Computing Critical Pairs in 2-Dimensional Rewriting Systems

Samuel Mimram

Rewriting Theory and Applications 2010

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The big plan

Rewriting theory has to be generalized
in higher dimensions!

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In this talk, I will be interested in extending the procedures of **unification** in dimension 2.

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- *terms*: t, u, ...
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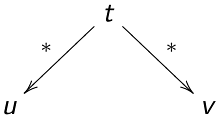

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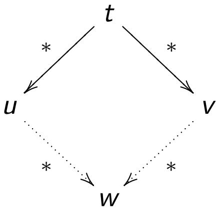

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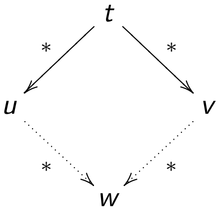

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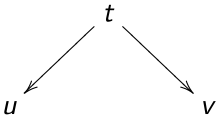

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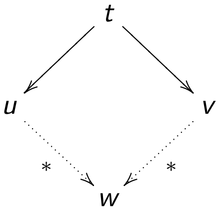

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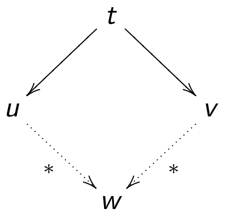

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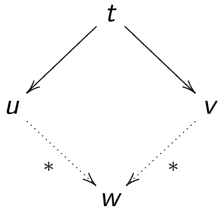

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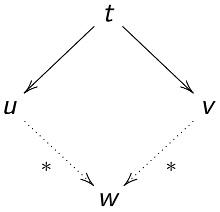

We only have to check *minimal* and *non-trivial* such t:
critical pairs
which can be computed using a **unification algorithm**

So, we want to compute **critical pairs**
in higher dimensions

What is dimension?

	Geometry	Rewriting systems
0	$\cdot$	$\cdot x$

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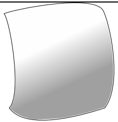
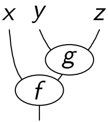
2		

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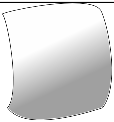
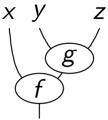
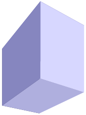
2		
3		???
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Why rewriting systems?

- *Presentations*: giving finite descriptions of monoids, etc.
- *Coherence*: e.g. MacLane's coherence theorem for monoidal categories
- ...

This has to be generalized to n-categories

Presentations of monoids

A **presentation**

$$\langle G \mid R \rangle$$

of a monoid M consists of

- a set G of *generators*
- a set $R \subseteq G^* \times G^*$ of *relations*

such that

$$M \qquad \cong \qquad G^* / \equiv_R$$

Example

- $\mathbb{N} \cong \langle a \mid \ \rangle$
- $\mathbb{N}/2\mathbb{N} \cong \langle a \mid aa = 1 \rangle$
- $\mathbb{N} \times \mathbb{N} \cong \langle a, b \mid ba = ab \rangle$
- $\mathfrak{S}_n \cong \langle \sigma_1, \ldots, \sigma_n \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\ \sigma_i^2 = 1,\ \sigma_i \sigma_j = \sigma_j \sigma_i \rangle$
- ...

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1. Orient R to get a string rewriting system.
2. Show that the rewriting system is terminating.
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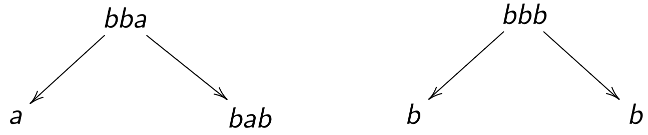

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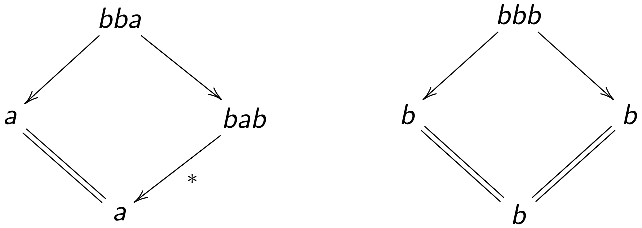
Critical pairs are joinable:



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Example $\mathbb{N} \times (\mathbb{N}/2\mathbb{N}) \overset{?}{\cong} \langle a, b \mid ba \to ab,\ bb \to 1 \rangle$

Normal forms are:

a^n and $a^n b$

They are in bijection with $\mathbb{N} \times (\mathbb{N}/2\mathbb{N})$!

Presentations of Lawvere theories

String rewriting systems correspond to
presentations of monoids.

Presentations of Lawvere theories

Term rewriting systems correspond to presentations of Lawvere theories.

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A **Lawvere theory** is a category $\mathcal{C}$

- whose objects are integers
- which is cartesian
- whose cartesian product is given on objects by addition

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TRS (Σ_2, Σ_3) induce LT whose morphisms $m \to n$ are n-uples of terms with variables $x_1, \ldots, x_m$, considered modulo relations.

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Example

Consider the TRS of commutative monoids: $\Sigma_2 = \{m : 2,\ e : 0\}$

$$\Sigma_3 = \left\{ \begin{array}{ll} \alpha : & m(m(x,y),z) \to m(x,m(y,z)) \\ \lambda : & m(e,x) \to x \\ \rho : & m(x,e) \to x \\ \gamma : & m(x,y) \to m(y,x) \end{array} \right\}$$

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It presents the Lawvere theory whose morphisms $M : m \to n$ are $(m \times n)$-matrices with coefficients in $\mathbb{N}$.

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Use rewriting theory!

n-categories

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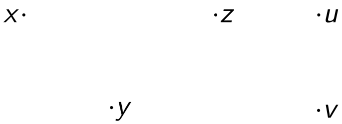
0-category:



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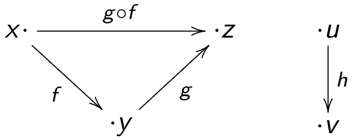
1-category:



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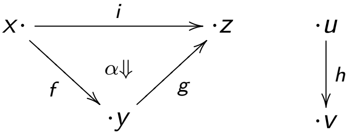

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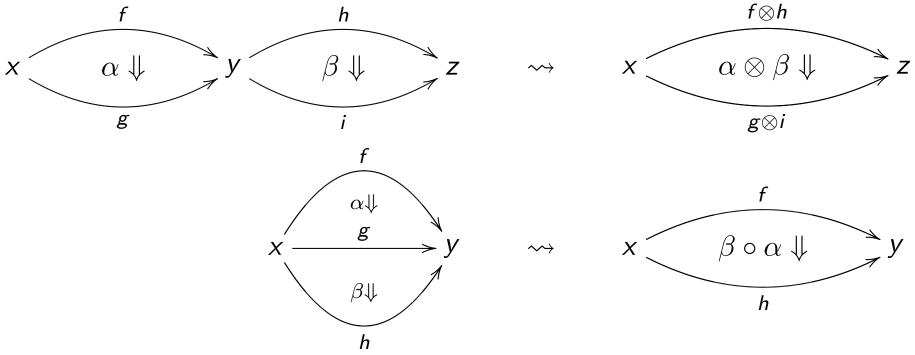
2 compositions in a 2-category:



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dimension	rewr. syst.	presents
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set $=$ 0-category

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monoid $=$ 1-category with only one object

Generalization: $\xrightarrow{a} \xrightarrow{b} \quad \rightsquigarrow \quad x \xrightarrow{a} y \xrightarrow{b} y$

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dimension	rewr. syst.	presents
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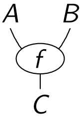
1	string	1-category
2	term	cartesian category

Lawvere th. $=$ cartesian category with $\mathbb{N}$ as objects

Generalization:  $\rightsquigarrow$ 

Presenting *n*-categories

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dimension	rewr. syst.	presents
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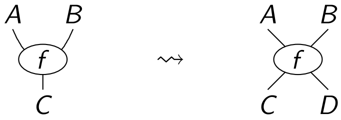
1	string	1-category
2	term	monoidal category

cartesian category = monoidal category in which every object is a comonoid

Generalization:



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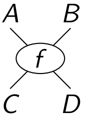
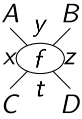
2	term	2-category

monoidal category $\quad = \quad$ 2-category with only one object

Generalization: 

$\rightsquigarrow$ 

Polygraphs

A 0-signature

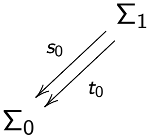
Σ_0

Example

signature

x y

A 0-rewriting system

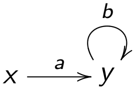


Example

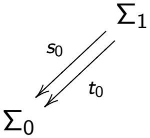
signature

x y

rules

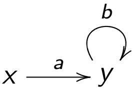


A 1-signature = a 0-rewriting system

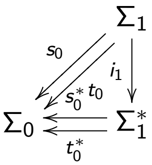


Example

signature



A 1-signature generates a *category*



Example

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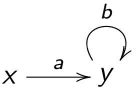
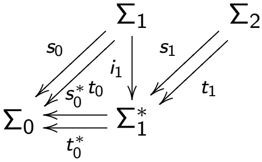


terms

$$x \xrightarrow{a} y \xrightarrow{b} y \xrightarrow{b} y$$

Polygraphs

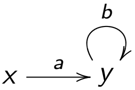
A 1-rewriting system



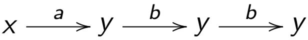
such that $s_0^* \circ s_1 = s_0^* \circ t_1$ and $t_0^* \circ s_1 = t_0^* \circ t_1$

Example

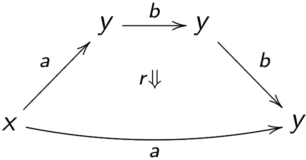
signature



terms

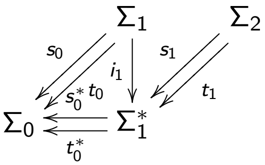


rules



Polygraphs

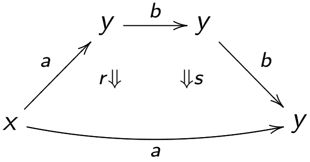
A 2-signature = a 1-rewriting system



such that $s_0^* \circ s_1 = s_0^* \circ t_1$ and $t_0^* \circ s_1 = t_0^* \circ t_1$

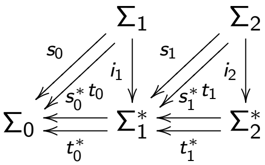
Example

signature



Polygraphs

A 2-signature generates a 2-category



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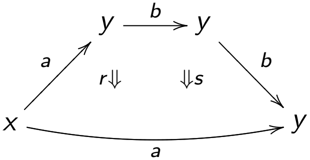
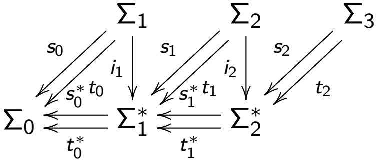


Polygraphs

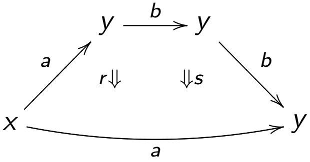
A 2-rewriting system



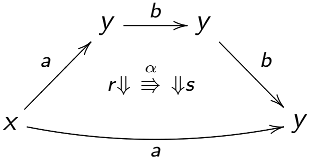
such that $s_1^* \circ s_2 = s_1^* \circ t_2$ and $t_1^* \circ s_2 = t_1^* \circ t_2$

Example

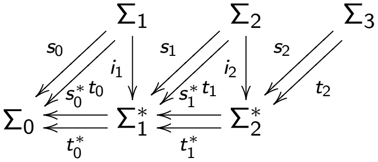
signature



rules



A 2-rewriting system



Right notion of *n-rewriting system*:

n-**polygraph**

[Street76,Power90,Burroni93]

A presentation of **Bij**

The category **Bij** has

- objects are integers $[n] = \{0, \dots, n-1\}$
- morphisms $f : [m] \to [n]$ are bijections

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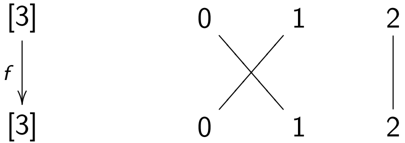

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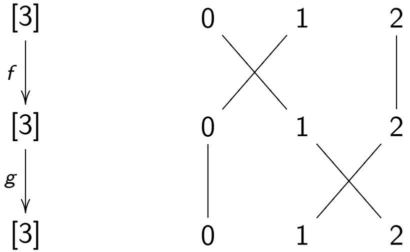

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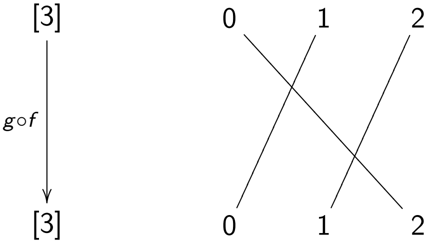

Vertical composition $\circ$

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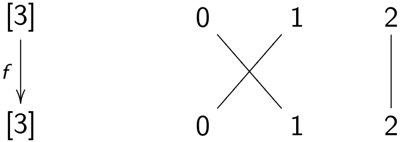

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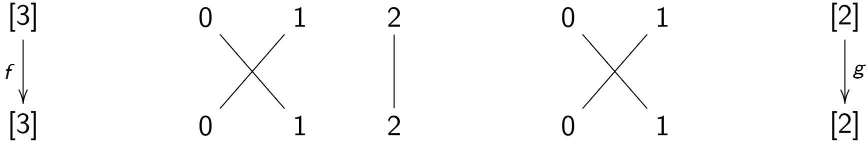

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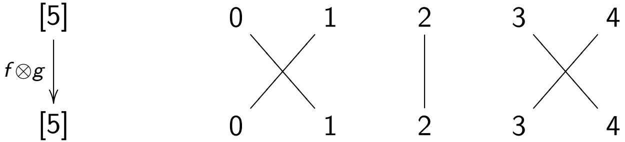

Horizontal composition $\otimes$

We want to give a presentation of **Bij**, i.e. describe it as

- a free category on sets of typed generators for 0-, 1- and 2-cells
- quotiented by relations between 2-cells in the generated 2-category

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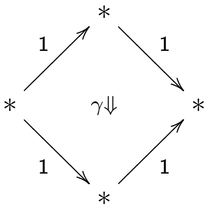

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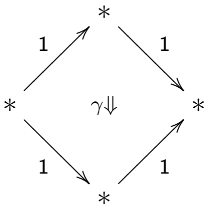

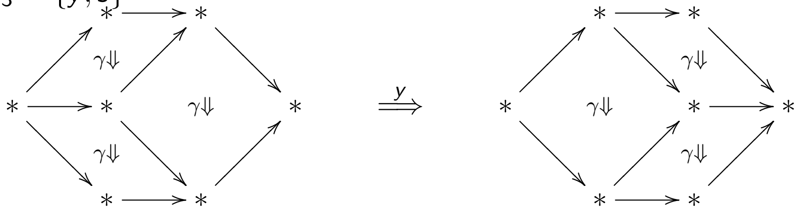

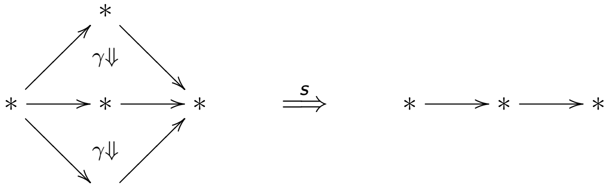

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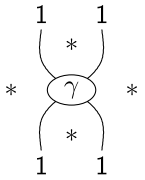

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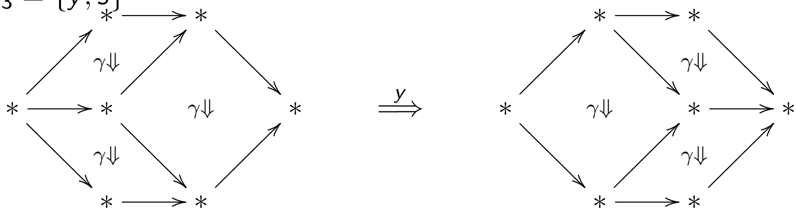

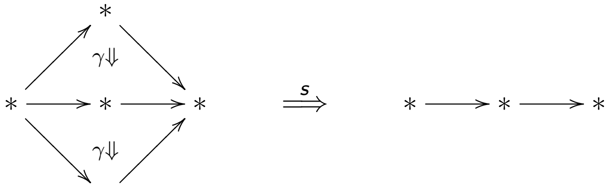

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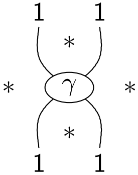

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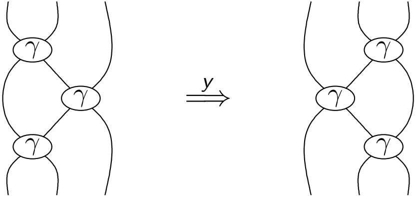
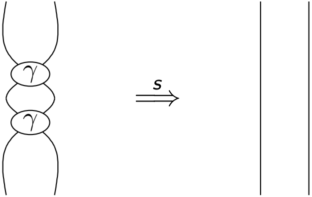
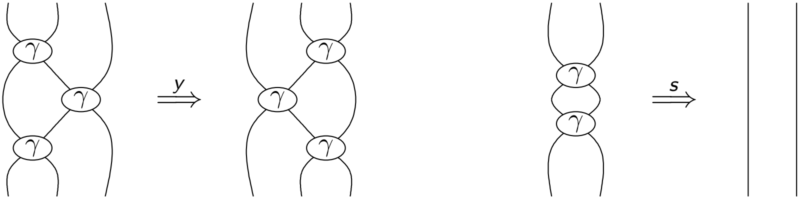


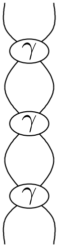


A presentation of **Bij**

The rules



of the rewriting system induce critical pairs



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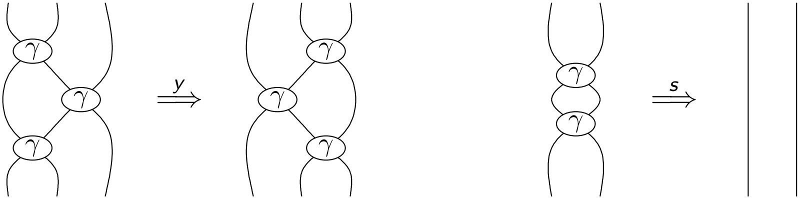

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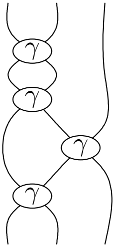

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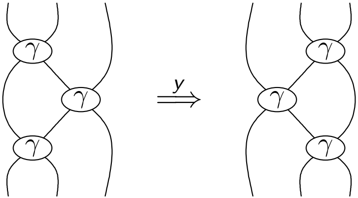

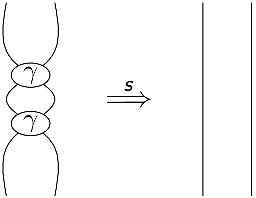

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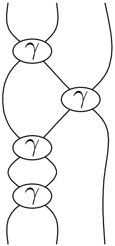

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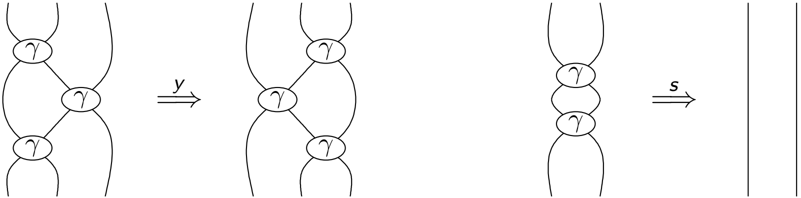

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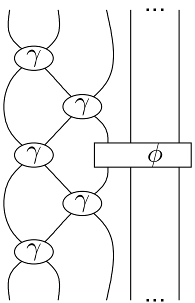
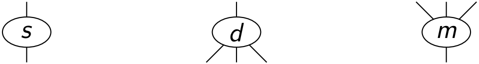


Two problems with critical pairs

Consider the 2-rewriting system Σ with

$$\Sigma_0 = \{*\} \qquad \Sigma_1 = \{1\} \qquad \Sigma_2 = \{s : 1 \to 1,\ d : 1 \to 3,\ m : 3 \to 1\}$$

the generators for 2-cells are drawn respectively as



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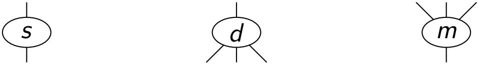

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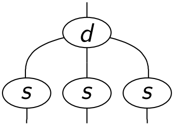
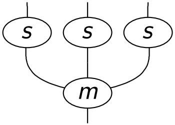


$\Rightarrow \ldots$ and



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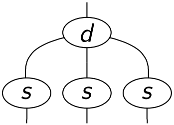
Two problems with critical pairs

The two rules



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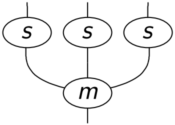
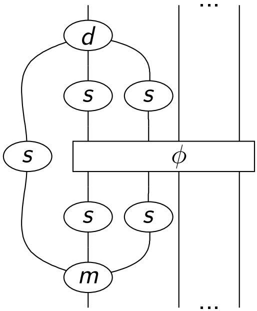


$\Rightarrow \ldots$

induce an infinite number of critical pairs:

variables on the border:



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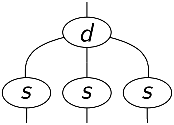

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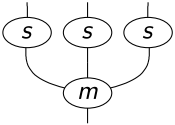

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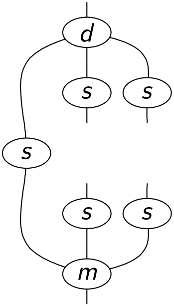
variables on the border:
use compact morphisms!



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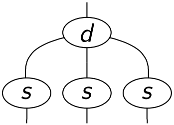

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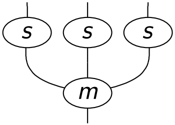

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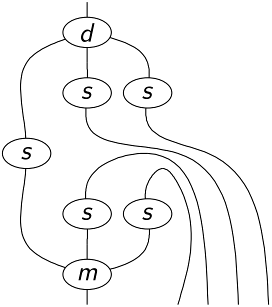

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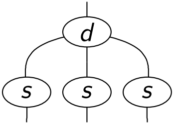

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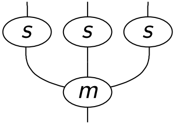
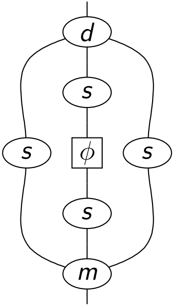


$\Rightarrow \dots$

induce an infinite number of critical pairs:

variables inside:



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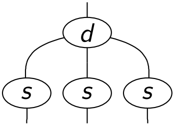

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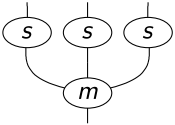
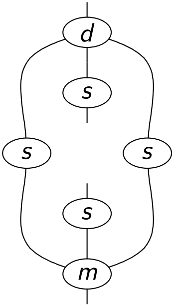


$\Rightarrow \dots$

induce an infinite number of critical pairs:

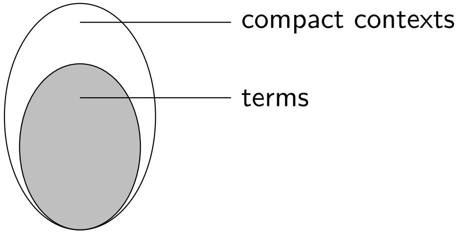
variables inside:
use contexts!
(= terms with metavariables)



Back to a finite number of critical pairs

Theorem

The 2-category of "terms" generated by a signature can be embedded into the ***multicategory of compact contexts****.*



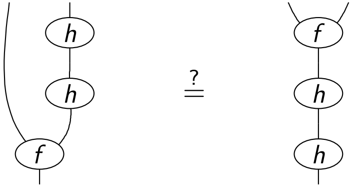
In other words, there is a finite number of *generating families* of critical pairs in those rewriting systems.

Unification in TRS

Suppose that we have a TRS

$f : 2 \qquad h : 1 \qquad\qquad f(x, h(h(y))) \Rightarrow \ldots \qquad h(h(f(x, y))) \Rightarrow \ldots$

In order to generate critical pairs,
we unify a subterm of $f(x, h(h(y)))$ with $h(h(f(x, y)))$



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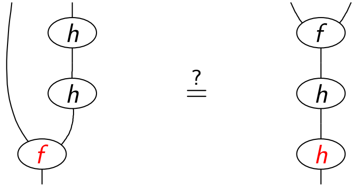

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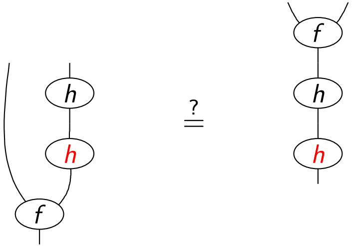

Unification in TRS

Suppose that we have a TRS

$f : 2 \qquad h : 1 \qquad\qquad f(x, h(h(y))) \Rightarrow \dots \qquad h(h(f(x, y))) \Rightarrow \dots$

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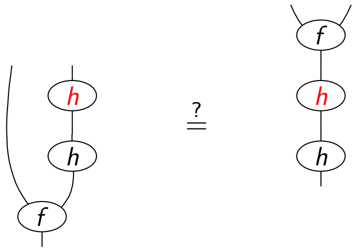

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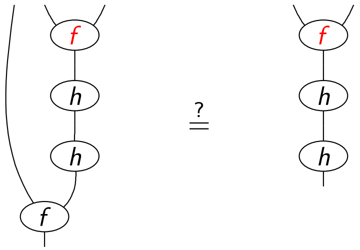
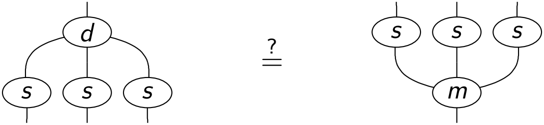


Unification in 2-rewriting systems

We want to compute the critical pairs generated by



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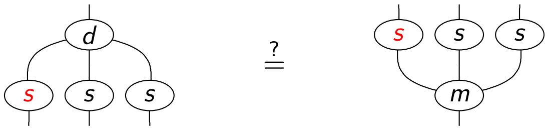

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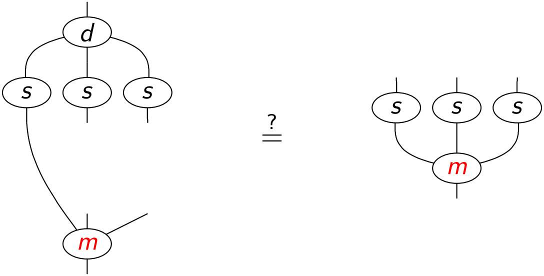

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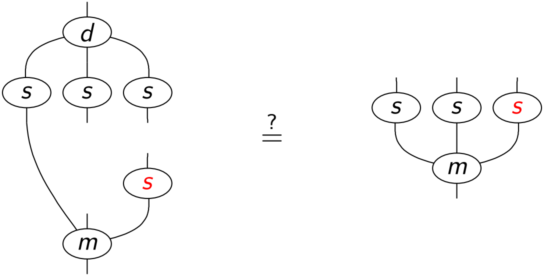

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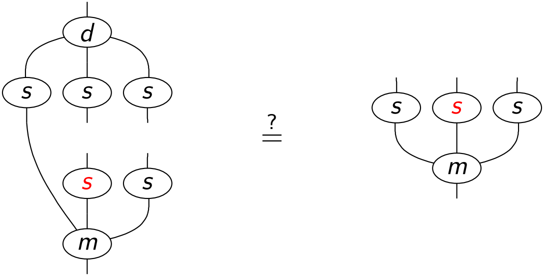

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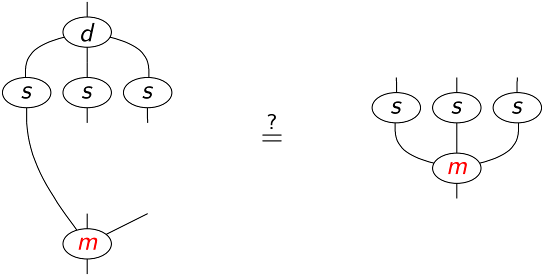

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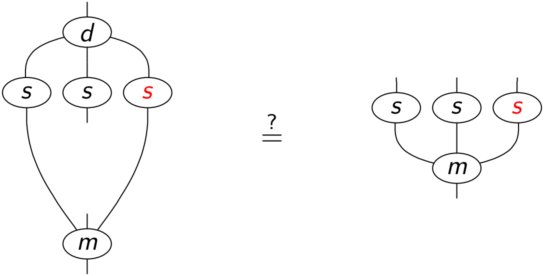

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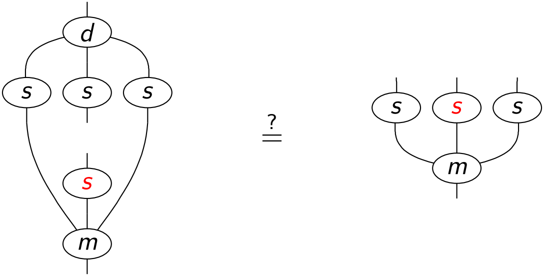

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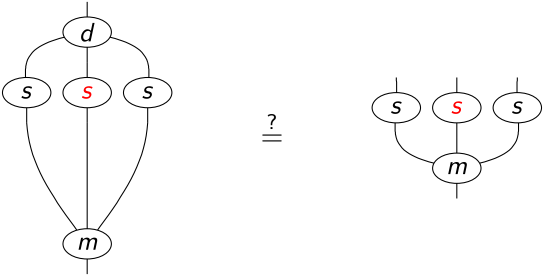

Under the carpet

- Precise definition of compact contexts
- Formal algebraic definition of morphisms in free categories
- Precise statement of the algorithm

TODO

- Generalize termination techniques to this setting (cf. [Guiraud])
- Generalize to higher dimensions
- Automated tools for studying categories
- ...