Tietze Equivalences as Weak Equivalences

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1 Introduction

Presentations can be used to provide small descriptions of monoids, from which one can perform useful computation (e.g. homology groups). In this context, a very important tool is the notion of Tietze transformation, which are elementary transformations on presentations preserving the presented monoid (they can for instance be used in order to transform a presentation into a convergent one and use rewriting techniques). Moreover, these transformations are complete in the sense that any two presentations of a same monoid are related by Tietze transformations. Originally, those transformations were rather studied for groups instead of monoids [9], but this is of little importance here, see [6, Proposition 2.1] for a recent exposition.

Our goal is not to prove new results, but rather to replace existing ones in the abstract setting of model categories, as a first step in order to generalize to other structures than monoids and groups (we have in mind recent generalizations, such as the transformations for coherent presentations of categories [4, Section 2.1]). The starting point is the observation that Tietze transformations look like trivial cofibration in a model structure on presentations in which the two presentations would be weakly equivalent when they present the same monoid. We thus begin by constructing a model structure on the category presentation with suitably chosen morphisms (it turns out that we need to allow some sort of degeneracies), where the weak equivalences are the ones mentioned before. We show that the Tietze transformations can then be interpreted as a pseudo-generating family of trivial cofibrations: they generate trivial cofibrations with fibrant codomain. Finally, the classical proof of completeness for Tietze transformations proceeds by constructing some kind of "cospan" of Tietze transformations between two presentations of the same monoid: we explain how to reconstruct this proof by abstract arguments coming based on our model structure.

2 Presentations of monoids

We begin by recalling the notion of presentation of monoid, and construct a category of those.

2.1 Monoid. A *monoid* $(M, \cdot, 1)$ consists of a set M equipped with a binary operation $- \cdot - : M \times M \to M$, the *multiplication*, and an element $1 \in M$, the *unit* such that multiplication is associative and the unit acts as a neutral element: for every $u, v, w \in M$,

$$(u \cdot v) \cdot w = u \cdot (v \cdot w) \qquad\qquad 1 \cdot u = u \cdot 1.$$

A *morphism* $f : M \to N$ between monoids is a function such that $f(1) = 1$ and $f(u \cdot v) = f(u) \cdot f(v)$ for every elements $u, v \in M$. We write **Mon** for the resulting category.

2.2 Free monoid. Given a set X, a *word* is a finite sequence $u = a_1 \dots a_n$ of elements of X. The *length* of such a word is $|u| = n$. We write X^* for the *free monoid* generated by X: its elements are words over X, multiplication uv of two words is their concatenation, and the unit is the empty word, noted 1. This construction extends as a functor from sets to monoids, which is left adjoint to the obvious forgetful functor.

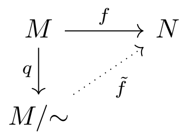
2.3 Congruence. Given a monoid M, a *congruence* is an equivalence relation $\approx$ such that, for every $u, u', v, v' \in M$,

$$u \approx u' \quad \text{and} \quad v \approx v' \qquad \text{implies} \qquad uv \approx u'v'.$$

This is equivalent to requiring that, for every $u, v, v', w \in M$,

$$v \approx v' \qquad \text{implies} \qquad uvw \approx uv'w.$$

2.4 Quotient monoid. Given a binary relation $\sim$ on a monoid M, one can define a *quotient monoid* $M/{\sim}$, equipped with a morphism $q : M \to M/{\sim}$ such that for every morphism $f : M \to N$ such that $u \sim v$ implies $f(u) = f(v)$, for $u, v \in M$, there exists a unique morphism $\tilde{f}$ making the following diagram commute:



Lemma 1. Given a quotient monoid $q : M \to M/{\sim}$, the function q is surjective and such that $q(u) = q(v)$ implies $u \approx v$, for $u, v \in M$, where $\approx$ is the smallest congruence generated by $\sim$.

Concretely, $M/{\sim}$ consists in equivalence classes of elements of M under the congruence $\approx$ generated by $\sim$, with multiplication and unit induced by those of M. This generated congruence can be described as follows:

Lemma 2. Given a binary relation $\sim$ on a monoid M, the congruence $\approx$ it generates is such that $u \approx v$ if and only if there exists $n \in \mathbb{N}$ and $u_i, v_i, v'_i, w_i \in M$, for $1 \leq i \leq n$, such that

- $u = u_1 v_1 w_1$ and $u_n v'_n w_n = v$,
- $v_i \sim v'_i$ or $v'_i \sim v_i$ for $1 \leq i \leq n$,
- $u_i v'_i w_i = u_{i+1} v_{i+1} w_{i+1}$ for $1 \leq i < n$.

2.5 Presentation. A *presentation* $\mathsf{P} = \langle \mathsf{P}_1 \,|\, \mathsf{P}_2 \rangle$ consists of

- a set P_1 of *generators*,
- a set $\mathsf{P}_2 \subseteq \mathsf{P}_1^* \times \mathsf{P}_1^*$ of *relations*.

Such a presentation is *finite* when both the sets P_1 and P_2 are. A relation $(u, v) \in \mathsf{P}_1^*$ is generally denoted by "$u \Rightarrow v$" and we write $\overset{\mathsf{P}}{=}$ for the smallest congruence generated by P_2. A *morphism* $f : \mathsf{P} \to \mathsf{Q}$ between presentations is a function $f : \mathsf{P}_1 \to \mathsf{Q}_1$ such that, for every $u \Rightarrow v \in \mathsf{P}_2$, we have $f(u) \Rightarrow f(v) \in \mathsf{Q}_2$. A *subpresentation* P' of P is a presentation equipped with a morphism $\mathsf{P}' \to \mathsf{P}$ whose underlying function is an inclusion. We write **Pres** for the category of presentations and their morphisms. Note that, by definition, there is a forgetful functor **Pres** $\to$ **Set** sending a presentation P to its set P_1 of generators.

2.6 Presented monoid. The monoid $\overline{\mathsf{P}}$ *presented* by a presentation P is the quotient monoid

$$\overline{\mathsf{P}} = \mathsf{P}_1^* / \mathsf{P}_2$$

i.e., the quotient of the free monoid P_1^* by the congruence $\overset{\mathsf{P}}{=}$ generated by P_2. We often write

$$q^{\mathsf{P}} : \mathsf{P}_1^* \to \overline{\mathsf{P}}$$

for the quotient morphism and, given $u \in \mathsf{P}_1^*$, we write $\overline{u}$ for its equivalence class $q^{\mathsf{P}}(u)$. More generally, we say that a monoid M is *presented* by P when M is isomorphic to $\overline{\mathsf{P}}$, what we sometimes write

$$M \simeq \langle \mathsf{P}_1 \mid \mathsf{P}_2 \rangle.$$

This construction extends as a functor **Pres** $\to$ **Mon**.

Example 3. We have the following presentations:

$$\mathbb{N} \simeq \langle a \mid \rangle \qquad\qquad \mathbb{N} \times \mathbb{N} \simeq \langle a, b \mid ab \Rightarrow ba \rangle$$
$$\mathbb{N}/2\mathbb{N} \simeq \langle a \mid aa \Rightarrow 1 \rangle \qquad\qquad \mathbb{Z} \simeq \langle a, b \mid ab \Rightarrow 1, ba \Rightarrow 1 \rangle.$$

2.7 Standard presentation. To any monoid M, one can associate a presentation $\langle M \rangle$, called the *standard presentation* of M, defined by

$$\mathsf{P}_1 = \{ \underline{a} \mid a \in M \}$$
$$\mathsf{P}_2 = \{ \underline{a}_1 \ldots \underline{a}_n \Rightarrow \underline{b}_1 \ldots \underline{b}_m \mid a_1 \ldots a_n = b_1 \ldots b_m \} .$$

This construction extends as a functor **Mon** $\to$ **Pres**. It can be used to show that any monoid admits at least one presentation:

Lemma 4. Given a monoid M, its standard presentation is a presentation of M: $\overline{\langle M \rangle} \simeq M$.

Lemma 5. The presentation functor is left adjoint to the standard presentation functor

$$\mathbf{Pres} \underset{\langle - \rangle}{\overset{\overline{-}}{\rightleftarrows}} \mathbf{Mon} \quad (\perp)$$

the counit of the adjunction being an isomorphism.

2.8 Reflexive presentations. A presentation P is *reflexive* when for every word $u \in \mathsf{P}_1^*$ there is a relation $u \Rightarrow u \in \mathsf{P}_2$. We write **rPres** for the full subcategory of **Pres** on reflexive presentations.

Lemma 6. The expected forgetful functor admits a left adjoint

$$\mathbf{Pres} \rightleftarrows \mathbf{rPres} \quad (\perp)$$

sending a presentation P to the presentation Q with

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{ u \Rightarrow u \mid u \in \mathsf{P}_1^* \}$$

and **rPres** is equivalent to the Kleisli category of the monad on **Pres** induced by the adjunction.

Lemma 7. The category **rPres** is equivalent to the category whose objects are presentations (not necessarily reflexive) and a morphism $f : \mathsf{P} \to \mathsf{Q}$ is a function $f : \mathsf{P}_1 \to \mathsf{Q}_1$ such that for every relation $u \Rightarrow v \in \mathsf{P}_2$ we have either $f(u) \Rightarrow f(v) \in \mathsf{Q}_2$ or $f(u) = f(v)$.

In the following, when describing concrete examples of reflexive presentations, we generally omit mentioning reflexivity relations (or, alternatively, the description of morphisms given by previous lemma could be considered).

Remark 8. The standard presentation is clearly reflexive and thus the adjunction of Lemma 5 restricts to an adjunction between reflexive presentations and monoids.

3 Tietze equivalences of presentations

3.1 Equivalence between presentations. There is a very natural notion of equivalence of presentations: two presentations can be considered as *equivalent* when they present isomorphic monoids. In order to provide a concrete and amenable description of this relation, Tietze has introduced a family of transformations on presentations which characterize the equivalence. Those were originally formulated in the context of presentations of groups [9].

We begin with a simpler but useful characterization of the equivalence:

Lemma 9. Two presentations P and Q are such that $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$ if and only if there is a cospan of presentations

$$\mathsf{P} \xrightarrow{f} \mathsf{R} \xleftarrow{g} \mathsf{Q}$$

such that the induced monoid morphisms $\overline{f} : \overline{\mathsf{P}} \to \overline{\mathsf{R}}$ and $\overline{g} : \overline{\mathsf{Q}} \to \overline{\mathsf{R}}$ are isomorphisms.

Proof. If there is a cospan as above then we have $\overline{\mathsf{P}} \simeq \overline{\mathsf{R}} \simeq \overline{\mathsf{Q}}$ and P and Q are thus equivalent. Conversely, suppose that P presents the monoid M, i.e., there is an isomorphism $\overline{\mathsf{P}} \to M$. Under the adjunction of Lemma 5, this induces a map $f : \mathsf{P} \to \langle M \rangle$ such that $\overline{f} : \overline{\mathsf{P}} \to \overline{\langle M \rangle} = M$. Similarly, we can construct a map $g : \mathsf{P} \to \langle M \rangle$. □

3.2 Tietze transformation. The *elementary Tietze transformations* are the following transformations producing a new presentation Q from a presentation P:

(T1) *adding a derivable generator*: given a new generator $a \notin \mathsf{P}_1$ and word $u \in \mathsf{P}_1^*$, we define the presentation Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \sqcup \{a\} \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{u \Rightarrow a\}\,,$$

(T2) *adding a derivable relation*: given two words $u, v \in \mathsf{P}_1^*$ such that $u \overset{\mathsf{P}}{=} v$, we define the presentation Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{u \Rightarrow v\}\,.$$

It is easy to see that those transformations preserve the presented monoids:

Lemma 10. Given an elementary Tietze transformation from P to Q, we have an isomorphism $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$.

A *Tietze transformation* from P to Q consists in a finite sequence of presentations $\mathsf{P} = \mathsf{P}^0, \mathsf{P}^1, \mathsf{P}^2, \ldots, \mathsf{P}^n = \mathsf{Q}$ such that for every i with $0 \leq i < n$ there is an elementary Tietze transformation from P^i to P^{i+1}. In this situation, we sometimes write

$$\mathsf{P} \rightsquigarrow \mathsf{Q}$$

Note that contrarily to the usual convention, we do not allow here removing generators or relations.

The transformation (T2) can be replaced by the following four transformations:

(T2r) *reflexivity*: given $u \in \mathsf{P}_1^*$, we define Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{u \Rightarrow u\},$$

(T2s) *symmetry*: given $u, v \in \mathsf{P}_1^*$ such that $u \Rightarrow v \in \mathsf{P}_2$, we define Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{v \Rightarrow u\},$$

(T2t) *transitivity*: given $u, v, w \in \mathsf{P}_1^*$ such that $u \Rightarrow v, v \Rightarrow w \in \mathsf{P}_2$, we define Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{u \Rightarrow w\}.$$

(T2c) *context*: given $u, v, v', w \in \mathsf{P}_1^*$ such that $v \Rightarrow v' \in \mathsf{P}_2$, we define Q by

$$\mathsf{Q}_1 = \mathsf{P}_1 \qquad\qquad \mathsf{Q}_2 = \mathsf{P}_2 \cup \{uvw \Rightarrow uv'w\},$$

The resulting systems are the same in the following sense:

Lemma 11. The following assertions are equivalent: there is a Tietze transformation from P to Q

(i) using (T1) and (T2),

(ii) using (T1), (T2r), (T2s), (T2t) and (T2c).

In the following, unless otherwise mentioned, we use the second set of Tietze transformations which are easier to work with because they are more "atomic".

3.3 Tietze equivalence. A *Tietze equivalence* from P to Q is a finite sequence of presentations $\mathsf{P} = \mathsf{P}^0, \mathsf{P}^1, \mathsf{P}^2, \ldots, \mathsf{P}^n = \mathsf{Q}$ such that for every i with $0 \leq i < n$ there is a Tietze transformation from P^i to P^{i+1} or from P^{i+1} to P^i. Two presentations are *Tietze equivalent* when there is a Tietze equivalence between them. Otherwise said, the Tietze equivalence is the smallest equivalence relation relating any two presentations between which there is an (elementary) Tietze transformation. By Lemma 10 above, Tietze equivalence preserves the presented monoids. For finite presentations, the converse holds:

Theorem 12. Given two finite presentations P and Q, we have $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$ if and only if P and Q are Tietze equivalent.

Proof. The right-to-left implication follows from Lemma 10. For the left-to-right implication, suppose given an isomorphism $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$. For the sake of simplicity we suppose that we actually have $\mathsf{P} = \mathsf{Q}$ and more generally that Tietze equivalent presentation give rise to identical presented monoids (the proof without this assumption can be constructed from the one below by inserting isomorphisms at required places). Given a generator $a \in \mathsf{P}_1$, by Lemma 1, there exists an element $u \in \mathsf{Q}_1^*$ such that $q^{\mathsf{P}}(a) = q^{\mathsf{Q}}(u)$. We write a^{Q} for a choice of such an element. Dually, given $b \in \mathsf{Q}_1$, we write $b^{\mathsf{P}} \in \mathsf{P}_1^*$ for a word such that $q^{\mathsf{P}}(b^{\mathsf{P}}) = q^{\mathsf{Q}}(b)$. We generalize this notation to words $u = a_1 \ldots a_n \in \mathsf{P}_1^*$, by setting $u^{\mathsf{Q}} = a_1^{\mathsf{Q}} \ldots b_n^{\mathsf{Q}}$ (and we define v^{Q} for $v \in \mathsf{Q}_1^*$ similarly). Note that, for $u \in \mathsf{P}_1^*$, we have

$$(u^{\mathsf{Q}})^{\mathsf{P}} \overset{\mathsf{P}}{=} u \tag{1}$$

(and dually). We construct a presentation R by

$$\mathsf{R}_1 = \mathsf{P}_1 \sqcup \mathsf{Q}_1 \qquad\qquad \mathsf{R}_2 = \mathsf{P}_2 \sqcup \mathsf{Q}_2 \sqcup \mathsf{R}_2^{\mathsf{P}} \sqcup \mathsf{R}_2^{\mathsf{Q}}$$

where

$$\mathsf{R}_2^{\mathsf{P}} = \left\{ a^{\mathsf{Q}} \Rightarrow a \,\middle|\, a \in \mathsf{P}_1 \right\} \qquad\qquad \mathsf{R}_2^{\mathsf{Q}} = \left\{ b^{\mathsf{P}} \Rightarrow b \,\middle|\, b \in \mathsf{Q}_1 \right\}$$

We now construct a Tietze transformation from P to R. Dually, we will be able to construct a Transformation from Q to R and we will be able to conclude that P and Q are Tietze equivalent:

$$\mathsf{P} \rightsquigarrow \mathsf{R} \leftsquigarrow \mathsf{Q}.$$

By using Tietze transformations (T1), starting from P, we can add each generator $b \in \mathsf{Q}_1$ along with the relation $b^{\mathsf{P}} \Rightarrow b$, thus obtaining a transformation

$$\mathsf{P} = \langle \mathsf{P}_1 \,|\, \mathsf{P}_2 \rangle \rightsquigarrow \mathsf{P}' = \langle \mathsf{P}_1, \mathsf{Q}_1 \,|\, \mathsf{P}_2, \mathsf{R}_2^{\mathsf{Q}} \rangle.$$

Note that, given a word $u \in \mathsf{Q}_1^*$, we have $u^{\mathsf{P}} \overset{\mathsf{P}'}{=} u$. Therefore, given $a \in \mathsf{P}_1$, we have $a^{\mathsf{Q}} \overset{\mathsf{P}'}{=} (a^{\mathsf{Q}})^{\mathsf{P}} \overset{\mathsf{P}'}{=} a$ by (1). By using Tietze transformations (T2) we can add each derivable relation $a^{\mathsf{Q}} \Rightarrow a$ to P' thus reaching the presentation R:

$$\mathsf{P} \rightsquigarrow \mathsf{P}' \rightsquigarrow \mathsf{R}.$$

□

Remark 13. The proof above uses Tietze transformations (T1) and (T2). The proof can be performed by using the other set of transformations given by Lemma 11, at the cost of having to take a slightly bigger R.

Remark 14. Note that the theorem only holds for finite presentations. We will see however that it can easily be generalized to presentations of arbitrary cardinality by allowing the Tietze transformations to add *sets* of derivable generators and *sets* of derivable relations (instead of only one), see Theorem 59.

The proof of Theorem 12 constructs a "cospan" of Tietze transformations. We will see that it can be constructed by using tools coming from model categories.

3.4 An example. Consider the presentations

$$\langle a \mid \rangle \qquad \text{and} \qquad \langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb \rangle.$$

Both present the additive monoid $\mathbb{N}$, and indeed there is a Tietze equivalence between them:

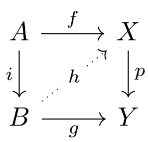
$$\begin{aligned}
\langle a \mid \rangle &\to \langle a, b \mid 1 \Rightarrow b \rangle && \text{(T1)}\\
&\to \langle a, b \mid 1 \Rightarrow b, b \Rightarrow bb \rangle && \text{(T2c)}\\
&\to \langle a, b \mid 1 \Rightarrow b, b \Rightarrow bb, 1 \Rightarrow bb \rangle && \text{(T2t)}\\
&\to \langle a, b \mid 1 \Rightarrow b, b \Rightarrow bb, 1 \Rightarrow bb, bb \Rightarrow b \rangle && \text{(T2s)}\\
&\leftarrow \langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb, bb \Rightarrow b \rangle && \text{(T2t)}\\
&\leftarrow \langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb \rangle && \text{(T2s)}
\end{aligned}$$

Also note that both presentations are "minimal": there is no way to remove a derivable generator or a relation without changing the presented monoid. In particular, starting from the second presentation, we have to add relations first in order to be able remove the generator b and all the relations.

4 Model categories

In this section, we recall elementary definitions and facts about model categories which we will use in the following and refer the reader to classical textbooks for details [5].

4.1 Lifting properties. Suppose fixed a category. A morphism $p : X \to Y$ has the *right lifting property*, or *rlp*, with respect to a morphism $i : A \to B$ when for every morphisms $f : A \to X$ and $g : B \to Y$ such that $p \circ f = g \circ i$ there exists a morphism $h : B \to X$ making the following diagram commute:



In this situation, we also say that i has the *left lifting property*, or *llp*, with respect to f, and write $i \boxslash p$. Given two classes $\mathcal{L}$ and $\mathcal{R}$, we write $\mathcal{L} \boxslash \mathcal{R}$ whenever $i \boxslash p$ for every $i \in \mathcal{L}$ and $p \in \mathcal{R}$. We also write $\mathcal{L}^{\boxslash}$ (resp. ${}^{\boxslash}\mathcal{R}$) for the class of morphism with the rlp (resp. llp) with respect to $\mathcal{L}$ (resp. $\mathcal{R}$).

Lemma 15. Given classes $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{R}$ and $\mathcal{R}'$ of morphisms,

$$\begin{array}{lll}
\mathcal{L} \subseteq {}^{\boxslash}(\mathcal{L}^{\boxslash}) & ({}^{\boxslash}(\mathcal{L}^{\boxslash}))^{\boxslash} = \mathcal{L}^{\boxslash} & \mathcal{L} \subseteq \mathcal{L}' \text{ implies } \mathcal{L}^{\boxslash} \supseteq \mathcal{L}'^{\boxslash},\\
\mathcal{R} \subseteq ({}^{\boxslash}\mathcal{R})^{\boxslash} & {}^{\boxslash}(({}^{\boxslash}\mathcal{R})^{\boxslash}) = {}^{\boxslash}\mathcal{R} & \mathcal{R} \subseteq \mathcal{R}' \text{ implies } {}^{\boxslash}\mathcal{R} \supseteq {}^{\boxslash}\mathcal{R}'.
\end{array}$$

Lemma 16. We suppose the category cocomplete. A class of the form $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ contains isomorphisms and is closed under

– copropducts: for any family $(i_k : A_k \to B_k)_{k \in K}$ of morphisms in $\mathcal{L}$, the morphism

$$\coprod_{k \in K} i_k : \coprod_{k \in K} A_k \to \coprod_{k \in K} B_k$$

is also in the class,

– pushouts: for any morphism $i : A \to B$ in $\mathcal{L}$ and morphism $f : A \to A'$, for any pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow{\scriptstyle j} \\ B & \longrightarrow & B' \end{array}$$

the morphism j also belongs to $\mathcal{L}$,

– countable compositions: for any diagram

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} \cdots \tag{2}$$

consisting of morphisms $f_k : A_k \to A_{k+1}$ in $\mathcal{L}$ for $k \in \mathbb{N}$, the canonical morphism

$$A_0 \to \operatorname{colim}_k A_k$$

also belongs to $\mathcal{L}$,

– retracts: given a morphism $i : A \to B$ and two retracts $r \circ s = \mathrm{id}_{A'}$ and $r' \circ s' = \mathrm{id}_{B'}$, any morphism $j : A' \to B'$ for which there is a commutative diagram

$$\begin{array}{ccccc} & & \mathrm{id}_{A'} & & \\ A' & \xrightarrow[s]{} & A & \xrightarrow[r]{} & A' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ B' & \xrightarrow{s'} & B & \xrightarrow{r'} & B' \\ & & \mathrm{id}_{B'} & & \end{array} \tag{3}$$

also belongs to $\mathcal{L}$.

Dually, any class for the form $\mathcal{L}^{\boxslash}$ contains isomorphisms and is closed under products, pullbacks, countable compositions and retracts.

Given a class $\mathcal{I}$ of morphisms, the class $\mathcal{I}$-cell of *$\mathcal{I}$-cellular extensions* is defined as the smallest class of morphisms closed under sums, pushouts and countable compositions (note that we do not require closure under retracts).

Lemma 17. A morphism is an $\mathcal{I}$-cellular extension if and only if it is a composite of pushouts of sums of elements of $\mathcal{I}$.

Lemma 18. Given a class $\mathcal{I}$ of morphisms, the class of $\mathcal{I}$-cellular extensions is included in ${}^{\boxslash}(\mathcal{I}^{\boxslash})$.

Proof. By Lemma 15, we have $\mathcal{I}$ included in ${}^{\boxslash}(\mathcal{I}^{\boxslash})$ and, by Lemma 16, this class is closed under sums, pushouts and countable compositions. ☐

Lemma 19 (Retract lemma). Given a factorization $f = p \circ i$ such that $f \boxslash p$, f is a retract of i. Dually, given a factorization $f = p \circ i$ such that $i \boxslash f$, f is a retract of p.

Proof. Since $f \boxslash p$, we have a map h such that

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ Z & = & Z \end{array}$$

and the map f is thus a retract of i:

$$\begin{array}{ccccc} X & = & X & = & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[h]{} & Y & \xrightarrow[p]{} & Z \end{array}$$

as claimed. □

4.2 Weak factorization system. A *weak factorization system* on a category is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that

– every morphism f factors as $f = p \circ i$ with $i \in \mathcal{L}$ and $p \in \mathcal{R}$,

– $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Remark 20. From Lemma 16 and Lemma 19, the second condition can be equivalently be replaced by the two following conditions

– $\mathcal{L} \boxslash \mathcal{R}$,

– the classes $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

One of the main technique in order to construct weak factorization systems is due to the following proposition [5, Section 2.1.2]. The notion of locally finitely presentable category is recalled in Section 5.5.

Proposition 21 (Small object argument). Suppose that the category is cocomplete and locally finitely presentable. For any class $\mathcal{I}$ of morphisms, $({}^{\boxslash}(\mathcal{I}^{\boxslash}), \mathcal{I}^{\boxslash})$ is a weak factorization system. Moreover, every morphism f factors as $f = p \circ i$ where $i \in {}^{\boxslash}(\mathcal{I}^{\boxslash})$ is an $\mathcal{I}$-cellular extension and $p \in \mathcal{I}^{\boxslash}$. Moreover, every element of ${}^{\boxslash}(\mathcal{I}^{\boxslash})$ is a retract of an $\mathcal{I}$-cellular extension.

4.3 Model category. A *model category* is a category equipped with three classes of morphisms

– $\mathcal{C}$: cofibrations,

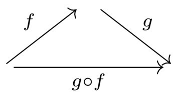
– $\mathcal{W}$: weak equivalences,

– $\mathcal{F}$: fibrations

such that

– the category is complete and cocomplete,

– weak equivalences satisfy the 2-out-of-3 property: given a diagram



if two morphisms belong to $\mathcal{W}$ then so does the third,

– $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ forms a weak factorization system,

– $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ forms a weak factorization system.

An object X is *cofibrant* when the initial morphism $\varnothing \to X$ is a cofibration, and *fibrant* when the terminal morphism $X \to 1$ is a fibration.

From previous section, we can expect that the weak factorization system can be generated as lifting completions of some classes. Indeed, many model categories are *cofibrantly generated* (also sometimes called *combinatorial* since we work here with locally presentable categories) [5, Theorem 2.1.19]:

Proposition 22. In a locally presentable complete and cocomplete category, suppose given a class $\mathcal{W}$ of morphisms satisfying the 2-out-of-3 property and two sets $\mathcal{I}$ and $\mathcal{J}$ of morphisms such that the inclusions

$$\mathcal{I}^{\boxslash} \subseteq \mathcal{J}^{\boxslash} \cap \mathcal{W} \qquad\qquad {}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq {}^{\boxslash}(\mathcal{I}^{\boxslash}) \cap \mathcal{W}$$

hold, one of them being an equality. Then we have a model category with $\mathcal{W}$ as weak equivalences, ${}^{\boxslash}(\mathcal{I}^{\boxslash})$ as cofibrations and $\mathcal{J}^{\boxslash}$ as fibrations. In this case, the elements of $\mathcal{I}$ as $\mathcal{J}$ are respectively called *generating cofibrations* and *generating trivial cofibrations*.

5 A model structure on reflexive presentations

Our aim is to construct a model structure on the category of reflexive presentations where weak equivalences correspond to presenting isomorphic categories and trivial cofibrations are Tietze transformations. The general strategy here is to use Proposition 22 and thus to satisfy all the required hypothesis: in particular, we want to show the equality $\mathcal{I}^{\boxslash} = \mathcal{J}^{\boxslash} \cap \mathcal{W}$. Unless otherwise mentioned, all the presentations considered in this section are supposed to be reflexive; the reason for this shall be discussed in Section 8.1. We first study some of the properties of the category of reflexive presentations.

5.1 Colimits. The category **rPres** has coproducts. Namely, given two presentations P and Q, their coproduct $\mathsf{P} \sqcup \mathsf{Q}$ is given by

$$(\mathsf{P} \sqcup \mathsf{Q})_1 = \mathsf{P}_1 \sqcup \mathsf{Q}_1 \qquad\qquad (\mathsf{P} \sqcup \mathsf{Q})_2 = \mathsf{P}_2 \sqcup \mathsf{Q}_2$$

and the argument generalizes to show that the category has small coproducts. In particular, the initial object $\varnothing$ is the empty presentation, with $\varnothing_1 = \emptyset$ and $\varnothing_2 = \emptyset$. Suppose given two morphisms of presentations

$$\mathsf{P} \overset{f}{\underset{g}{\rightrightarrows}} \mathsf{Q}$$

Their coequalizer is the presentation R whose set of generators is the coequalizer

$$\mathsf{P}_1 \underset{g}{\overset{f}{\rightrightarrows}} \mathsf{Q}_1 \overset{h}{\dashrightarrow} \mathsf{R}_1$$

i.e., the quotient set $\mathsf{R}_1 = \mathsf{Q}_1/{\sim}$ under the smallest equivalence relation such that $f(a) \sim f(b)$ for $a \in \mathsf{P}_1$, the function h being the quotient map, and the set of relations is

$$\mathsf{R}_2 = \{h^*(u) \Rightarrow h^*(v) \,|\, u \Rightarrow v \in \mathsf{Q}_1\}\,.$$

The category is thus cocomplete. In particular, the pushout of a diagram

$$\mathsf{Q}^1 \xleftarrow{f^1} \mathsf{P} \xrightarrow{f^2} \mathsf{Q}^2$$

is the presentation R whose set R_1 of generators is the pushout of the underlying sets of generators, with cocoone maps $h^1 : \mathsf{Q}^1 \to \mathsf{R}$ and $h^2 : \mathsf{Q}^2 \to \mathsf{R}$, and relations

$$R_2 = \left\{h^1(u) \Rightarrow h^1(v) \,\middle|\, u \Rightarrow v \in \mathsf{Q}^1_2\right\} \cup \left\{h^2(u) \Rightarrow h^2(v) \,\middle|\, u \Rightarrow v \in \mathsf{Q}^2_2\right\}.$$

Note that the forgetful functor **rPres** $\to$ **Set** preserves colimits.

5.2 Limits. The product $\mathsf{P} \times \mathsf{Q}$ of two reflexive presentations P and Q has generators $(\mathsf{P} \times \mathsf{Q})_1 = \mathsf{P}_1 \times \mathsf{Q}_1$ and the set $(\mathsf{P} \times \mathsf{Q})_2$ of relations is

$$\left\{(a_1, a'_1) \ldots (a_m, a'_m) \Rightarrow (b_1, b'_1) \ldots (b_n, b'_n) \,\middle|\, \begin{array}{l} a_1 \ldots a_m \Rightarrow b_1 \ldots b_n \in \mathsf{P}_2 \\ a'_1 \ldots a'_m \Rightarrow b'_1 \ldots b'_n \in \mathsf{Q}_2 \end{array}\right\}.$$

This generalizes to small products. In particular, the terminal presentation 1 has one generator a and all relations of the form $a^m \Rightarrow a^n$ for $m, n \in \mathbb{N}$. Given two morphisms of polygraphs

$$\mathsf{P} \underset{g}{\overset{f}{\rightrightarrows}} \mathsf{Q}$$

their equalizer R is given by

$$\mathsf{R}_1 = \{a \in \mathsf{P}_1 \,|\, f(a) = g(a)\}$$

i.e., this is the equalizer of the underlying sets, and relations are

$$\mathsf{R}_2 = \{u \Rightarrow v \in \mathsf{P}_2 \,|\, f^*(u) = g^*(u) \text{ and } f^*(v) = g^*(u)\}\,.$$

The category is thus complete and the forgetful functor **rPres** $\to$ **Set** preserves limits.

5.3 Monomorphisms. A monomorphism $f : \mathsf{P} \to \mathsf{Q}$ is a morphism whose underlying function $f : \mathsf{P}_1 \to \mathsf{Q}_1$ is injective, i.e., the forgetful functor **rPres** $\to$ **Set** reflects monomorphisms. In this sense, the monomorphisms of presentations inherit the properties of those of the categories of sets. For instance,

Lemma 23. In **rPres**, monomorphisms are stable under coproducts, pushouts and countable compositions.

Proof. The forgetful functor to sets preserves coproducts, pushouts and countable compositions, and reflects monomorphisms. ☐

Remark 24. These stability conditions are not generally true in a category. As a counter-example, in the category of commutative rings, the inclusion $i : \mathbb{Z} \to \mathbb{Q}$ is a mono, but the sum (which is here the tensor product, and corresponds to the usual tensor product of $\mathbb{Z}$-modules)

$$\mathrm{id}_{\mathbb{Z}/2} \otimes i : \mathbb{Z}/2 = \mathbb{Z}/2 \otimes \mathbb{Z} \to \mathbb{Z}/2 \otimes \mathbb{Q} = 1$$

is not a mono. It is however the case that monomorphisms are stable under pushout in a topos (and, more generally, an adhesive category).

5.4 Epimorphisms. Similarly, an epimorphism $f : \mathsf{P} \to \mathsf{Q}$ is a morphism whose underlying function $\mathsf{P}_1 \to \mathsf{Q}_1$ is a surjection.

5.5 Local presentability. We refer to [1] for a detailed presentations of the notions introduced here. An object X of a category $\mathcal{C}$ is *finitely presentable* when the representable functor

$$\mathrm{Hom}(X, -) : \mathcal{C} \to \mathbf{Set}$$

preserves filtered limits: this means that for a diagram $(Y_i)_{i \in I}$ indexed by a filtered category I, the canonical morphism

$$\mathrm{colim}_i \, \mathrm{Hom}(X, Y_i) \to \mathrm{Hom}(X, \mathrm{colim}_i \, Y_i)$$

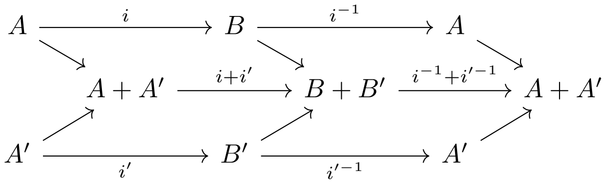
is an isomorphism. In particular, finitely presentable presentations objects are precisely the finite presentations.

A locally small category $\mathcal{C}$ is *locally finitely presentable* when it is cocomplete and there is a set of finitely presentable objects such that every object of $\mathcal{C}$ is a filtered colimit of objects in this set. In the case of the category of presentations, every presentation is the filtered colimit of its finite subpresentations, and the category **rPres** is thus locally finitely presentable (and **Mon** as well).

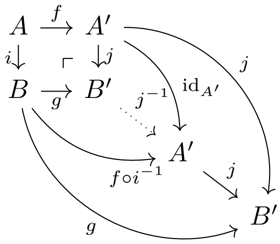
5.6 Weak equivalences. We write $\mathcal{W}$ for the class of morphisms $f : \mathsf{P} \to \mathsf{Q}$ such that the induced morphism $\overline{f} : \overline{\mathsf{P}} \to \overline{\mathsf{Q}}$ is an isomorphism. Many of the properties of isomorphisms are thus reflected on weak equivalences:

Lemma 25. The class $\mathcal{W}$ satisfies the 2-out-of-3 property and is closed under coproducts, pushouts, countable compositions and retracts.

Proof. The class of isomorphisms in any category satisfies the 2-out-of-3 property. Isormorphisms are closed under sums



and pushouts



Consider a countable composition of isomorphisms $f_i : A_i \to A_{i+1}$ as in (2). There is a cocone on A_0 consisting of the morphisms

$$f_0^{-1} \circ f_1^{-1} \circ \ldots \circ f_{i-1}^{-1} : A_i \to A_0$$

which is easily seen to be universal and the composite is thus (isomorphic to) id_{A_0}. Consider a retract j of an isomorphism i as in (3). We claim that the morphism $j' = s \circ i^{-1} \circ r'$ is the inverse of j. One has

$$\begin{aligned} j' \circ j &= s \circ i^{-1} \circ r' \circ j \\ &= s \circ i^{-1} \circ i \circ r \\ &= s \circ r \\ &= \mathrm{id}_{A'} \end{aligned} \qquad \begin{aligned} j \circ j' &= j \circ s \circ i^{-1} \circ r' \\ &= s' \circ i \circ i^{-1} \circ r' \\ &= s' \circ r' \\ &= \mathrm{id}_{B'} \end{aligned}$$

□

5.7 Generating cofibrations. Consider the presentation

$$\mathsf{G} = \langle a \mid \rangle.$$

Given $m, n \in \mathbb{N}$, we introduce notations for the following presentations:

$$\mathsf{G}^n = \langle a_1, \ldots, a_n \mid \rangle \qquad \mathsf{R}^{m,n} = \langle a_1, \ldots, a_{m+n} \mid a_1 \ldots a_m \Rightarrow a_{m+1} \ldots a_{m+n} \rangle.$$

We write $\mathcal{I}$ for the class of morphisms, called *generating cofibrations*, consisting of the obvious inclusions of presentations

$$g : \varnothing \hookrightarrow \mathsf{G} \qquad\qquad r^{m,n} : \mathsf{G}^{m+n} \hookrightarrow \mathsf{R}^{m,n}$$

for some $m, n \in \mathbb{N}$.

5.8 Cofibrations. We write $\mathcal{C} = {}^{\boxslash}(\mathcal{I}^{\boxslash})$ for the class of morphisms whose elements are called *cofibrations*. Note that, given a presentation P, the pushouts

are respectively the polygraph obtained from P by adding a generator and a relation (between two words of P_1^* specified by f).

Lemma 26. Every presentation P is cofibrant, in the sense that the initial morphism $\varnothing \hookrightarrow \mathsf{P}$ is a cofibration.

Proof. By Proposition 21, it is enough to show that the initial morphism $\varnothing \hookrightarrow \mathsf{P}$ can be obtained as a composite of pushouts of generating cofibrations. Given a relation $u \Rightarrow v \in \mathsf{P}_2$, we have a canonical inclusion

$$\mathsf{G}^{|u|+|v|} \xrightarrow{r^{|u|+|v|}} \mathsf{R}^{|u|,|v|}$$

and a canonical inclusion

$$\mathsf{G}^{|u|+|v|} \longrightarrow \coprod_{a \in \mathsf{P}_1} \mathsf{G}.$$

By summing those morphisms over relations $(u, v) \in \mathsf{P}_2$, and post-composing (the vertical morphism) with the codiagonal

$$\coprod_{(u,v) \in \mathsf{P}_2} \coprod_{a \in \mathsf{P}_1} \mathsf{G} \longrightarrow \coprod_{a \in \mathsf{P}_1} \mathsf{G},$$

we obtain a diagram

$$\begin{array}{ccc} \coprod_{(u,v)\in\mathsf{P}_2} \mathsf{G}^{|u|+|v|} & \longrightarrow & \coprod_{(u,v)\in\mathsf{P}_2} \mathsf{R}^{|u|,|v|} \\ \downarrow & & \\ \coprod_{a\in\mathsf{P}_1} \mathsf{G} & & \end{array}$$

whose pushout is precisely P. Finally, we consider the composite of morphism

$$\varnothing \longrightarrow \coprod_{a \in \mathsf{P}_1} \mathsf{G} \longrightarrow \mathsf{P}$$

where the second morphism is constructed in the cocone of the pushout. Intuitively, this composite expresses the fact that any presentation can be constructed from the empty one by first adding all the generators, and then adding all the relations. □

The construction given in the above proof easily generalizes to show:

Lemma 27. Any monomorphism $f : \mathsf{P} \to \mathsf{Q}$ is a cofibration (and, in fact, an $\mathcal{I}$-cellular extension).

Conversely, one has:

Lemma 28. Cofibrations are monomorphisms.

Proof. The generating cofibrations are monomorphisms. Moreover, monomorphisms are closed under coproducts, under pushouts and countable compositions by Lemma 23. By Proposition 21, cofibrations are thus retracts of monomorphisms. We conclude using the fact that monomorphisms are closed under retracts. Namely, suppose given a retract j of a monomorphism i, as in (3), and two morphisms h_1, h_2 such that $j \circ h_1 = j \circ h_2$, we have

$$\begin{aligned} j \circ h_1 &= j \circ h_2 \\ s' \circ j \circ h_1 &= s' \circ j \circ h_2 \\ i \circ s \circ h_1 &= i \circ s \circ h_2 \\ s \circ h_1 &= s \circ h_2 \\ r \circ s \circ h_1 &= r \circ s \circ h_2 \\ h_1 &= h_2 \end{aligned}$$

and we conclude. □

Corollary 29. The class $\mathcal{C}$ of cofibrations is the class of monomorphisms in **rPres**.

5.9 Trivial fibrations. The morphisms in the class $\mathcal{I}^{\boxslash}$ are called *trivial fibrations*. From the lifting property with respect to the generators we immediately deduce,

Lemma 30. The morphisms $f : \mathsf{P} \to \mathsf{Q}$ in $\mathcal{I}^{\boxslash}$ are those

- whose underlying function $f : \mathsf{P}_1 \to \mathsf{Q}_1$ is surjective, and
- such that for every $u, v \in \mathsf{P}_1^*$, $f^*(u) \Rightarrow f^*(v) \in \mathsf{Q}_2$ implies $u \Rightarrow v \in \mathsf{P}_2$.

Lemma 31. Trivial fibrations are weak equivalences: $\mathcal{I}^{\boxslash} \subseteq \mathcal{W}$.

Proof. Since $f : \mathsf{P}_1 \to \mathsf{Q}_1$ is surjective, we have that $\overline{f} : \overline{\mathsf{P}} \to \overline{\mathsf{Q}}$ is also surjective. We have to show that it is also injective in order to conclude. Suppose given $u, v \in \mathsf{P}_1^*$ such that $f^*(u) \overset{\mathsf{Q}}{=} f^*(v)$: we have a sequence

$$f^*(u) = w_0 \Leftrightarrow w_1 \Leftrightarrow \ldots \Leftrightarrow w_n = f^*(v)$$

where, for $0 \leq i < n$, there is a decomposition of w_i and w_{i+1} as

$$w_i = t_i u_i v_i \quad \text{and} \quad w_{i+1} = t'_i u'_i v'_i \qquad \text{with} \qquad u_i \Rightarrow u'_i \in \mathsf{Q}_2 \quad \text{or} \quad u_i \Leftarrow u'_i \in \mathsf{Q}_2.$$

Moreover, since Q is reflexive, we can always suppose that this sequence is non-empty, i.e., $n > 0$: we can replace the empty sequence by the reflexivity relation $f^*(u) \Rightarrow f^*(u)$. By surjectivity, for $0 \leq i \leq n$, there are words t_i^{P}, u_i^{P}, v_i^{P}, $t_i'^{\mathsf{P}}$, $u_i'^{\mathsf{P}}$, $v_i'^{\mathsf{P}}$ in P_1^* whose image under f is respectively t_i, u_i, v_i, t'_i, u'_i, v'_i, and we may moreover assume $t_0^{\mathsf{P}} u_0^{\mathsf{P}} v_0^{\mathsf{P}} = u$ and $t_{n-1}'^{\mathsf{P}} u_{n-1}'^{\mathsf{P}} v_{n-1}'^{\mathsf{P}} = v$. Finally, since f is a trivial fibration, we have $u_i \Rightarrow u'_i$ or $u_i \Leftarrow u'_i$ and we conclude that $u \overset{\mathsf{P}}{=} v$. ☐

From the results of Section 5.4, one has:

Lemma 32. Every trivial fibration is an epimorphism.

5.10 Trivial cofibrations. The class of *trivial cofibrations* is $\mathcal{C} \cap \mathcal{W}$ and consists of monomorphisms $f : \mathsf{P} \to \mathsf{Q}$ such that the induced morphism of monoids $\overline{f} : \overline{\mathsf{P}} \to \overline{\mathsf{Q}}$ is an isomorphism.

Lemma 33. A morphism $f : \mathsf{P} \to \mathsf{Q}$ is a trivial cofibration when

- f is a monomorphism,
- for every $a \in \mathsf{Q}_1$, there exists $u \in \mathsf{P}_1^*$ such that $f(u) \overset{\mathsf{Q}}{=} a$,
- for $u, v \in \mathsf{P}_1^*$ such that $f(u) \overset{\mathsf{Q}}{=} f(v)$, we have $u \overset{\mathsf{P}}{=} v$.

Proof. Suppose that f is a trivial cofibration. Since f is a cofibration, it is a monomorphism by Lemma 28. Given $a \in \mathsf{Q}_1$, since $\overline{f}$ is surjective there is $u \in \mathsf{P}_1^*$ such that $f(\overline{u}) = \overline{a}$, and we have $f(u) \overset{\mathsf{Q}}{=} a$. Given $u, v \in \mathsf{P}_1^*$ such that $f(u) \overset{\mathsf{Q}}{=} f(v)$, we have $\overline{f}(\overline{u}) = \overline{f}(\overline{v})$, thus $\overline{u} = \overline{v}$ in $\overline{\mathsf{P}}$ since $\overline{f}$ injective, and finally $u \overset{\mathsf{P}}{=} v$.

Conversely, suppose given a monomorphism $f : \mathsf{P} \to \mathsf{Q}$. Given $a \in \mathsf{Q}_1$, by hypothesis, there exists $u_a \in \mathsf{P}_1^*$ such that $\overline{f}(\overline{u}_a) = \overline{a}$. Therefore, given $\overline{v} \in \overline{\mathsf{Q}}$, for some $v = a_1 \dots a_n \in \mathsf{Q}_1^*$, we have

$$\overline{f}(\overline{u}_{a_1} \dots \overline{u}_{a_n}) = \overline{f}(\overline{u}_{a_1}) \dots \overline{f}(\overline{u}_{a_n}) = \overline{a}_1 \dots \overline{a}_n = \overline{v}$$

and $\overline{f}$ is thus surjective. Suppose given $u, v \in \mathsf{P}_1^*$ such that $\overline{f}(\overline{u}) = \overline{f}(\overline{v})$: we have $f(u) \overset{\mathsf{Q}}{=} f(v)$, thus $u \overset{\mathsf{P}}{=} v$ and finally $\overline{u} = \overline{v}$. □

Lemma 34. The class of trivial cofibrations satisfies ${}^\boxslash((\mathcal{C} \cap \mathcal{W})^\boxslash) = \mathcal{C} \cap \mathcal{W}$.

Proof. By Lemma 16 and Lemma 25, the class $\mathcal{C} \cap \mathcal{W}$ is closed under sums, pushouts, countable compositions and retracts. We conclude by Proposition 21. □

5.11 Fibrations. The class $\mathcal{F}$ of *fibrations* is determined by the two other classes: should there be a model structure, it is necessarily $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^\boxslash$. An explicit description of fibrant objects is given by Lemma 52 and Lemma 45.

5.12 A model structure. Finally, we have all the ingredients required to construct a model structure.

Theorem 35. There is a model structure on the category **rPres** of reflexive presentations with $\mathcal{W}$ as weak equivalences, $\mathcal{C} = {}^\boxslash(\mathcal{I}^\boxslash)$ as cofibrations and $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^\boxslash$ as fibrations.

Proof. We apply Proposition 22, with $\mathcal{J} = \mathcal{C} \cap \mathcal{W}$. The 2-out-of-3 property for $\mathcal{W}$ was shown in Section 5.6. We first show $\mathcal{I}^\boxslash \subseteq \mathcal{J}^\boxslash \cap \mathcal{W}$. We have $\mathcal{J} = \mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}$ thus $\mathcal{C}^\boxslash \subseteq \mathcal{J}^\boxslash$. Moreover, by Lemma 15 and Lemma 31, we have $\mathcal{C}^\boxslash \subseteq \mathcal{W}$. Thus $\mathcal{C}^\boxslash \subseteq \mathcal{J} \cap \mathcal{W}$. Finally, by Lemma 34, we have

$${}^\boxslash(\mathcal{J}^\boxslash) = \mathcal{J} = \mathcal{C} \cap \mathcal{W} = {}^\boxslash(\mathcal{I}^\boxslash) \cap \mathcal{W}$$

which concludes the proof. □

In fact, the situation considered here can be axiomatized as in the following theorem, due to Smith, see [2, Theorem 1.7]:

Theorem 36. In a locally presentable category, suppose given a subcategory $\mathcal{W}$ and a set $\mathcal{I}$ of morphisms such that

- $\mathcal{W}$ is closed under retracts and has the 2-out-of-3 property,
- $\mathcal{I}^\boxslash \subseteq \mathcal{W}$,
- ${}^\boxslash(\mathcal{I}^\boxslash) \cap \mathcal{W}$ is closed under pushouts and transfinite compositions,
- $\mathcal{W}$ satisfies the solution set condition at $\mathcal{I}$.

Then there is a cofibrantly generated model structure with ${}^\boxslash(\mathcal{I}^\boxslash)$ as cofibrations, $\mathcal{W}$ as weak equivalences and $({}^\boxslash(\mathcal{I}^\boxslash) \cap \mathcal{W})^\boxslash$ as fibrations.

We do not detail the solution set condition and simply note that it is always satisfied for small categories (which is the case of the categories considered here).

5.13 A Quillen functor. The category **Mon** can canonically be equipped with the *trivial model structure* where weak equivalences are isomorphisms and every morphism is both fibrant and cofibrant. The presentation functor **rPres** $\to$ **Mon** described in Section 2.6 is a left adjoint (Lemma 5 and Remark 8) which trivially preserves cofibrations and trivial cofibrations, and is thus a Quillen functor. Moreover, this functor reflects weak equivalences and, given a presentation P, the counit $\mathsf{P} \to \langle\overline{\mathsf{P}}\rangle$ of the adjunction is a weak equivalence: by [5, Corollary 1.3.16], the presentation functor is thus a Quillen equivalence. By [5, Proposition 1.3.13], this means that the derived functor induces, as expected, an equivalence of categories between the localization of **rPres** under weak equivalences and the one of **Mon** (which is **Mon** itself):

$$\mathrm{Ho}(\mathbf{rPres}) \cong \mathrm{Ho}(\mathbf{Mon}) \simeq \mathbf{Mon}.$$

6 Tietze transformations as trivial cofibrations

In Section 6.1 below, we introduce a class $\mathcal{J}$ of morphisms of reflexive presentations such that pushouts of morphisms in this class corresponds to elementary Tietze transformations. Contrarily to what one could expect, this family does not generate all trivial cofibrations: we have a strict inclusion ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subsetneq \mathcal{C} \cap \mathcal{W}$. However, we show that the two classes coincide for morphisms with fibrant codomain: we thus say that the class $\mathcal{J}$ is pseudo-generating, following the terminology of Simpson [7, Section 8.7].

6.1 Pseudo-generating trivial cofibrations. We write $\mathcal{J}$ for the class of morphisms of **rPres**, called *pseudo-generating trivial cofibrations*

$$\begin{gathered}
\langle a_1, \ldots, a_m \,|\, \rangle \hookrightarrow \langle a_1, \ldots, a_m, a_{m+1} \,|\, u \Rightarrow a_{m+1} \rangle \\
\langle a_1, \ldots, a_m \,|\, \rangle \hookrightarrow \langle a_1, \ldots, a_m \,|\, u \Rightarrow u \rangle \\
\langle a_1, \ldots, a_{m+n} \,|\, u \Rightarrow v \rangle \hookrightarrow \langle a_1, \ldots, a_{n+m} \,|\, u \Rightarrow v, v \Rightarrow u \rangle \\
\langle a_1, \ldots, a_{m+n+p} \,|\, u \Rightarrow v, v \Rightarrow w \rangle \hookrightarrow \langle a_1, \ldots, a_{n+m+p} \,|\, u \Rightarrow v, v \Rightarrow w, u \Rightarrow w \rangle \\
\langle a_1, \ldots, a_{m+n+p+q} \,|\, u \Rightarrow v \rangle \hookrightarrow \langle a_1, \ldots, a_{m+n+p+q} \,|\, wuw' \Rightarrow wvw' \rangle
\end{gathered}$$

for some $m, n, p \in \mathbb{N}$ with

$$\begin{aligned}
u &= a_1 \ldots a_m & \qquad w &= a_{m+n+1} \ldots a_{m+n+p} \\
v &= a_{m+1} \ldots a_{m+n} & \qquad w' &= a_{m+n+p+1} \ldots a_{m+n+p+q}
\end{aligned}$$

Lemma 37. Given a pseudo-generating cofibrations $j : \mathsf{P} \to \mathsf{Q}$ and a morphism of presentations $f : \mathsf{P} \to \mathsf{P}'$, consider the pushout $j' : \mathsf{P}' \to \mathsf{Q}'$ of j along f:

$$\begin{array}{ccc}
\mathsf{P} & \xrightarrow{f} & \mathsf{P}' \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\
\mathsf{Q} & \longrightarrow & \mathsf{Q}'
\end{array}$$

then there is an elementary Tietze transformation from P' to Q', and conversely every elementary Tietze transformation arises in this way.

Proof. Pushout of the five kinds of morphisms in $\mathcal{J}$ precisely give rise to the four kinds of Tietze transformations (T1), (T2r), (T2s), (T2t) and (T2c). $\square$

We are thus tempted to call *generalized Tietze transformation* a morphism in $\mathcal{J}$-cell. In particular, every element of $\mathcal{J}$ is itself a Tietze transformation and thus, by Theorem 12,

Lemma 38. Generating trivial cofibrations are weak equivalences: $\mathcal{J} \subseteq \mathcal{W}$.

Moreover, those morphisms are monomorphisms and thus, by Lemma 27,

Lemma 39. The pseudo-generating trivial cofibrations are cofibrations: $\mathcal{J} \subseteq {}^{\boxslash}(\mathcal{I}^{\boxslash})$.

Remark 40. By general properties [5, Proposition 2.1.18], we have that morphisms in ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ are retracts of Tietze transformations. We do not know whether morphisms in ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ precisely Tietze transformations or not.

6.2 Morphisms in ${}^{\boxslash}(\mathcal{J}^{\boxslash})$. The following lemmas show that the morphisms in the class ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ are trivial cofibrations. We will however see in Section 6.4 that not every trivial cofibration is in this class, i.e., the inclusion is strict.

Lemma 41. We have ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq {}^{\boxslash}(\mathcal{I}^{\boxslash})$.

Proof. By Lemma 39, we have that $\mathcal{J} \subseteq {}^{\boxslash}(\mathcal{I}^{\boxslash})$. Thus, by Lemma 15, we have ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq {}^{\boxslash}(({}^{\boxslash}(\mathcal{I}^{\boxslash}))^{\boxslash}) = {}^{\boxslash}(\mathcal{I}^{\boxslash})$. $\square$

Lemma 42. We have ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq \mathcal{W}$.

Proof. By Lemma 37, a pushout of an element in $\mathcal{J}$ is an elementary Tietze transformation and thus a weak equivalence by Lemma 10. By Proposition 21, any element of ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ is a countable composition of elementary Tietze transformations, and thus a weak equivalence by Lemma 25. $\square$

Lemma 43. We have ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq \mathcal{C} \cap \mathcal{W}$.

Proof. By Lemmas 41 and 42. $\square$

6.3 Pseudo-fibrations. The morphisms in $\mathcal{J}^{\boxslash}$ are called *pseudo-fibrations*. A *pseudo-fibrant object* P is one such that the terminal morphism $\mathsf{P} \to 1$ is a pseudo-fibration.

Lemma 44. A presentation P is pseudo-fibrant when

- for every word $u \in \mathsf{P}_1^*$, there is a generator $a \in \mathsf{P}_1$ such that $u \Rightarrow a \in \mathsf{P}_2$,
- the relation P_2 on P_1^* is a congruence.

In particular, we have $u \overset{\mathsf{P}}{=} v$ if and only if $u \Rightarrow v \in \mathsf{P}_2$.

More generally, pseudo-fibrations can be described as follows:

Lemma 45. A morphism $f : \mathsf{P} \to \mathsf{Q}$ is a pseudo-fibration when

- for every $u \in \mathsf{P}_1^*$ and $b \in \mathsf{Q}_1$ such that $f(u) \Rightarrow b \in \mathsf{Q}_2$, there exists $a \in \mathsf{P}_1$ with $f^*(a) = b$ and $u \Rightarrow a \in \mathsf{P}_2$,
- for every $u \in \mathsf{P}_1^*$,

$$f^*(u) \Rightarrow f^*(u) \in \mathsf{Q}_2 \qquad \text{implies} \qquad u \Rightarrow u \in \mathsf{P}_2,$$

– for every $u, v \in \mathsf{P}_1^*$ with $u \Rightarrow v \in \mathsf{P}_2$,

$$f^*(v) \Rightarrow f^*(u) \in \mathsf{Q}_2 \qquad \text{implies} \qquad v \Rightarrow u \in \mathsf{P}_2,$$

– for every $u, v, w \in \mathsf{P}_1^*$ with $u \Rightarrow v \in \mathsf{P}_2$ and $v \Rightarrow w \in \mathsf{P}_2$,

$$f^*(u) \Rightarrow f^*(w) \in \mathsf{Q}_2 \qquad \text{implies} \qquad u \Rightarrow w \in \mathsf{P}_2,$$

– for every $u, v, w, w' \in \mathsf{P}_1^*$ with $u \Rightarrow v \in \mathsf{P}_2$,

$$f^*(wuw') \Rightarrow f^*(wvw') \in Q_2 \qquad \text{implies} \qquad wuw' \Rightarrow wvw' \in \mathsf{P}_2.$$

Lemma 46. Any fibration is a pseudo-fibration: $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\boxslash} \subseteq \mathcal{J}^{\boxslash}$.

Proof. By Lemma 43, we have ${}^{\boxslash}(\mathcal{J}^{\boxslash}) \subseteq \mathcal{C} \cap W$. Therefore, by Lemma 15, $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\boxslash} \subseteq ({}^{\boxslash}(\mathcal{J}^{\boxslash}))^{\boxslash} = \mathcal{J}^{\boxslash}$. □

Lemma 47. For any object P, there exists a pseudo-fibrant object $\tilde{\mathsf{P}}$, called a *pseudo-fibrant replacement* of P, together with a map $\mathsf{P} \to \tilde{\mathsf{P}}$ in ${}^{\boxslash}(\mathcal{J}^{\boxslash})$.

Proof. Use the small object argument (Proposition 21) to factor the terminal morphism $\mathsf{P} \to 1$ as a morphism in ${}^{\boxslash}(\mathcal{J}^{\boxslash})$ followed by a morphism in $\mathcal{J}^{\boxslash}$. □

6.4 $\mathcal{J}$ is not generating. Contrarily to what one might expect, the class $\mathcal{J}$ is not generating trivial cofibrations. This can be seen by observing that the following inclusion does not hold:

$$\mathcal{J}^{\boxslash} \cap \mathcal{W} \subseteq \mathcal{I}^{\boxslash}$$

For instance, consider the inclusion

$$\langle a \mid \rangle \to \langle a, b \mid b \Rightarrow bb, 1 \Rightarrow bb \rangle$$

which corresponds to the example developed Section 3.4. This morphism is a pseudo-fibration since the only relations to lift are the reflexivity relations (which are not noted here, see Section 2.8) and a weak equivalence since both presented monoids are $\mathbb{N}$. However, it is not a trivial fibration since it is not surjective on generators. The same example can be used to show that the inclusion

$${}^{\boxslash}(\mathcal{I}^{\boxslash}) \cap \mathcal{W} \subseteq {}^{\boxslash}(\mathcal{J}^{\boxslash})$$

does not hold either: the map above is a cofibration since it is a monomorphism and a weak equivalence, but it cannot be obtained as a retract of a composite of pushouts of sums of elements of $\mathcal{J}$. Namely, the generator b has to be added using a Tietze transformation (T1), but the relations are not of the right form. Intuitively, the relation $1 \Rightarrow b$ has to be added first, see Section 3.4.

Remark 48. As a simpler (but less convincing) example, consider the inclusion

$$\langle a \mid \rangle \to \langle a, b \mid b \Rightarrow aa \rangle$$

which is not an elementary Tietze transformation (because of the chosen orientation for the relation (T1)). Similarly, the inclusion

$$\langle a, b, c, d \mid aa \Rightarrow bb, bb \Rightarrow cc, cc \Rightarrow dd \rangle \to \langle a, b, c, d \mid aa \Rightarrow bb, bb \Rightarrow cc, cc \Rightarrow dd, aa \Rightarrow dd \rangle$$

is a pseudo-fibration and a weak equivalence, but not a trivial fibration one since the relation $aa \Rightarrow dd$ cannot be lifted.

6.5 $\mathcal{J}$ is pseudo-generating. It is interesting to note that the inclusions of previous section are satisfied if we restrict to fibrations whose codomain is fibrant. We begin by a reciprocal to Lemma 43:

Lemma 49. Any trivial cofibration $f : \mathsf{P} \to \mathsf{Q}$ with pseudo-fibrant codomain Q belongs to $\mathcal{J}$-cell, and thus to ${}^{\boxslash}(\mathcal{J}^{\boxslash})$.

Proof. Since i is a trivial cofibration, it is an injection and we have $\overline{\mathsf{P}} = \overline{\mathsf{Q}}$. For simplicity, we suppose that i is an inclusion. For every generator in $a \in \mathsf{Q}_1 \setminus \mathsf{P}_1$, there is a word $u_a \in \mathsf{P}_1^*$ such that $u_a \overset{\mathsf{Q}}{=} a$ and therefore $u_a \Rightarrow a \in \mathsf{Q}_2$ since Q is pseudo-fibrant (Q_2 is a congruence). Writing P^0 for P with the generator a and a relation $u_a \Rightarrow a$ added, for every $a \in \mathsf{Q}_1 \setminus \mathsf{P}_1$, we have a morphism $\mathsf{P} \to \mathsf{P}^0$ in $\mathcal{J}$-cell factoring f (the inclusion $\mathsf{P} \to \mathsf{P}^0$ can be expressed as a pushout of a coproduct of pseudo-generating trivial cofibrations of the first form). We write P^{i+1} for the presentation obtained from P^i by adding

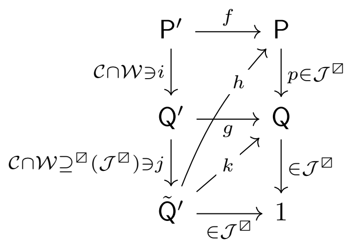
- a relation $u \Rightarrow u$ for every word u over P_1^i,
- a relation $v \Rightarrow u$ for every relation $u \Rightarrow v \in P_2^i$,
- a relation $u \Rightarrow w$ for every relations $u \Rightarrow v, v \Rightarrow w \in P_2^i$,
- a relation $wuw' \Rightarrow wvw'$ for every relation $u \Rightarrow v \in P_2^i$ and words w, w' over P_1^i.

There is a morphism $\mathsf{P}^i \to \mathsf{P}^{i+1}$ in $\mathcal{J}$-cell. Every generator of Q gets added at the first step and every relation of Q gets added at some step. Therefore $\mathsf{Q} = \operatorname{colim}_i \mathsf{P}^i$ and f belongs to $\mathcal{J}$-cell. □

Remark 50. The above proof essentially consists in using the small object argument to construct a factorization $f = h \circ g$ with $g \in \mathcal{J}$-cell and $h \in \mathcal{J}^{\boxslash}$, and observing that h can be chosen to be an identity when Q is pseudo-fibrant.

Lemma 51. Any pseudo-fibration $p : \mathsf{P} \to \mathsf{Q} \in \mathcal{J}^{\boxslash}$ with pseudo-fibrant target Q is a fibration (i.e., $p \in (\mathcal{C} \cap \mathcal{W})^{\boxslash}$).

Proof. Suppose given a trivial cofibration $i : \mathsf{P}' \to \mathsf{Q}' \in \mathcal{C} \cap \mathcal{W}$ and two morphisms $f : \mathsf{P}' \to \mathsf{P}$ and $g : \mathsf{Q}' \to \mathsf{Q}'$ such that $p \circ f = g \circ i$. By Lemma 47, we can consider a pseudo-fibrant replacement $\tilde{\mathsf{Q}}'$ of Q' together with the associated morphism $j : \mathsf{Q}' \to \tilde{\mathsf{Q}}'$ in ${}^{\boxslash}(\mathcal{J}^{\boxslash})$, and thus in $\mathcal{C} \cap \mathcal{W}$ by Lemma 43. By orthogonality, there is a map $k : \tilde{\mathsf{Q}}' \to \mathsf{Q}$ such that $k \circ j = g$. Finally, by Lemma 49 $(j \circ i) \boxslash p$, from which we deduce the existence of $h : \tilde{\mathsf{Q}}' \to \mathsf{P}$ such that $h \circ j \circ i = f$ and $p \circ h = k$.



Therefore the morphism $h \circ j : \mathsf{Q}' \to \mathsf{P}$ is a filler and thus $i \boxslash p$. □

Lemma 52. Pseudo-fibrant and fibrant objects coincide.

Proof. By Lemma 46, any fibrant object is pseudo-fibrant. Conversely, by Lemma 51, it suffices to check that the terminal object is pseudo-fibrant, which can be verified directly. □

Lemma 53. Given a monoid M, its standard presentation $\langle M\rangle$ is fibrant.

Proof. The presentation $\langle M\rangle$ satisfies the conditions of Lemma 44 and is thus pseudo-fibrant and thus fibrant by Lemma 52. □

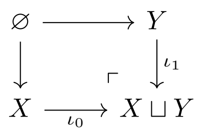
7 Tietze equivalences as cospans

In this section we reconstruct the proof of the Tietze theorem by showing that any two presentations of the same monoid can be related by a cospan of generalized Tietze transformations.

7.1 Coproduct. We often write $\iota_0, \iota_1 : X \sqcup X \to X$ for the canonical injections into a coproduct.

Lemma 54. In a model category, when X is cofibrant, the canonical injections $\iota_0 : Y \to Y \sqcup X$ and $\iota_1 : Y \to X \sqcup Y$ are cofibrations.

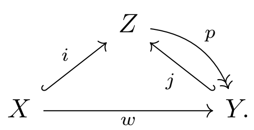
Proof. We have a pushout diagram



When X is cofibrant, the initial map into X is a cofibration, and the map ι_1 is thus also a cofibration, as a pushout of a cofibration (the other case is similar). □

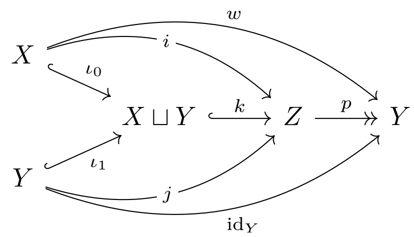
7.2 Weak equivalences as cospans. We now recall the contents of the proof of the celebrated Ken Brown lemma, which shows that every weak equivalence between cofibrant objects factors as a cospan of trivial cofibrations.

Lemma 55 (Ken Brown's lemma). In a model category, every weak equivalence $w : X \to Y$ between cofibrant objects X and Y factors as $w = p \circ i$ where i is a trivial cofibration and p a trivial fibration which admits a section by a trivial cofibration j:



Proof. We can factor the map $(w, \mathrm{id}_Y) : X \sqcup Y \to Y$ as a cofibration $k : X \sqcup Y \to Z$ followed by a trivial fibration $p : Z \to Y$. Since X and Y are cofibrant, by

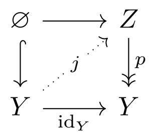
Lemma 54, the injections into $X \sqcup Y$ are cofibrations. We define $i = k \circ \iota_0$ and $j = k \circ \iota_1$:



The maps i and j are cofibrations as composites of cofibrations and are weak equivalences by the 2-out-of-3 property. □

Remark 56. In the previous lemma, the cospan (i, j) can be considered as a factorization of w, in the sense that we have $j \circ w = j \circ p \circ i = i$.

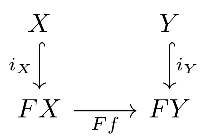
Remark 57. In a model category where monomorphisms are cofibrations (such as the case of interest here, see Lemma 28), a simpler argument can be given: since Y is cofibrant and p is a trivial fibration, the diagram



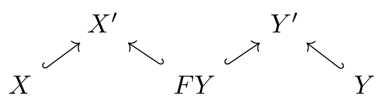
admits a filler $j : Y \to Z$, which is a section of p; moreover, since j is a monomorphism, it is a cofibration, and it is a weak equivalence by the 2-out-of-3 property.

Theorem 58. In a model category $\mathcal{M}$ in which every object is cofibrant, every isomorphism in $\mathrm{Ho}(\mathcal{M})$ is the localization of a cospan of trivial cofibrations.

Proof. Consider an isomorphism $f : X \to Y$ in $\mathrm{Ho}(\mathcal{M})$. We write $\mathcal{M}'$ for the full subcategory of $\mathcal{M}$ whose objects are fibrant. The fibrant replacement functor $F : \mathcal{M} \to \mathcal{M}'$ induces an equivalence between the homotopy categories [5, Proposition 1.2.3]. Moreover, $\mathrm{Ho}(\mathcal{M}')$ is a quotient of $\mathcal{M}'$ by homotopy equivalences [5, Theorem 1.2.10], the map Ff is thus a homotopy equivalence and thus a weak equivalence [5, Proposition 1.2.8]. The map f is thus the localization of a span of weak equivalences



where $i_X : X \to FX$ is the trivial cofibration associated to the fibrant replacement. By Lemma 55, we thus have two cospans of trivial cofibrations



and we conclude to the existence of one cospan of trivial cofibrations using the fact that trivial cofibrations are closed under pushouts. □

7.3 Tietze equivalences. We can now conclude with the abstract proof of the Tietze theorem.

Theorem 59. In the category **rPres**, two presentations P and Q are such that $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$ if and only if there is a cospan of generalized Tietze transformations (of morphisms in $\mathcal{J}$-cell) from P to Q.

Proof. Suppose given two presentations $\mathsf{P}, \mathsf{Q} \in$ **rPres** such that $\overline{\mathsf{P}} \simeq \overline{\mathsf{Q}}$. With the model structure introduced in Section 5, this can be rewritten ans $\mathrm{Ho}(\mathsf{P}) \simeq \mathrm{Ho}(\mathsf{Q})$, and therefore we deduce that there is a cospan of trivial cofibrations

$$\mathsf{P} \hookrightarrow \mathsf{R} \hookleftarrow \mathsf{Q}.$$

Up to taking a fibrant replacement of R and suppose that R is fibrant and thus pseudo-fibrant by Lemma 52. We deduce that this is a span of Tietze transformations by Lemma 49. Conversely, Tietze transformations are weak equivalences by Lemma 42 and thus P and Q become isomorphic after localizing under weak equivalences. □

8 Variants and extensions

Many variants of the situation considered here could be thought of and are left for future work.

8.1 Non-reflexive presentations. If we consider the category **Pres** of (non-necessarily reflexive) presentations, many of the constructions performed in previous section can still be carried over. However, Lemma 31 does not hold anymore, preventing the construction of a model category: the elements of $\mathcal{I}^{\boxslash}$ are not necessarily weak equivalences. As a counter-example consider the morphism

$$\langle a, b \mid \rangle \to \langle c \mid \rangle.$$

It belongs to $\mathcal{I}^{\boxslash}$ since it satisfies the conditions of Lemma 30 (which still holds): it is surjective on generators and lifts every required relations since there are none. It is however not a weak equivalence since the monoids presented by the source and the target are respectively $\mathbb{N} * \mathbb{N}$ and $\mathbb{N}$ which are not isomorphic (the first one is not commutative for instance). We expect that there is however a semi-model structure in the sense of [8], whose cofibrations are generated by $\mathcal{I}$.

8.2 Multisets of relations. The notion of presentation can be modified in order to allow multiple relations with the same source and the same target: such a presentation P consists of a set P_1 of generators together with a set P_2 of relations equipped with source and target maps $\mathsf{s}, \mathsf{t} : \mathsf{P}_2 \to \mathsf{P}_1$. Here, an element $\alpha \in \mathsf{P}_2$ with $\mathsf{s}(\alpha) = u$ and $\mathsf{t}(\alpha) = v$ encodes a relation $u \Rightarrow v$. We expect that this modification does not significantly changes the situation studied here.

8.3 Presentations of categories. As a further generalization, one can consider presentations of categories. Such a presentation P of a category consists of a set P_0 of objects, a set P_1 of generators for morphisms equipped with source and target maps $\mathsf{s}_0, \mathsf{t}_0 : \mathsf{P}_1 \to \mathsf{P}_0$, and a set P_2 of relations equipped with source and target maps $\mathsf{s}_1, \mathsf{t}_1 : \mathsf{P}_2 \to \mathsf{P}_1^*$ such that $\mathsf{s}_0^* \circ \mathsf{s}_1 = \mathsf{s}_0^* \circ \mathsf{t}_1$ and $\mathsf{t}_0^* \circ \mathsf{s}_1 = \mathsf{t}_0^* \circ \mathsf{t}_1$. Here, P_1^* denotes the morphisms of the free category over the graph $(\mathsf{P}_0, \mathsf{P}_1)$ and the category presented by P is obtained by quotienting the morphisms of this free category under the congruence generated by P_2. The notion of presentation of monoid of Section 8.2, is the particular case where $\mathsf{P}_0 = \{\star\}$ is reduced to one element. We expect the proof of this paper to generalize to this setting.

8.4 Presentations of n-categories. This notion of presentation sketched in previous section, is a particular case of the notion of *polygraph*, see [3], which generalizes to present n-categories. It would be interesting to see whether the model structure extends to this case.

8.5 Presentations of groupoids. The notion of Tietze transformation was originally developed for presentations of groups. It would be interesting to generalize the model structure to this case (as well as generalizations of presentations of groupoids).

8.6 Coherent presentations. A notion of Tietze transformation for coherent presentations of categories is introduced in [4]. We would like to investigate this case, as well as, more generally, develop a notion of Tietze transformation for resolutions of categories by $(\infty, 1)$-polygraphs.

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