A Local View on Innocence

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Congratulations Paolo!
Part I

Game Semantics
Denotational semantics

What are those programs/proofs doing?
Denotational semantics

What are those *concurrent* programs/proofs doing?
An *interactive trace semantics*:

- types are interpreted by **games**
- programs are interpreted by **strategies**
A game 

\((M, \leq, \#, \lambda)\)

consists of

- a set of \textbf{moves} \(M\)
- a partial order \(\leq\) expressing \textbf{causal dependencies}
- a symmetric relation \(\#\) expressing \textbf{incompatibilities}
- a \textbf{polarization} of the moves \(\lambda : M \rightarrow \{O, P\}\)

\[
\begin{array}{c}
q \\
\hline
B = \\
\hline
T \quad \# \\
\hline
\end{array}
\]

\[T \quad \# \\
\hline
F \]

\[
\begin{array}{c}
q \\
\hline
B = \\
\hline
T \quad \# \\
\hline
F \\
\end{array}
\]

\[
\begin{array}{c}
q \\
\hline
B = \\
\hline
T \quad \# \\
\hline
F \\
\end{array}
\]
Strategies

- A **play** is a sequence of moves respecting the dependencies and the incompatibility.
- A **strategy** is a set of plays.
The game $\mathbb{B}$:
The game $B \otimes B$:
The game $\mathbb{B} \rightarrow \mathbb{B}$:
The strategy not

B → B
The strategy not

\[ B \implies B \]
The strategy not

\[ B \quad \rightarrow \quad B \]

\[ q \quad \rightarrow \quad q \]

\[ T \]

\[ F \]
The strategy not

\[ B \rightarrow B q \]

\[ q \]

\[ q \]

\[ F \]

\[ T \]
We get a model (of PCF / MLL / ...), but it’s not very informative (yet).
We have to characterize **definable** strategies
(= strategies which are the interpretation of a proof)
We have to characterize **definable** strategies
(= strategies which are the interpretation of a proof)

Two series of work laid the foundations of game semantics:
- fully abstract models of PCF [HON,AJM]
- fully complete models of MLL [AJ,HO]
extended later on (references, control, non-determinism, . . .)
Definable strategies

We have to characterize **definable** strategies
(= strategies which are the interpretation of a proof)

In these models, definable strategies are characterized using conditions such as **innocence**, bracketing, . . .

\[
\text{innocent strategy} \approx \text{strategy behaving like a } \lambda\text{-term}
\]
We want to:

- build a model of MLL adapted to concurrency
- reformulate innocence in this model using local principles
- extend innocence to get a fully complete model of MLL
- define a unifying framework: recover preexisting models
Part II

Innocence
In the framework of HO games:

- plays are alternating (and start with an Opponent move)
- strategies contain only even-length plays
- plays are pointed
- strategies $\sigma$ are deterministic:
  \[ s \cdot m \in \sigma \quad \text{and} \quad s \cdot n \in \sigma \quad \text{implies} \quad m = n \]
A strategy $\sigma$ is **innocent** when it reacts only according to its view of the play:

$$s \cdot m \in \sigma \quad \text{and} \quad \langle s \rangle = \langle t \rangle \quad \text{implies} \quad t \cdot m \in \sigma$$

Here, $\langle s \rangle \sqsubseteq s$ is the **view** of the play $s$. 

**Innocence**
Innocence

1st counter-example: uniformity

fun \ f \to \cdots \ f \ \cdots

(B \Rightarrow B) \Rightarrow B

\ q \ q
1\textsuperscript{st} counter-example: uniformity

\begin{align*}
\text{fun } f & \rightarrow \cdots f \text{ true} \cdots \\
(B & \Rightarrow B) \Rightarrow B
\end{align*}
Innocence

1st counter-example: uniformity

fun \( f \rightarrow \ldots f \ ? \ldots \)

\[(\mathcal{B} \rightarrow B) \rightarrow B\]
Innocence

1\textsuperscript{st} counter-example: uniformity

\[
\text{fun } f \to \cdots f \ ? \cdots
\]

\[
(B \ \Rightarrow \ B) \ \Rightarrow \ B
\]
Innocence

2nd counter example: history-freeness

\[
\begin{array}{c}
\mathbb{B} \times \mathbb{B} \\
q \\
( \\
T \\
\) \\
\mathbb{B} \times \mathbb{B} \\
F \\
\) \\
\mathbb{B} \times \mathbb{B} \\
F \\
\) \\
\end{array}
\]
Decomposing innocence

\[ A \Rightarrow B = !A \rightarrow B \]

A linear decomposition of innocence:

- !A is many copies of A which are handled uniformly
  \[ !A = \bigotimes_{\omega} A/ \approx \]

- innocent strategies on \( A \rightarrow B \) are history-free
Part III

Asynchronous games
Implementations of conjunction

\[ B \otimes B \rightarrow B \]

left conjunction

\[ q_L \]

\[ q_R \]

\[ T_L \]

\[ F_R \]

\[ F \]
Implementations of conjunction

right conjunction

$B \otimes B \to B$

$q$

$q_R$

$q_L$

$F_R$

$T_L$

$F$
Implementations of conjunction

\[ B \otimes B \rightarrow B \]

parallel conjunction

\[ q_L \quad q_R \]

\[ T_L \quad F_R \quad F \]
Implementations of conjunction

parallel conjunction

$q_L$

$q_R$

$q$

$T_L$

$F_R$

$F$
From plays to Mazurkiewicz traces
Playing on asynchronous graphs

partial order vs asynchronous graphs

\[
\begin{align*}
    m & \quad \sim \quad n \\
    \downarrow & \quad \sim \quad \uparrow \\
    n & \quad \sim \quad m
\end{align*}
\]

Definition
asynchronous graph = graph + homotopy tiles
The asynchronous graph of a game

- **Position**: downward-closed set of compatible moves.
- **Play**: path from the initial position $\emptyset$. 

---

**Diagram**

- $\text{bool}$
- $q$
  - $T$ ➔ $\emptyset$
  - $\#$ ➔ $\{q\}$
  - $F$ ➔ $\{q, T\}$
- $q$
  - $T$ ➔ $\{q, T\}$
  - $F$ ➔ $\{q, F\}$
The asynchronous graph of a game

\[ q \rightarrow \emptyset \]

\[ \cdot \cdot \cdot \]

\[ q \]

\[ \cdot \]

\[ \{q\} \]
The asynchronous graph of a game
The asynchronous graph of a game
Strategies of conjunction

The game $\text{bool} \otimes \text{bool} \vdash \text{bool}$ contains eight subgraphs:

```
* ⊗ * → *

* ⊗ * → q

q ⊗ * → q

qL → qR

q ⊗ q → q

qL → qR

q ⊗ q → q

qL → qR

q ⊗ F → q

qL → qR

q ⊗ F → q

qL → qR

F ⊗ * → q

F ⊗ F → F
```
Strategies of conjunction

Left implementation of conjunction:
Strategies of conjunction

Right implementation of conjunction:

\[
\begin{align*}
&T \otimes * \rightarrow q \\
&q \otimes * \rightarrow q \\
&T \otimes q \rightarrow q \\
&T \otimes F \rightarrow q \\
&T \otimes F \rightarrow F
\end{align*}
\]
Strategies of conjunction

Parallel implementation of conjunction:

\[
\begin{array}{c}
\ast \otimes \ast \rightarrow \ast \\
q \\
\ast \otimes \ast \rightarrow q \\
q_L \rightarrow q_R \\
q \otimes q \rightarrow q \\
q \otimes q \rightarrow q \\
q \otimes F \rightarrow q \\
T \otimes F \rightarrow q \\
T \otimes F \rightarrow q \\
T \otimes T \rightarrow q \\
T \otimes F \rightarrow F \\
T \otimes F \rightarrow F \\
\end{array}
\]
We want to be non-alternating!
MALL with lifts

- We consider here MALL formulas:

\[
\begin{align*}
\Gamma, A, B & \quad \Rightarrow \quad \Gamma, A \mathcal{R} B \quad (\mathcal{R}) \\
\Gamma, A & \quad \Rightarrow \quad \Gamma, A \mathcal{L} B \quad (\mathcal{L}) \\
\Gamma, A & \quad \Rightarrow \quad \Gamma, A \mathcal{R} B \quad (\mathcal{R}) \\
\Gamma, A & \quad \Rightarrow \quad \Gamma, A \mathcal{L} B \quad (\mathcal{L})
\end{align*}
\]
MALL with lifts

• We consider here MALL formulas:

\[
\begin{align*}
\vdash \Gamma, A, B \\
\vdash \Gamma, A \otimes B
\end{align*}
\]

\[
\begin{align*}
\vdash \Gamma, A \quad \vdash \Gamma, B \\
\vdash \Gamma, A \& B
\end{align*}
\]

\[
\begin{align*}
\vdash \Gamma, A \\
\vdash \Gamma, A \oplus B
\end{align*}
\]

• with explicit lifts:

\[
\begin{align*}
\vdash \Gamma, A \\
\vdash \Gamma, \forall x.A
\end{align*}
\]

\[
\begin{align*}
\vdash [t/x] \Gamma, A \\
\vdash \Gamma, \exists x.A
\end{align*}
\]
MALL with lifts

- We consider here MALL formulas:

\[
\frac{\Gamma, A, B}{\Gamma, A \uplus B} \quad \frac{\Gamma_1, A \quad \Gamma_2, B}{\Gamma_1, \Gamma_2, A \ominus B}
\]

\[
\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \& B} \quad \frac{\Gamma, A}{\Gamma, A \oplus B}
\]

- with explicit lifts:

\[
\frac{\Gamma, A}{\Gamma, \uparrow A} \quad \frac{\Gamma, A}{\Gamma, \downarrow A}
\]
The formula corresponding to booleans is

$$B = \uparrow(\downarrow 1 \oplus \downarrow 1)$$
The formula corresponding to booleans is

\[ \mathbb{B} = \uparrow(\downarrow 1 \oplus \downarrow 1) \]
From formulas to games

The formula corresponding to booleans is

\[ B = \uparrow (\downarrow 1 \oplus \downarrow 1) \]
From formulas to games

The formula corresponding to booleans is

\[ B = (\downarrow 1 \oplus \downarrow 1) \]

\[ \begin{array}{c}
\ast \\
\downarrow q \\
\downarrow q \\
\downarrow T \\
\downarrow F \\
\end{array} \]

\[ \begin{array}{c}
T \\
F \\
\end{array} \]
From proofs to strategies

The game associated to $\uparrow A$ is of the form

\[
\uparrow \\
\downarrow \\
A
\]
From proofs to strategies

The game associated to $\uparrow A \otimes \uparrow B = \uparrow A \Rightarrow \uparrow B$ is of the form

\[
\begin{array}{c}
\uparrow \\
\uparrow \\
A \\
\end{array}
\quad
\begin{array}{c}
\uparrow \\
\uparrow \\
B \\
\end{array}
\]
From proofs to strategies

The game associated to $\uparrow A \otimes \uparrow B = \uparrow A \uplus \uparrow B$ is of the form

The corresponding asynchronous graph contains
From proofs to strategies

Three proofs of \( \uparrow A \nleftrightarrow \uparrow B \):
From proofs to strategies

Three proofs of $\uparrow A \not\leftrightarrow \uparrow B$:

\[
\begin{align*}
\vdash & \quad A, \uparrow B \\
\vdash & \quad \uparrow A, \uparrow B \\
\vdash & \quad \uparrow A, \uparrow B
\end{align*}
\]
From proofs to strategies

Three proofs of $\uparrow A \not\equiv \uparrow B$:

\[
\begin{align*}
\vdash A, B \\
\vdash A, \uparrow B \quad (\uparrow) \\
\vdash \uparrow A, \uparrow B \quad (\uparrow)
\end{align*}
\]
From proofs to strategies

Three proofs of $\uparrow A \not\leftrightarrow \uparrow B$:

$$\vdash A, B \quad \vdash A, \uparrow B \quad \vdash \uparrow A, \uparrow B$$
From proofs to strategies

Three proofs of $\uparrow A \not\leftrightarrow \uparrow B$:

\[ \vdash \uparrow A, \uparrow B \]
From proofs to strategies

Three proofs of $\uparrow A \not\Vdash \uparrow B$:

\[
\frac{\vdash \uparrow A, B}{\vdash \uparrow A, \uparrow B} \quad (\uparrow)
\]
From proofs to strategies

Three proofs of $\uparrow A \nleftrightarrow \uparrow B$:

\[
\begin{align*}
& \vdash A, B \\
& \vdash \uparrow A, \uparrow B (\uparrow) \\
& \vdash \uparrow A, \uparrow B (\uparrow)
\end{align*}
\]
From proofs to strategies

Three proofs of $\uparrow A \& \uparrow B$:

\[
\vdash A, B \\
\vdash \uparrow A, B (\uparrow) \\
\vdash \uparrow A, \uparrow B (\uparrow)
\]

\[
\vdash \uparrow A, \uparrow B \\
\vdash \uparrow A (\uparrow) \\
\vdash \uparrow B (\uparrow)
\]

\[
\vdash *, *, \uparrow B \\
\vdash \uparrow A (\uparrow) \\
\vdash *, *, \uparrow *
\]

\[
\vdash *, *, \uparrow * \\
\vdash \uparrow *, \uparrow * \\
\vdash *, \uparrow *
\]
From proofs to strategies

Three proofs of $\uparrow A \not\supset \uparrow B$:

\[
\vdash A, B \\
\vdash \uparrow A, \uparrow B \quad (\uparrow, \uparrow)
\]
From proofs to strategies

Three proofs of $\uparrow A \otimes \uparrow B$:

\[
\vdash A, B \\
\vdash \forall x. A, \forall y. B (\uparrow, \uparrow)
\]
Proofs explore formulas

\[
\begin{array}{ll}
\text{play} & = \text{exploration of the formula} \\
\text{proof} & = \text{exploration strategy}
\end{array}
\]
From sequentiality to causality

A game induces an asynchronous graph:
From sequentiality to causality

Conversely, one needs the Cube Property
The Cube Property

Theorem

*Homotopy classes of paths are generated by a partial order on moves.*
Definition
An asynchronous game is a pointed asynchronous graph satisfying the Cube Property.

Definition
A strategy $\sigma : A$ is a prefix closed set of plays on the asynchronous graph $A$. 
How do we characterize “good” strategies?
Definition

A strategy \( \sigma \) is \textbf{positional} when its plays form a subgraph of the game:

\[
\sigma \ni \exists s \xrightarrow{u} y \quad \text{and} \quad \exists s \xrightarrow{\sim t} \exists t \xrightarrow{u} y \quad \text{and} \quad \exists x \in \sigma \quad \text{implies} \quad \exists x \in \sigma
\]
Ingenuous strategies

We consider strategies which

1 are positional,
Ingenuous strategies

We consider strategies which

1. are **positional**, 

2. satisfy the **Cube Property**, 

We consider strategies which
① are positional,
② satisfy the Cube Property,
③ satisfy

\[
\begin{align*}
\sigma \exists m &\quad n \in \sigma \\
y_1 &\sim y_2 \\
\sigma \exists n &\quad m \in \sigma \\
y_1 &\sim y_2 \\
\sigma \exists n &\quad m \in \sigma \\
y_1 &\sim y_2
\end{align*}
\]
Ingenuous strategies

We consider strategies which

1. are **positional**, 
2. satisfy the **Cube Property**, 
3. satisfy . . .
4. are **deterministic**: 

\[
\begin{array}{c}
\sigma \ni m \\
y_1
\end{array} \quad \begin{array}{c}
n \in \sigma \\
y_2
\end{array}
\]

implies

\[
\begin{array}{c}
\sigma \ni m \\
y_1
\end{array} \sim \begin{array}{c}
\sigma \ni n \\
z
\end{array} \quad \begin{array}{c}
m \in \sigma \\
y_2
\end{array}
\]

where \( m \) is a Proponent move.
Ingenuous strategies

The game $\mathbb{B} \otimes \mathbb{B} \rightarrow \mathbb{B}$:

$\begin{array}{ccc}
\mathbb{B} & \otimes & \mathbb{B} \\
& & \rightarrow \\
q_L & \downarrow & q_R \\
& & \downarrow \\
T_L & & F_R \\
& & \downarrow \\
& & F
\end{array}$
The left conjunction

\[ B \otimes B \rightarrow B \]

\[ q_L \leftarrow q_R \rightarrow q \]

\[ T_L \rightarrow F_R \rightarrow F \]
Ingenuous strategies

The parallel conjunction

\[
\begin{array}{c}
B \otimes B \to B \\
q_L \leftarrow \quad q_R \leftarrow q \\
T_L \downarrow \quad F_R \downarrow F
\end{array}
\]
Property

Asynchronous games and strategies form a \(*\)-autonomous category (which is compact closed).
This category still has “too many” strategies!

\[ A \otimes B = A \Join B \]
In the spirit of the relational model, a strategy $\sigma$ should be characterized by its set $\sigma^\circ$ of halting positions.

**Definition**

A **halting position** of a strategy $\sigma$ is a position $x$ such that there is no Player move $m : x \rightarrow y$ that $\sigma$ can play.
The game $\text{bool} \otimes \text{bool}$ contains the subgraph:
The pair true $\otimes$ false:
The left biased pair true $\otimes$ false:
Definition
An ingenuous strategy $\sigma$ is **courteous** when it satisfies

$$
\begin{align*}
\sigma \ni m & \implies x \ni n \\
\sigma \ni n & \implies m
\end{align*}
$$

where $m$ is a Player move.

Theorem
A courteous ingenuous strategy $\sigma$ is characterized by its set $\sigma^\circ$ of halting positions.
Concurrent strategies

The halting positions of such a strategy $\sigma : A$ are precisely the fixpoints of a closure operator on the positions of $A$.

- We thus recover the model of concurrent strategies.
- A semantical counterpart of the focalization property: strategies can play all their Player moves in one “cluster” of moves.
Theorem

An innocent strategy $\sigma : A$ is a strategy which is

1. ingenuous
2. courteous
3. receptive: if $s \in \sigma$ and $s \cdot m$ is a play then $s \cdot m \in \sigma$
4. sequential

$+ \text{ dual property}$
Sequentiality, which is an asynchronous counterpart of alternation, schedules composition and ensures that it will perform correctly.
Without sequentiality...

The operation $(−)^\circ$ from the category of games and courteous ingenuous strategies to the category of relations is not functorial!
This mismatch is essentially due to deadlock situations occurring during the interaction.
The scheduling criterion

the left conjunction:

\[ B \otimes B \rightarrow B \]
The scheduling criterion

The right boolean composed with the left conjunction:

\[ B \otimes B \rightarrow B \]
The scheduling criterion

Two kinds of tensors: \( \otimes \) and \( \odot \).

\[
B \otimes B \rightarrow B = B^* \odot B^* \odot B
\]
The scheduling criterion

Two kinds of tensors: $\otimes$ and $\mathcal{B}$.
The scheduling criterion

Two kinds of tensors: $\otimes$ and $\mathfrak{g}$.

$B \otimes B$

$F$

$q$

$q$

$T$
The scheduling criterion

Two kinds of tensors: $\otimes$ and $\mathcal{S}$. 

\[ B \otimes B \]
Functoriality

Theorem

Strategies which are

- ingenuous
- courteous
- receptive
- and satisfy the scheduling criterion

compose and satisfy

\[(\sigma; \tau)^\circ = \sigma^\circ; \tau^\circ\]
Full completeness

The criterion only detects *oriented cycles*.

\[(A^* \otimes B^*) \not\cong (B \otimes A)\]
The criterion only detects oriented cycles.

\[(A^* \otimes B^*) \not\sim (B \otimes A)\]
The criterion only detects *oriented cycles*.

\[(A^* \otimes B^*) \not\sim (B \otimes A)\]
Full completeness for MLL + MIX

If we implement atoms by

\[
\begin{array}{c}
\downarrow \\
\uparrow \\
\uparrow
\end{array}
\]
Full completeness for MLL + MIX

If we implement axioms by

we get a fully complete model of MLL + MIX, similar to the AJ model.
Conclusion

We have:

• a game semantics adapted to concurrency
• an unifying framework in which we recover
  • innocent strategies
  • game semantics
  • concurrent games
  • the relational model
  • event structure semantics

In the future:

• extend this model (exponentials in particular)
• a local presentation of the correctness criterion
• typing of concurrent processes (CCS without deadlocks)
Thanks!