C}$,

– functions symbols

$$\ulcorner f \urcorner \quad : \quad (A_1, \ldots, A_n) \quad \to \quad A$$

are morphisms

$$f \quad : \quad A_1 \times \ldots \times A_n \quad \to \quad A$$

in $\mathcal{C}$,

– relation symbols

$$\ulcorner R \urcorner \quad : \quad (A_1, \ldots, A_n)$$

are subobjects

$$R \quad \hookrightarrow \quad A_1 \times \ldots \times A_n$$

There is a canonical Σ-structure in $\mathcal{C}$, defined in the (really) obvious way.

When $\mathcal{C}$ is a cartesian (or geometric / etc.) category, we can define a theory $T_\mathcal{C}$ on Σ by imposing axioms

– of categories

$$\vdash_{x:A} \ulcorner \mathrm{id}_A \urcorner(x) = x \qquad\qquad \vdash_{x:A} \ulcorner g \circ f \urcorner(x) = \ulcorner g \urcorner(\ulcorner f \urcorner(x))$$

– of terminal objects

$$\vdash \exists(x : 1)\top \qquad\qquad \vdash_{x:1, y:1} x = y$$

– of cartesian products

$$\ldots$$

so that $\mathrm{Mod}_{T_\mathcal{C}}(\mathcal{D}) = \mathbf{Cart}(\mathcal{C}, \mathcal{D})$. In [4], they even take as axioms all the $\ulcorner \phi \urcorner \vdash_\Gamma \ulcorner \psi \urcorner$ such that for every morphism $h : c \to \Gamma$, $\phi \circ h = \mathbf{Top}_c$ implies $\psi \circ h = \mathbf{Top}_c$ ($\mathbf{Top}_c : c \to \Omega$ is the truth value at c).

4 The syntactic category

Syntax and semantics Recall that we have

- sorts $A \in \mathcal{S}$
- terms

$$t \ ::= \ x \mid f(t_1, \ldots, t_n)$$

- formulas

$$\begin{array}{rll} \phi, \psi \ ::= & R(t_1, \ldots, t_n) \mid t_1 = t_2 \mid \top \mid \phi \wedge \psi & \text{Horn formulas} \\ & \mid \exists x.\phi & \text{regular} \\ & \mid \bot \mid \phi \vee \psi & \text{coherent} \\ & \mid \displaystyle\bigvee_{i \in I} \phi_i & \text{geometric} \end{array}$$

- judgments $\phi \vdash_{\vec{x}:\vec{A}} \psi$
- interpretation: of formulas $[\![\phi \vdash_{\vec{x}} \psi]\!]$ and sequents

$$\begin{tikzcd}
{[\![\vec{x}.\phi]\!]} \arrow[rr, hook] \arrow[dr, hook] & & {[\![\vec{x}.\psi]\!]} \arrow[dl, hook'] \\
& {[\![\vec{x} : \vec{A}]\!]} &
\end{tikzcd}$$

- given a category $\mathcal{C}$, we write $\mathrm{Mod}_T(\mathcal{C})$ for the category of models M of T in $\mathcal{C}$.

We can characterize morphisms from $\vec{x}.\phi$ to $\vec{y}.\psi$, when we have a substitution as the morphisms from ϕ to the pullback:

$$\begin{tikzcd}
\phi \arrow[r] \arrow[dr, hook] \arrow[rr, dotted, bend left] & {} \arrow[r, dotted] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \psi \arrow[d, hook] \\
& \vec{x} \arrow[r, "\langle t_i \rangle"'] & \vec{y}
\end{tikzcd}$$

For instance consider the theory of categories with

- sorts: O, F
- operations:

$$x : O \vdash \mathrm{id}_x : F \qquad x : F \vdash sx : O \qquad x : F \vdash tx : O$$

- relations: $T(x, y, z)$
- axioms
  - $sx = ty \vdash \exists z.T(x, y, z)$

- $T(x, y, z) \vdash sx = ty \wedge tx = tz \wedge sy = sz$
- $T(x, y, z) \wedge T(x, y, z') \vdash z = z'$
- $\ldots$

A T-model M in a cartesian category $\mathcal{C}$ (with all finite limits) is a category:

$$\begin{tikzcd} sx = tz \arrow[rr, hook, "\sim"] \arrow[dr, hook] & & \exists z.T(x,y,z) \arrow[dl, hook'] \\ & x.F \times y.F & \end{tikzcd}$$

the upper left object is the one on which composition is defined.

$$\begin{tikzcd} T(x,y,z) \wedge T(x,y,z') \arrow[r, hook] \arrow[d, hook] & z = z' \arrow[dl, hook'] \\ x.F \times y.F \times z.F \times z.F & \end{tikzcd}$$

Lemma. Suppose that T is horn and

$$\phi(\vec{x}, y) \wedge \phi[z/y] \vdash y = z$$

is derivable. Then in every T-model the following morphism is a mono:

$$[\![\phi]\!] \hookrightarrow \vec{A} \times B \xrightarrow{(\vec{x})} \vec{A}$$

i.e.,

$$\begin{tikzcd} \phi \arrow[r, equal] \arrow[d, hook] & {} \arrow[d, hook] \\ \vec{A} \times B \arrow[r, "\vec{x}"'] & \vec{A} \end{tikzcd}$$

This is shown by remarking that f is a mono iff we have a pullback square

$$\begin{tikzcd} {} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner"] & {} \arrow[d, "f"] \\ {} \arrow[r, "f"'] & {} \end{tikzcd}$$

Here:

$$\begin{tikzcd} \phi \wedge \phi[z/y] \arrow[r, hook] \arrow[d, hook] & \phi[z/y] \arrow[r] \arrow[d, hook] & \phi \arrow[d, hook] \\ \phi \arrow[r, hook] \arrow[d] & x.\vec{A} \times y.B \times z.B \arrow[r] \arrow[d] & x.\vec{A} \times y.B \arrow[d] \\ \phi \arrow[r, hook] & x.\vec{A} \times y.B \arrow[r] & \vec{A} \end{tikzcd}$$

Lemma. Suppose in T that

$$\theta \vdash_{\vec{x},\vec{y}} \phi \wedge \psi \qquad \theta \wedge \theta[\vec{z}/\vec{y}] \vdash_{\vec{x},\vec{y},\vec{z}} \vec{y} = \vec{z} \qquad \phi \vdash_{\vec{x}} \exists \vec{y}.\theta$$

then in every T-model there exists a dotted arrow

$$\begin{tikzcd} \vec{x}.\phi \arrow[r, dotted] \arrow[d, hook] & \vec{y}.\psi \arrow[d, hook] \\ \vec{x} & \vec{y} \end{tikzcd}$$

The syntactic category. The category C_T has

- objects: $\vec{x}.\phi$ modulo α-conversion,
- morphisms: $\theta : \vec{x}.\phi \to \vec{y}.\psi$ such that the above relations are derivable, quotiented by $\theta = \theta'$ whenever $\theta \dashv\vdash \theta'$ is derivable.

The identity on $\vec{x}.\phi$ is

$$\vec{x}.\phi \xrightarrow{\phi \wedge x = x'} \vec{x}'\phi[x'/x]$$

and composition is

$$\vec{x}.\phi \xrightarrow{\theta} \vec{y}.\psi \xrightarrow{\gamma} \vec{z}.\chi \qquad = \qquad \vec{x}.\phi \xrightarrow{\exists \vec{y}.\theta \wedge \gamma} \vec{z}.\chi$$

This category C_T is cartesian:

- terminal object is $[].\top$
- $\vec{x}.\phi \times \vec{y}.\psi$ is

$$\begin{tikzcd} & \vec{x}\vec{y}.\phi \wedge \psi \arrow[dl, "\phi\wedge\psi"'] \arrow[dr, "\phi\wedge\psi"] & \\ \vec{x}.\phi & & \vec{y}.\psi \end{tikzcd}$$

- equalizers:

$$\vec{x}.\exists \vec{y}\theta \wedge \gamma \xrightarrow{\exists y.\theta \wedge \gamma} \vec{x}\phi \underset{\gamma}{\overset{\theta}{\rightrightarrows}} \vec{y}.\psi$$

Completeness. $C_T \Vdash (\phi \vdash \psi)$ iff $\phi \vdash \psi$ is derivable.

Classification. $\mathrm{Mod}_T(\mathcal{E}) \cong \mathbf{Cart}(C_T, \mathcal{E})$.

Sketches. Suppose given C a category and K a set of cocones of C. A presheaf $X \in \hat{C}$ is a *K-model* for every $k \in K$,

$$\hat{C}(ytk, X) \quad \xrightarrow{\sim} \quad \hat{C}(\operatorname{colim} k, X)$$

i.e.,

$$\begin{tikzcd} \operatorname{colim} k \arrow[r] \arrow[d] & X \\ ytk \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

For instance, given a site (C, J), we take $K = J$, a K-model is a sheaf.

Gabriel-Ulmer: there is a reflection

$$\begin{tikzcd} \mathrm{Mod}_K \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \hat{C} \arrow[l, dotted, shift right, "L"'] \end{tikzcd}$$

and moreover $\mathrm{Mod}_K = \hat{C}[W^{-1}]$.

Theorem (G-U): for every $\mathcal{E}$ cocomplete,

$$\mathbf{Cocont}(\mathrm{Mod}_K, \mathcal{E}) \quad \cong \quad K\text{-}\mathbf{Cocont}(C, \mathcal{E})$$

Proof

$$\begin{tikzcd} & C \arrow[r, "F"] \arrow[dl] \arrow[d, hook] & \mathcal{E} \\ \mathrm{Mod}_K \arrow[r, hook, shift right] \arrow[r, phantom, "\perp"] & \hat{C} \arrow[l, shift right, "L"'] \arrow[ur, "- \otimes_C F"'] & \end{tikzcd}$$

For instance, C_T^{op}, $K = \{\text{finite colimits cocones}\}$. We have

$$\begin{aligned} \mathbf{Cocont}(\mathrm{Mod}_K, \mathcal{E}) &\cong K\text{-}\mathbf{Cocont}(C, \mathcal{E}) \\ &\cong \mathrm{Mod}_T(\mathcal{E}) \\ &\cong \mathbf{Cart}(C_T, \mathcal{E}) \\ &\cong \mathbf{Flat}(C_T, \mathcal{E}) \\ &\cong \mathbf{Geom}(\mathcal{E}, \hat{C}_T) \end{aligned}$$

Doctrines. The idea of doctrine is that instead of doing each time the work for all the various frameworks (regular, geometric, etc.), we can parametrize the constructions by a “2-theory”, called a “doctrine”. A first idea is that a doctrine should be a 2-monad on **Cat** whose algebras are expected categories (regular, geometric, etc.). This works for the above examples, but this is not really suitable since morphisms are not right (we sometimes want to consider morphisms which change the domain for instance).

5 The classifying topos

Representable functor. A functor $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is *representable* when there exists $A \in \mathcal{C}$ and an isomorphism:

$$\phi \quad : \quad \mathrm{Hom}(-, A) \quad \simeq \quad P \quad : \quad \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

(note that ϕ is part of the data of the representation along with A).

By the Yoneda lemma, the (iso)morphism $\phi : \mathrm{Hom}(-, A) \to P$ is entirely specified by an element $u \in PA$, called the *universal element*.

More generally, it happens that we want to represent a pseudofunctor

$$P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$$

in which case we require an equivalence of categories

$$\mathbf{Cat}(-, \mathcal{A}) \quad \cong \quad P$$

Subobject classifyer. There are two equivalent definitions of a *subobject classifyer* in a category $\mathcal{C}$ with finite limits:

1. it is a monomorphism $\top : 1 \hookrightarrow \Omega$ such that for every monomorphism $m : U \hookrightarrow X$ there exists a unique morphism χ_m forming a pullback diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook, "m"'] & 1 \arrow[d, "\top"] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

2. it is an object Ω such that there is an isomorphism

$$\phi_X \quad : \quad \mathrm{Sub}(X) \quad \simeq \quad \mathrm{Hom}(X, \Omega)$$

which is natural in X, (otherwise said, the functor $\mathrm{Sub} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable by Ω).

Proof. In the first situation, we construct ϕ_X which to a subobject $m : U \hookrightarrow X$, ϕ_X associates χ_m. Given a morphism $f : Y \to X$, we have two pullbacks squares

$$\begin{tikzcd} V \arrow[r] \arrow[d, hook, "\mathrm{Sub}(f)(m)=f^*(m)"'] & U \arrow[r] \arrow[d, hook, "m"'] & 1 \arrow[d, "\top"] \\ Y \arrow[r, "f"'] & X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

and the outer is thus also a pullback, showing naturality. Conversely, in the second situation, in order to define $\top$, the only canonical thing we can consider is

$$\phi_\Omega \quad : \quad \mathrm{Sub}(\Omega) \quad \simeq \quad \mathrm{Hom}(\Omega, \Omega)$$

and take $\top : \Omega_0 \hookrightarrow \Omega$ to be the subobject associated to $\mathrm{id}_\Omega \in \mathrm{Hom}(\Omega, \Omega)$ (note that we do not know that $\Omega_0 = 1$ yet). By naturality of ϕ, we have

$$\begin{tikzcd} \mathrm{Sub}(\Omega) \arrow[r, "\phi_\Omega"] \arrow[d, "\mathrm{Sub}(f)"'] & \mathrm{Hom}(\Omega, \Omega) \arrow[d, "\mathrm{Hom}(f,\Omega)"] \\ \mathrm{Sub}(X) \arrow[r, "\phi_X"'] & \mathrm{Hom}(X, \Omega) \end{tikzcd} \qquad \begin{tikzcd} \top \arrow[r, mapsto] \arrow[d, mapsto] & \mathrm{id}_\Omega \arrow[d, mapsto] \\ f^*(\top) \arrow[r, mapsto] & f \end{tikzcd} \qquad \begin{tikzcd} \top \arrow[r, mapsto] \arrow[d, mapsto] & \mathrm{id}_\Omega \arrow[d, mapsto] \\ m \arrow[r, mapsto] & \phi_X(m) \end{tikzcd}$$

which we apply to a choice of morphism $f : X \to \Omega$ or subobject $m : U \hookrightarrow X$, i.e.,

$$\chi_{f^*(\top)} = f \qquad\qquad \chi_m^*(\top) = \chi_m$$

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook, "f^*(\top)"'] & \Omega_0 \arrow[d, "\top"] \\ X \arrow[r, "f=\chi_{f^*(\top)}"'] & \Omega \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r] \arrow[d, hook, "m=\chi_m^*(\top)"'] & \Omega_0 \arrow[d, "\top"] \\ X \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

uniqueness existence

Otherwise said, every monomorphism m is the pullback of **Top** a unique function f as in

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook, "m"'] & \Omega_0 \arrow[d, "\top"] \\ X \arrow[r, "f"'] & \Omega \end{tikzcd}$$

Finally, in order to show that Ω_0 is the terminal object, consider the case $m = \mathrm{id}_X$. We obtain the existence of a morphism $f : X \to \Omega_0$ making a pullback square as on the left

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & \Omega_0 \arrow[d, "\top"] \\ X \arrow[r, "\top\circ f"'] & \Omega \end{tikzcd} \qquad\qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, equal] & \Omega_0 \arrow[d, "\top"] \\ X \arrow[r, "\top\circ g"'] & \Omega \end{tikzcd}$$

Given another morphism $g : X \to \Omega_0$, we have a pullback square as on the right, and therefore $\top \circ f = \top \circ g$, and thus $f = g$ since $\top$ is a mono, i.e., Ω_0 is the terminal object. □

Note that in a good definition the subobject classifier $\top : 1 \hookrightarrow \Omega$ should not be required to be monomorphism from the terminal object: it can be deduced that the source is necessarily the terminal object.

This says that when we have a subobject classifyer we can "turn arrows backwards", i.e., a monomorphism m as on the left can be seen as a function χ_m as on the right

$$\begin{tikzcd} U \arrow[d, hook, "m"'] \\ X \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \Omega \\ X \arrow[u, "\chi_m"] \end{tikzcd}$$

and conversely. Namely, consider the situation in **Set**:

- to a point in $x \in X$, χ_m associates all the preimages of x under m, which is either empty or reduced to one element, and can thus be encoded as a truth value in $\Omega = \{\emptyset, \star\}$,
- conversely, the set U can be expressed as the collection of its points

$$U \quad = \quad \bigsqcup_{y \in U} \{\star\} \quad = \quad \bigsqcup_{y \in \{x \mid f(x) = \star\}} \{\star\}\,.$$

Also, note that there is a "universal monic" (the truth $\top$) such that every other monic can be obtained by pullback, which corresponds to the identity function. This way of "turning morphisms backwards" generalizes to many other kinds of situations.

Sets and families. The case of the subobject classifier in **Set**, can be extended in a more intensional way in order to classify functions instead of monos. Consider the category **SET** of large sets and functions. There is a bijection

$$\begin{tikzcd} U \arrow[d, "f"'] \\ X \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \mathbf{Set} \\ X \arrow[u, "\chi_f"] \end{tikzcd}$$

between

- functions to X with small fibres, i.e., $f^{-1}(x)$ is a set for every $x \in X$,

– families of sets indexed by X

i.e.,

$$\mathrm{SmallFun}(X) \quad \simeq \quad \mathbf{SET}(X, \mathbf{Set})$$

(where, on the left, we consider functions up to isomorphism). Namely, to a function $f : U \to X$ we associate the family of its fibers $\chi_f : X \to \mathbf{Set}$ which to $x \in X$ associates $f^{-1}(x)$. Conversely, to a family $g : X \to \mathbf{Set}$, we associate the canonical projection

$$\bigsqcup_{x \in X} X \quad \to \quad X$$

The classifying morphism is the one corresponding to $\mathrm{id}_{\mathbf{Set}}$, i.e., the canonical projection

$$\mathbf{Set}_\star = \bigsqcup_{X \in \mathbf{Set}} X \quad \to \quad \mathbf{Set}$$

On the left, we have the collection of pointed sets, i.e., sets with a distinguished element. Indeed, given $g : X \to \mathbf{Set}$, the pullback

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "f"'] & \mathbf{Set}_\star \arrow[d, "\top"] \\
X \arrow[r, "g"'] & \mathbf{Set}
\end{tikzcd}$$

gives

$$U \quad = \quad \bigsqcup_{x \in X} g(x)$$

and f the canonical projection. In the following, we will see that we often need to have a condition on fibers (here, small fibers) to obtain a good correspondence. Again, note that we only classify functions up to permuting the elements in a fiber, which is not very good: contrarily to the case of monos, there can be multiple ways in which two functions are identified.

Preshseaves and discrete fibrations. We can go from sets to categories as follows. A *discrete opfibration* is functor $F : C \to B$ which has the "unique future lifting property": for every $c \in C$ such that for every morphism $f : Fc \to b$ in B there exists a unique $f' : c \to b'$ in C such that $Ff' = f$:

$$\begin{tikzcd}
C \arrow[d, "F"'] \\
B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
c \arrow[r, dotted, "f'"] \arrow[d, mapsto] & b' \arrow[d, mapsto] \\
Fc \arrow[r, "f"'] & b
\end{tikzcd}$$

We have a correspondence between discrete opfibrations to B (with small fibers) and covariant presheaves on B

$$\begin{tikzcd}
C \arrow[d, "F"'] \\
B
\end{tikzcd}
\quad \leftrightsquigarrow \quad
\begin{tikzcd}
\mathbf{Set} \\
B \arrow[u, "P"]
\end{tikzcd}$$

(there is a contravariant version with discrete fibrations). Namely, to a discrete opfibration $F : C \to B$, we associate the presheaf P such that

– for $b \in B$, $Pb = F^{-1}(b)$, i.e., the set of objects of C which are sent to b by F,

– for $f : a \to b$, $Pf : Pa \to Pb$ is the function which to an element $c \in Pa$ (i.e., $c \in C$ with $Fc = a$) associates the target b' of the unique morphism $f' : c \to b'$ with $Ff' = f$.

Conversely, suppose given a presheaf $P : B \to \mathbf{Set}$. The *category of elements* of P, noted $\mathrm{El}(P)$, is the category whose

– objects are the pairs (b, x) with $b \in B$ and $x \in Pb$,

– morphisms $(b, x) \to (c, y)$ are morphisms $f : b \to c$ in B such that $Pf(x) = y$.

There is an obvious projection functor

$$\pi_P \quad : \quad \mathrm{El}(P) \quad \to \quad B$$

which is easily seen to be a discrete opfibration. The classifying discrete opfibration is

$$\mathbf{Set}_\star \quad \to \quad \mathbf{Set}$$

the forgetful functor from the category of pointed sets to sets, sometimes called the *universal* **Set***-bundle*. This means that the category of elements of P can be obtained as the pullback

$$\begin{tikzcd} \mathrm{El}(P) \arrow[r] \arrow[d, "\pi_P"'] & \mathbf{Set}_\star \arrow[d] \\ B \arrow[r, "P"'] & \mathbf{Set} \end{tikzcd}$$

Note that other classical definitions of the category of elements include (we should think about whether they generalize to other situations):

– $\mathrm{El}(P)$ is the comma category $\star \downarrow P$ where $\star : 1 \to \mathbf{Set}$ is the functor from the terminal category to $\mathbf{Set}$ picking the one-element set $\{\star\}$, i.e., the "lax pullback"

$$\begin{tikzcd} \mathrm{El}(P) \arrow[r] \arrow[d, "\pi_P"'] \arrow[dr, phantom, "\swarrow"] & 1 \arrow[d, "\star"] \\ B \arrow[r, "P"'] & \mathbf{Set} \end{tikzcd}$$

– $\mathrm{El}(P)$ is the comma category Y/P where $Y : B \to \hat{B}$ is the Yoneda embedding and P is seen as a functor $1 \to \mathbf{Set}^C$, i.e., the "lax pullback" or in the other direction?

$$\begin{tikzcd} \mathrm{El}(P) \arrow[r, "\pi_P"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & B \arrow[d, "Y"] \\ \star \arrow[r, "P"'] & \hat{B} \end{tikzcd}$$

Pseudofunctors and fibrations. Replacing **Set** by the 2-category **Cat**, we obtain a correspondence

$$\begin{tikzcd} C \arrow[d, "F"'] & \leftrightsquigarrow & \mathbf{Cat} \\ B & & B \arrow[u, "P"] \end{tikzcd}$$

between Grothendieck fibrations $C \to B$ and pseudofunctors $B \to \mathbf{Cat}$ (i.e., "functors" such that composition is preserved up to isomorphism only). There is an analogous of the category of elements which is called the *Grothendieck construction*:

$$\int \quad : \quad \mathbf{Cat}(C, \mathbf{Cat}) \quad \to \quad \mathbf{Cat}/C^{\mathrm{op}}$$

which gives rise to an equivalence of 2-categories when we corestrict to fibrations.

Homotopy type theory. In homotopy type theory, a type

$$\vdash \quad A : \mathsf{Type}$$

denotes an object in the ambient $(\infty, 1)$-category while a dependent type

$$a : A \quad \vdash \quad B(a) : \mathsf{Type}$$

denotes a morphism $B \to A$, seen as a fibration or a bundle. There is a corresponding classifying map

$$B \quad : \quad A \quad \to \quad \mathsf{Type}$$

which can be obtained by an internal $(\infty, 1)$-Grothendieck construction, the total space being

$$\sum_{a:A} B(a).$$

The universal type bundle is the canonical projection

$$\sum_{A:\mathsf{Type}} A \quad \to \quad \mathsf{Type}.$$

Covering spaces. A *covering space* Y for a topological space X (path connected and locally path connected) is a morphism $p : Y \to X$ such that every $x \in X$ admits a neighborhood U such that $p^{-1}(U) = \bigsqcup_{i \in I} U_i$ with each U_i being an open of Y which is homeomorphic to U. Informally, Y can be obtained by partly "delooping" X. A *covering map* f between covering spaces Y and Z of X is an homeomorphism making the expected triangle commute:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

The *universal covering space* is the (unique up to homeomorphism) covering space $\tilde{X}$ of X which is path connected and simply connected.

A covering space $p : Y \to X$ has the *homotopy lifting property*: there is a unique lift

$$\begin{tikzcd} Z \arrow[r, "\tilde{f}_0"] \arrow[d, "\langle \mathrm{id}_Z, 0\rangle"'] & Y \arrow[d, "p"] \\ Z \times [0,1] \arrow[r, "f"'] \arrow[ur, dotted] & X \end{tikzcd}$$

In particular, we can lift paths and homotopic paths will lift to homotopic paths.

In school [1], we learn that

> *for X connected, covering maps $Y \to X$ correspond to subgroups of $\pi_1(X)$ for a connected X*

When X is not connected this can be generalized to

> *for X connected, covering maps $Y \to X$ with fiber F correspond to transitive actions of $\pi_1(X)$ on F.*

Namely, a transitive action is the same as a subgroup: given a subgroup $H \subseteq \pi_1(X)$, we define $F = \pi_1(X)/H$ with the obvious action, and given an transitive action $\phi : \pi_1(X) \times F \to F$, the subgroup is the stabilizer of a point $H = \{\phi(g, x) \mid g \in \pi_1(X)\}$. This generalizes to non-connected X:

> *covering maps $Y \to X$ with fiber F correspond to actions of $\pi_1(X) \to \mathrm{Aut}(F)$.*

We can even handle different fiber as follows. Given a covering space $p : X \to Y$, its *monodromy* is the functor

$$\mathrm{Fib}_p \quad : \quad \Pi_1(X) \quad \to \quad \mathbf{Set}$$

where $\Pi_1(X)$ is the fundamental groupoid of X (objects: points of X, morphisms: homotopy classes of paths) defined by

- $\mathrm{Fib}_p(x) = p^{-1}(x)$,
- $\mathrm{Fib}_p(\gamma)$ is the endpoint of the lifting (well-defined up to homotopy) of the path γ.

This extends as a functor

$$\mathrm{Fib} \quad : \quad \mathrm{Cov}(X) \quad \to \quad \mathbf{Set}^{\Pi_1(X)}$$

The fundamental groupoid $\Pi_1(Y)$ of the total space Y can be recovered as the category of elements of Fib_p (up to equivalence of categories):

$$\begin{tikzcd} \Pi_1(Y) = \mathrm{El}(\mathrm{Fib}_p) \arrow[r] \arrow[d] & \mathbf{Set}_\star \arrow[d] \\ \Pi_1(X) \arrow[r, "\mathrm{Fib}_p"'] & \mathbf{Set} \end{tikzcd}$$

We can build a functor

$$\mathrm{Rec} \quad : \quad \mathbf{Set}^{\Pi_1(X)} \quad \to \quad \mathrm{Cov}(X)$$

which to $F : \Pi_1(X) \to \mathbf{Set}$ associates the space $\bigsqcup_{x \in \Pi_1(X)} Fx$ appropriately topologized. This gives rise to an adjoint equivalence between $\mathrm{Cov}(X)$ and $\mathbf{Set}^{\Pi_1(X)}$: this is the *fundamental theorem of covering spaces*.

The categorical version of all this is that fibrations of groupoids $E \to B$ are classified by functors to groupoids $B \to \mathbf{Gpd}$.

Fiber bundles. A *fiber bundle* is a morphism in **Top**:

$$p \quad : \quad E \quad \to \quad B$$

Given a topological group G, a *G-principal bundle* is a bundle equipped with a left action

$$\mu \quad : \quad G \times E \quad \to \quad E$$

which

- preserves the fibers

  $$p \circ \mu(g, y) \quad = \quad y$$

  for $g \in G$ and $y \in E$, or equivalently we have a coequalizer diagram

  $$G \times E \underset{\pi_E}{\overset{\mu}{\rightrightarrows}} E \xrightarrow{p} B$$

- and is locally trivial: there exists a covering of B by a family of open sets U_i equipped with homeomorphisms

  $$\phi_i \quad : \quad G \times U_i \quad \xrightarrow{\sim} \quad p^{-1}(U_i)$$

  such that

  $$p \circ \phi_i(g, x) = x \qquad\qquad \phi_i(gh, x) = \mu(g, \phi_i(h, x))$$

This definition implies that the action of G on each fiber $p^{-1}(x)$ is both

- free: $\mu(g, y) = y$ implies $g = e$, and
- transitive: for $y, y' \in p^{-1}(x)$ there exists $g \in G$ such that $gy = y'$.

When G is discrete, this is equivalent to requiring that we have a *G-torsor* $p : E \to B$, i.e., an *étale map* (every point $e \in E$ has a neighborhood V such that pV is open an $p|_V : V \to pV$ is an homeomorphism) such that

- each fiber $E_x = p^{-1}(x)$ is non-empty,
- the action $G \times E_x \to E_x$ on each fiber is both free and transitive.

A typical example is given by coverings $p : E \to B$. In this case, the image $p(\pi_1(E))$ is a normal subgroup N of $\pi_1(B)$ and we have an action of the group $G = \pi(E)/N$ making E a principal bundle for G.

Note that each fiber looks like G excepting that we have "forgotten the unit element": for every element $e \in E_x$, we have an isomorphism $\mu(-, e) : G \xrightarrow{\sim} E_x$ but there is no canonical such choice. Now, we have an adjunction

$$\begin{tikzcd} \mathbf{Top}/X = \mathbf{Bund} \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\top"] & \mathbf{Set}^{\mathcal{O}(X)^{\mathrm{op}}} \arrow[l, bend left, "\Lambda"] \end{tikzcd}$$

(each bundle is sent to the sheaf of cross sections and each presheaf is sent to the bundle of germs) which restricts to an equivalence of categories

$$\mathbf{Etale}(X) \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \mathbf{Sh}(X)$$

So that every G-torsor over X can be expressed as a sheaf F on X together with a natural left action G on F such that

– the stalk F_x at each point x is non-empty,

– each induced action $\mu_x : G \times F_x \to F_x$ is free and transitive.

We thus arrive at a definition which can be generalized to any topos. Given a discrete group G and $\gamma : \mathcal{E} \to \mathbf{Set}$ a geometric morphism, a *G-torsor* over $\mathcal{E}$ is an object T of $\mathcal{E}$ equipped with a left action $\mu : \gamma^*(G) \times T \to T$ for which

– the canonical map $T \to 1$ is an epi,

– the action induces an isomorphism in $\mathcal{E}$

$$\langle \mu, \pi_2 \rangle \quad : \quad \gamma^*(G) \times T \quad \to \quad T \times T$$

In this sense, a principal G-bundle over a space X is the same as a G-torsor in the sheaf topos $\mathbf{Sh}(X)$.

The topos $\mathbf{B}G$ of right G-sets contains a canonical G-torsor U_G over $\mathbf{B}G$. G acts on the right on itself, thus the right action $U_G \times G \to U_G$. The inverse image of $\gamma : \mathbf{B}G \to \mathbf{Set}$ is the functor $\gamma^* : \mathbf{Set} \to \mathbf{B}G$ equipping a set with the trivial right G-action, and we define the left action

$$\mu \quad : \quad \gamma^*(G) \times U_G \quad \to \quad U_G$$

simply by multiplication. This topos classifies G-torsors: there is an equivalence of categories

$$\mathbf{Tor}_G(\mathcal{E}) \quad \cong \quad \mathbf{Geom}(\mathcal{E}, \mathbf{B}G)$$

which is natural in $\mathcal{E}$; the universal G-torsor is U_G.

Cohomology. Given a group G and $n \in \mathbb{N}$, the n-th *Eilenberg-MacLane space* or *classifying space* $K(G, n)$ is the space (unique up to weak homotopy equivalence) such that

$$\pi_n(K(G, n)) = G \qquad\qquad \pi_i(K(G, n)) = 1$$

for $i \neq n$. For instance,

$$K(\mathbb{Z}, 1) = S^1 \qquad\qquad K(\mathbb{Z}, 2) = \mathcal{C}P^\infty$$

Homotopy classes of maps into $K(G, n)$ correspond to (singular) cohomology groups:

$$H^n(X, G) \quad = \quad [X, K(G, n)]$$

As usual, with $X = K(G, n)$ we can consider the cohomology class γ_n in $H^n(K(G, n), G)$ corresponding to the identity in $[K(G, n), K(G, n)]$. A morphism $f : X \to K(G, n)$ gives rise to the cohomology class $f^*(\gamma_n)$ in $H^n(X, G)$, where

$$f^* \quad : \quad H^n(K(G, n), G) \quad \to \quad H^n(X, G)$$

is the morphism induced by f in cohomology. This space can be defined as the geometric realization of the simplicial set such $S \in \hat{\Delta}$ whose i simplices are the elements of $Z^n(\Delta^i, G)$, the n-dimensional cocycles on the i-dimensional simplex Δ^i.

More generally, Brown's representability theorem characterizes representable functors on CW-complexes. A functor

$$F \quad : \quad \mathrm{Ho}(\mathbf{Top}_\star)^{\mathrm{op}} \quad \to \quad \mathbf{Set}$$

where $\mathrm{Ho}(\mathbf{Top}_\star)$ is the homotopy category of pointed connected topological spaces (or equivalently CW-complexes, which are cofibrant objects), is representable if and only if

1. it sends coproducts (wedge sums) to products:
$$F(\vee_i X_i) \quad \simeq \quad \prod_i F(X_i)$$
2. it sends weak pushouts to weak pullbacks (weak means here the usual definition with existence but not unicity, weak pushouts on the homotopy category coincide with homotopy pushouts).

In a more concrete way, each $K(G, n)$ is isomorphic to the loop-space of the next one

$$K(G, n) \quad \simeq \quad \Omega K(G, n+1)$$

More generally, we define a spectrum as a sequence of space (K_n) such that there is a weak homotopy equivalence $K_n \xrightarrow{\sim} \Omega K_{n+1}$. In this case, the maps $[X, K_n] = [X, \Omega K_{n+1}]$ have a structure of group induced by the fact that we have a loop space on the right and a Eckmann-Hilton argument shows that $[X, K_n] = [X, \Omega^2 K_{n+2}]$ is actually an abelian group. Each K_n should thus be thought of as a "homotopy abelian group", i.e., one whose laws are only validated up to homotopy. If (K_n) is an spectrum then

$$X \quad \mapsto \quad h^n(X) = [X, K_n]$$

defines a reduced cohomology theory and the converse is actually true: the Brown representabiliy theorem says that every cohomology theory can be obtained from some spectrum in this way. We recall that a (reduced) *cohomology theory* consists of contravariant functors h^n from CW-complexes to $\mathbf{Ab}$ together with coboundary morphisms

$$\delta \quad : \quad h^n(A) \quad \to \quad h^{n+1}(X/A)$$

for each CW-pair (X, A) such that

1. $f \quad \simeq \quad g : X \to Y$ implies $h^n(f) = h^n(g) : h^n(X) \to h^n(Y)$,
2. for each CW-pair (X, A) there is a long exact sequence
$$\cdots \xrightarrow{\delta} h^n(X/A) \xrightarrow{h^n(q)} h^n(X) \xrightarrow{h^n(i)} h^n(A) \xrightarrow{\delta} h^{n+1}(X/A) \xrightarrow{h^{n+1}(q)} \cdots$$
where $q : X \to X/A$ is the quotient map and $i : A \to X$ is the inclusion,
3. for a wedge sum $X = \bigvee_i X_i$ with inclusions $i_i : X_i \to X$, the product map
$$\prod_i i_i \quad : \quad h^n(X) \quad \to \quad \prod_i h^n(X_i)$$
is an isomorphism for each n.

A cohomology theory is *ordinary* when $h^n(S^0) \simeq 0$ for $n > 0$. The cohomology theories arising from spectra do not necessarily validate this axiom.

Classifying toposes. Given a theory T, its *classifying topos* $\mathcal{B}(T)$ is a topos such that models of T in a topos $\mathcal{E}$ correspond to geometric morphisms from $\mathcal{E}$ to $\mathcal{B}(T)$, i.e., we have an equivalence of categories

$$\mathrm{Mod}_T(\mathcal{E}) \quad \cong \quad \mathbf{Geom}(\mathcal{E}, \mathcal{B}(T))$$

which is natural in $\mathcal{E}$. Otherwise said, we classify the pseudofunctor

$$\mathrm{Mod}_T \quad : \quad \mathbf{Topos}^{\mathrm{op}} \quad \to \quad \mathbf{CAT}$$

In particular, the *universal model* U_T is the one corresponding to the identity $\mathrm{id}_{\mathcal{B}(T)}$. This model satisfies that for every T-model M in $\mathcal{E}$ there exists a geometric morphism $f : \mathcal{E} \to \mathcal{B}(T)$ (unique up to isomorphism) such that $M \cong f^*(U_T)$:

$$\begin{tikzcd} T \arrow[r] \arrow[d, "M"'] & U_T \arrow[d] \\ \mathcal{E} \arrow[r, "f"'] & \mathcal{B}(T) \end{tikzcd}$$

(I guess that the above diagram only really makes sense if we consider the syntactic topos associated to T).

The object classifier. Consider the theory T with one sort and no operation or relation symbol. We thus have

$$\mathrm{Mod}_T(\mathcal{E}) \quad = \quad \mathcal{E}$$

We want to construct a topos $\mathcal{S}[U]$ and an equivalence of categories

$$\mathcal{E} \quad \cong \quad \mathbf{Geom}(\mathcal{E}, \mathcal{S}[U])$$

The notation here is meant to be reminiscent of the polynomial algebra $\Bbbk[X]$ for which there is an isomorphism

$$A \quad \simeq \quad \mathbf{Alg}_{\Bbbk}(\Bbbk[X], A)$$

If we take $\mathcal{S}[U] = \hat{\mathcal{C}}$ for some category $\mathcal{C}$, we have that

$$\mathbf{Geom}(\mathcal{E}, \hat{\mathcal{C}}) \quad \simeq \quad \mathbf{Flat}(\mathcal{C}, \mathcal{E})$$

Now, if $\mathcal{C}$ has finite limits

$$\mathbf{Flat}(\mathcal{C}, \mathcal{E}) \quad \simeq \quad \mathbf{Lex}(\mathcal{C}, \mathcal{E})$$

Finally, if we take $\mathcal{C}$ to be the category with finite limits freely generated by a point, we have

$$\mathbf{Lex}(\mathcal{C}, \mathcal{E}) \quad \simeq \quad \mathbf{Cat}(1, \mathcal{E})$$

Concretely, it can be shown that such a $\mathcal{C}$ is $\mathbf{Fin}^{\mathrm{op}}$, the opposite of the category of finite sets and functions and

$$\mathcal{S}[U] \quad = \quad \mathbf{Set}^{\mathbf{Fin}}$$

where the universal object in $\mathbf{Set}^{\mathbf{Fin}}$ is the inclusion $\mathbf{Fin} \to \mathbf{Set}$.

One way to see that **Fin** has the required property is to recall that the free cocompletion of a category C is $\hat{C}$, and the finite cocompletion is the full subcategory on "finite presheaves" (= finite colimits of representables). Here, the subcategory of $\hat{1}$ is clearly **Fin**. This can also be shown by hand. First remark that a functor $F : \mathbf{Fin} \to \mathcal{C}$ which preserves finite coproducts preserves finite colimits. Namely, given a coequalizer

$$A \rightrightarrows B \longrightarrow C$$

we have $A = \coprod_{a \in A} \{\star\}$ and similarly for other objects. Since F preserves finite coproducts, the above diagram is sent to

$$\textstyle\coprod_{a \in A} X \rightrightarrows \coprod_{b \in B} X \longrightarrow \coprod_{c \in C} X$$

with $X = F\{\star\}$. This is a coequalizer iff for every object Y we have an equalizer diagram

$$\mathbf{Fin}(C, \mathcal{C}(X,Y)) \longrightarrow \mathbf{Fin}(B, \mathcal{C}(X,Y)) \rightrightarrows \mathbf{Fin}(A, \mathcal{C}(X,Y))$$

which is the case because the above diagram is a coequalizer. Finally, a functor F is determined on each set X by $FX = \coprod F(\{\star\})$.

Rings. In any category $\mathcal{C}$ with products, we can define a ring object R as a diagram

$$1 \underset{1}{\overset{0}{\rightrightarrows}} R \underset{\times}{\overset{+}{\leftleftarrows}} R \times R$$

satisfying the obvious axioms. We can thus define category $\mathrm{Ring}(\mathcal{C})$ of rings in $\mathcal{C}$. Any left exact functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $\mathrm{Ring}(\mathcal{C}) \to \mathrm{Ring}(\mathcal{D})$. Therefore, a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ induces $f^* : \mathrm{Ring}(\mathcal{E}) \to \mathrm{Ring}(\mathcal{F})$.

Now, we want to construct a *classifying ring* $\mathcal{R}$, i.e., an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathcal{R}) \quad \cong \quad \mathrm{Ring}(\mathcal{E})$$

As before, we can take

$$\mathcal{R} \quad = \quad \mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$$

where $\mathcal{D}$ is the free category with finite limits on a ring object, i.e.,

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}) \quad \cong \quad \mathbf{Flat}(\mathcal{D}, \mathcal{C}) \quad \cong \quad \mathbf{Lex}(\mathcal{D}, \mathcal{C}) \quad \cong \quad \mathrm{Ring}(\mathcal{C})$$

We now show that

$$\mathcal{D} \quad = \quad \mathbf{fpRing}^{\mathrm{op}}$$

the opposite of the category of finitely presented rings, i.e., of the form

$$\mathbb{Z}[X_1, \ldots, X_n]/(P_1, \ldots, P_k)$$

In particular, the canonical ring object is $\mathbb{Z}[X]$. In the category **fpRing**:

- $\mathbb{Z}[X]$ is the initial object,
- coproduct is given by tensor product:

$$\mathbb{Z}[X_i]/(P_j) \otimes \mathbb{Z}[Y_i]/(Q_j) \quad = \quad \mathbb{Z}[X_i, Y_i]/(P_j, Q_j)$$

– a morphism $\phi : \mathbb{Z}[X_i]/(P_j) \to \mathbb{Z}[Y_i]/(Q_j)$ is determined by a family of functions $\phi_i = \phi(X_i)$ such that $P_j(\phi_1, \ldots, \phi_n) = 0$ for every j, and the coequalizer of two morphisms ϕ and ψ as above is

$$\mathbb{Z}[Y_i]/(Q_j, \phi_k - \psi_k)$$

The category $\mathbf{fpRing}^{\mathrm{op}}$ thus has finite limits and has a ring object

$$\mathbb{Z} \underset{1}{\overset{0}{\leftleftarrows}} \mathbb{Z}[X] \underset{X \cdot Y}{\overset{X+Y}{\rightrightarrows}} \mathbb{Z}[X,Y] \simeq \mathbb{Z}[X] \otimes \mathbb{Z}[Y]$$

(drawn in $\mathbf{fpRing}$). Finally, we can show that we have an equivalence of categories

$$\begin{array}{rcl} \mathbf{Lex}(\mathbf{fpRing}^{\mathrm{op}}, \mathcal{C}) & \cong & \mathrm{Ring}(\mathcal{C}) \\ F & \mapsto & F(\mathbb{Z}[X]) \end{array}$$

explicitly. In the other direction, given a ring R, we have to define a left exact $F_R : \mathbf{fpRing} \to \mathcal{C}$, and there is only one way we can do this really:

– we have to have $F_R(\mathbb{Z}[X]) = R$ for the equivalence to work,

– since F_R preserves products

$$F_R(\mathbb{Z}[X_1, \ldots, X_n]) \quad = \quad F_R(\mathbb{Z}[X_1]) \times \ldots \times F_R(\mathbb{Z}[X_n]) \quad = \quad R^n$$

– since F_R preserves equalizers $F_R\left(\mathbb{Z}[X_1, \ldots, X_n]/(P_1, \ldots, P_k)\right)$ the equalizer

$$F_R\left(\mathbb{Z}[X_i]/(P_j)\right) \longrightarrow R^n \underset{0}{\overset{\langle P_i \rangle}{\rightrightarrows}} R^k$$

– and similarly for morphisms.

The proof can be completed by manual checks.

Note that the same proof would work for any (essentially) algebraic theory other than rings...

Simplicial sets. The topos $\hat{\Delta}$ of simplicial sets classifies linear orders with distinct bottom and top:

$$\mathbf{Orders}(\mathcal{E}) \quad \cong \quad \mathbf{Geom}(\mathcal{E}, \hat{\Delta})$$

with universal order $\triangle(-, [1])$. The equivalence associates, to an order in $\mathcal{E}$, the associated nerve / realization in $\mathcal{E}$.

Geometric theories. Given a theory with no axioms (i.e., a signature), a left-exact functor $F : \mathcal{E} \to \mathcal{F}$ preserves products, monos and pullbacks and thus induces a morphism

$$\mathrm{Mod}_T(F) \quad : \quad \mathrm{Mod}_T(\mathcal{E}) \quad \to \quad \mathrm{Mod}_T(\mathcal{F})$$

for instance, the interpretation of

$$T \quad : \quad (A, B)$$

in $\mathcal{E}$

$$[\![R]\!]_{\mathcal{E}} \quad \hookrightarrow \quad [\![A]\!]_{\mathcal{E}} \times [\![B]\!]_{\mathcal{E}}$$

is sent to

$$F[\![R]\!]_{\mathcal{E}} \quad \hookrightarrow \quad F([\![A]\!]_{\mathcal{E}} \times [\![B]\!]_{\mathcal{E}}) \quad \simeq \quad F[\![A]\!]_{\mathcal{E}} \times F[\![B]\!]_{\mathcal{E}} \quad = \quad [\![A]\!]_{\mathcal{F}} \times [\![B]\!]_{\mathcal{F}}$$

A geometric morphism

$$f \quad : \quad \mathcal{F} \quad \to \quad \mathcal{E}$$

thus induces, via f^*, a morphism

$$\mathrm{Mod}_T(f) \quad : \quad \mathrm{Mod}_T(\mathcal{E}) \quad \to \quad \mathrm{Mod}_T(\mathcal{F})$$

In presence of axioms this is not generally well defined because the validity of formulas is not generally preserved, but it is the case for

– open morphisms,

– geometric theories.

Given a geometric morphism $f : \mathcal{F} \to \mathcal{E}$, $f^* : \mathcal{E} \to \mathcal{F}$ is left exact and thus preserves monos hence induces a functor why???

$$f^*c \quad : \quad \mathrm{Sub}_{\mathcal{E}}(c) \quad \to \quad \mathrm{Sub}_{\mathcal{F}}(f^*c)$$

f is *open* when this functor admits a left adjoint (this generalizes the situation where a morphism of spaces sends open set to open sets). Those preserve the validity of formulas by induction (the open assumption is required to show that the interpretation of $\forall$, $\Rightarrow$ and $\neg$ commutes to the interpretation).

Instead of geometric morphisms, we can also restrict to geometric theories. A *geometric formula* is one formed out of relation symbols, $=$, $\top$, $\bot$, $\vee$, $\wedge$, $\exists$ and $\bigvee$. A geometric theory is one containing only formulas of the form $\phi \vdash \psi$ with ϕ and ψ geometric.

Syntactic category. Suppose fixed a geometric theory T (it would work for other flavors than geometric). The syntactic category $\mathcal{C}_T$ has as objects the geometric formulas in context $\vdash_\Gamma \phi$ up to α-equivalence (a free variable in ϕ is bound in Γ) and a morphism

$$(\vdash_\Gamma \phi) \quad \to \quad (\vdash_\Delta \psi)$$

is a geometric formula

$$\vdash_{\Gamma,\Delta} \theta$$

which is provably functional in T, i.e., the following sequents are derivable

$$\begin{array}{rcll} \theta & \vdash_{\Gamma,\Delta} & \phi \wedge \psi & \text{domain and codomain are respected} \\ \phi & \vdash_{\Gamma} & (\exists \vec{y})\theta & \text{there is an image} \\ \theta \wedge \theta[\vec{y}'/\vec{y}] & \vdash_{\Gamma,\Delta,\Delta'} & \vec{y} = \vec{y}' & \text{images are unique} \end{array}$$

where we implicitly suppose that the variables in Γ and Δ are disjoint (which we can do because of the α-equivalence) and write $\vec{x}$ (resp. $\vec{y}$) for the variables of Γ (resp. Δ), and Δ' is Δ with y_i replaced by y_i'. Moreover, the formulas in morphisms are considered up to provable equivalence in the theory T, i.e., we identify two morphisms θ and θ' (with the above type) such that

$$\theta \vdash_{\Gamma,\Delta} \theta' \qquad \text{and} \qquad \theta' \vdash_{\Gamma,\Delta} \theta.$$

Identities are

$$\theta \quad \vdash_{\Gamma,\Gamma'} \quad \phi \wedge (\vec{x} = \vec{x}')$$

and composition of

$$\vdash_\Gamma \phi \xrightarrow{\theta} \vdash_\Delta \psi \xrightarrow{\rho} \vdash_\Sigma \chi$$

is given by

$$(\exists \vec{y})(\theta \wedge \rho).$$

This category is a geometric category.

Proof. Let us give the main constructions.

- The terminal object is given by $\vdash \top$: namely, given an object $\vdash_\Gamma \phi$, the morphism $\phi : (\vdash_\Gamma \phi) \to (\vdash \top)$ is the only possible one.
- Given morphisms

  $$\theta_1 : (\vdash_\Gamma \phi) \to (\vdash_{\Delta_1} \phi_1) \qquad\qquad \theta_2 : (\vdash_\Gamma \phi) \to (\vdash_{\Delta_2} \phi_2)$$

  their product is given on objects as $\vdash_{\Delta_1,\Delta_2}$ and on morphisms by $\theta_1 \wedge \theta_2$.
- The equalizer of $\theta, \theta' : (\vdash_\Gamma \phi) \to (\vdash_\Delta \psi)$ is $\theta \wedge \theta'$.
- The image factorization of a morphism $\theta : (\vdash_\Gamma \phi) \to (\vdash_\Delta \psi)$ is given by

  $$\vdash_\Gamma \phi \xrightarrow{\theta} \vdash_\Delta (\exists \vec{x})\theta \xhookrightarrow{(\exists \vec{y})\theta} \vdash_\Delta \psi$$

- Pullbacks correspond to substitutions.
- etc. ☐

We have a *universal model* of T in $\mathcal{C}_T$ where the interpretation of

- a sort A is $\vdash_{x:A} \top$
- a function symbol $f : (A_1, \ldots, A_n) \to A$ is

  $$(\vdash_{x_1:A_1,\ldots,x_n:A_n} \top) \longrightarrow (\vdash_{y:A} \top)$$

- a relation symbol $R : (A_1, \ldots, A_n)$ is

  $$(\vdash_{x_1:A_1,\ldots,x_n:A_n} R) \xrightarrow{R} (\vdash_{x_1:A_1,\ldots,x_n:A_n} \top)$$

For any formula $\vdash_\Gamma \phi$, its interpretation is the subobject $(\vdash_\Gamma \phi) \hookrightarrow (\vdash_\Gamma \top)$. A geometric sequent is provable iff its satisfied in this model, from which we can immediately deduce the completeness theorem.

It can be shown that we have

$$\mathbf{Geom}(\mathcal{C}, \mathcal{C}_T) \quad \cong \quad \mathrm{Mod}_T(\mathcal{C})$$

However, $\mathcal{C}_T$ is not a topos in general. In order to obtain one, we have to "complete" the category. The *geometric topology* is the topology whose covering sieves are those which contain small covering families. We recall that a *covering family* is a family of cofinal arrows such that the union of their image is the maximal subobject. Concretely, we take here families ϕ_i such that the following formula holds:

$$\psi(y) \vdash_{\Gamma,\Delta} \bigvee_i (\exists \vec{x})(\phi_i(\vec{x}, \vec{y}))$$

i.e., formally, every element in the codomain is in the image of some ϕ_i. This topology is *subcanonical*, i.e., all representable presheaves are sheaves. We thus have a Yoneda embedding $Y : \mathcal{C}_T \hookrightarrow \mathbf{Sh}(\mathcal{C}_T, J)$. Finally, by Diaconescu's theorem,

$$\mathbf{Geom}(\mathcal{E}, \mathcal{C}_T) \quad \cong \quad \mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}_T, J)) \quad \cong \quad \mathrm{Mod}_T(\mathcal{E})$$

6 Higher toposes

Lex localizations. Given a topos on a site $(\mathcal{C}, J)$ the inclusion $i : \mathbf{Sh}(\mathcal{C}, J) \hookrightarrow \hat{\mathcal{C}}$ admits a left adjoint (the *sheafification functor*) which preserves finite limits:

$$\begin{tikzcd} \mathbf{Sh}(\mathcal{C}, J) \arrow[r, hook, shift right, "i"'] \arrow[r, phantom, "\perp"] & \hat{\mathcal{C}} \arrow[l, shift right] \end{tikzcd}$$

and moreover, up to equivalence, every sheaf category can be obtained in this way. Sheaves $\mathcal{E}$ on $\mathcal{C}$ correspond to left exact localizations $\mathcal{E} \hookrightarrow \hat{\mathcal{C}}$. Moreover, in this situation, the inclusion $i : \mathbf{Sh}(\mathcal{C}, J) \hookrightarrow \hat{\mathcal{C}}$ is always an accessible functor (preserves κ-filtered colimits for some cardinal κ). For higher toposes, this condition is not implied anymore so that we have to require it.

An ∞-*topos* $\mathcal{X}$ is an accessible left exact reflective sub-$(\infty, 1)$-category of an $(\infty, 1)$-category of $(\infty, 1)$-presheaves:

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, shift right, "i"'] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Spaces}) \arrow[l, shift right, "\mathrm{lex}"'] \end{tikzcd}$$

References

[1] John C Baez and Michael Shulman. Lectures on n-categories and cohomology. In *Towards higher categories*, pages 1–68. Springer, 2010.

[2] Michael Barr. Toposes without points. *Journal of Pure and Applied Algebra*, 5(3):265–280, 1974.

[3] Olivia Caramello. *Theories, Sites, Toposes.* Oxford University Press, 2018.

[4] Joachim Lambek and Philip J Scott. *Introduction to higher-order categorical logic*, volume 7. Cambridge University Press, 1988.

[5] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.