Polygraphs in homotopy type theory

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8-9 June 2023 / Métayer days

Polygraphs

The research of François

revolves around **polygraphs**.

A future for polygraphs

I will try to present an overview of recent results around

polygraphs in **homotopy type theory**

(which I recently started working on)

Some recent investigations around polygraphs

The general plan:

1. polygraphs for ω-categories are not right from a topological pov
2. we can define polygraphs for ∞-groupoids in HoTT
3. we can adapt traditional (rewriting) methods in this setting
4. we have new powerful methods to construct polygraphs

This is an overview: most of what I will present is not due to me (excepting errors).

We are currently investigating this with Émile Oleon.

Part I

Traditional polygraphs are not right

Polygraphs as free ω-categories

Polygraphs are (presentations of) free ω-categories, constructed from generators:

- 0-cells

$$x \qquad\qquad y \qquad\qquad z$$

- 1-cells

$$x \xrightarrow{f} x$$

- 2-cells

$$\begin{array}{ccccc} & & x & & \\ & \overset{f}{\nearrow} & \Downarrow \alpha & \overset{f}{\searrow} & \\ x & & \xrightarrow[f]{\qquad\qquad} & & x \end{array}$$

- etc.

This provides a good notion of presentation of category.

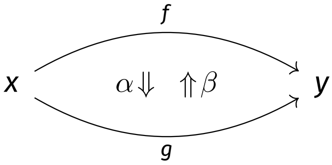
Equivalence between ω-categories

In an ω-category C, two cells $f, g : x \to y$ are **equivalent**, noted $f \sim g$, when there are cells

$$\alpha : f \Rightarrow g \qquad\qquad \beta : g \Rightarrow f$$

such that

$$\beta \circ \alpha \sim \mathrm{id}_f \qquad\qquad \alpha \circ \beta \sim \mathrm{id}_g$$



(this is a coinductive definition!)

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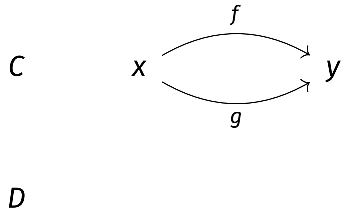

An *ω-functor* $F : C \to D$ is a **weak equivalence** when it is "surjective up to equivalence": given $f, g : x \to y$ and $\beta : Ff \Rightarrow Fg$ there is $\alpha : f \Rightarrow g$ such that $F\alpha = \beta$.

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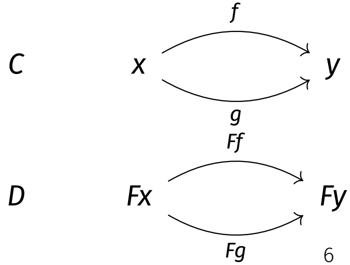

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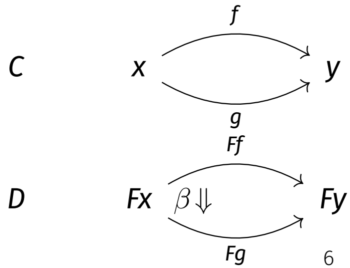

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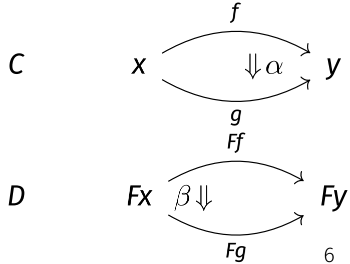

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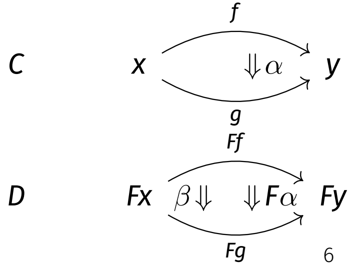

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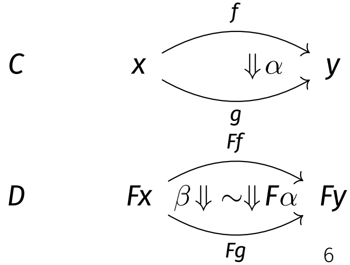

Resolutions of categories

In the 2003 article of François

Resolutions by polygraphs

it is shown that for every category C, there is a polygraph P such that

$$P \simeq C$$

Resolutions of categories

For instance, if we start from the category $B\mathbb{N}_2$

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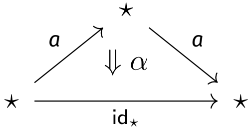

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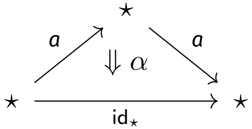
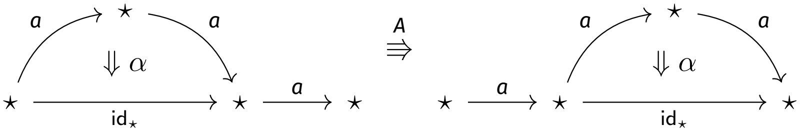


- one 3-cell



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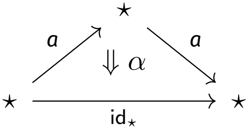

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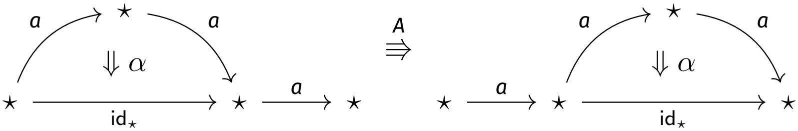


- and so on...

Polygraphs as cofibrant replacements

In the 2010 article by Lafont, Métayer, Worytkiewicz,

A folk model structure on omega-cat

they construct a model structure on $\mathbf{Cat}_\omega$ where

- weak equivalences are weak equivalences
- cofibrant objects are polygraphs

In particular, every ω-category C has a cofibrant replacement

$$P \overset{\sim}{\twoheadrightarrow} C$$

Polygraphic homology

Every polygraph P induces a chain complex

$$\mathbb{Z}P_0 \xleftarrow{d_0} \mathbb{Z}P_1 \xleftarrow{d_1} \mathbb{Z}P_2 \xleftarrow{d_2} \mathbb{Z}P_3 \xleftarrow{d_3} \cdots$$

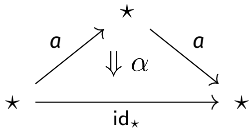
with our example

$$\mathbb{Z}\{\star\} \xleftarrow{d_0} \mathbb{Z}\{a\} \xleftarrow{d_1} \mathbb{Z}\{\alpha\} \xleftarrow{d_2} \mathbb{Z}\{A\} \xleftarrow{d_3} \cdots$$

with

$$d_1(\alpha) = -2a$$

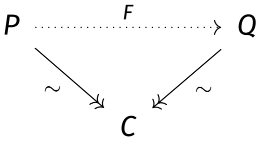
since



Polygraphic homology

We define the homology of an ω-category HC as the homology of the associated chain complex $H(\mathbb{Z}P)$ for some resolution P.

It does not depend on the choice of the resolution:



that's the point of using polygraphs!

However, the smaller the polygraph is, the simpler the calculations are!

Thomason equivalences

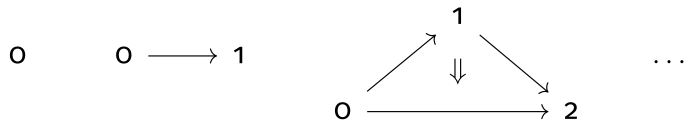
In 1987, Street has defined a functor

$$O : \Delta \to \mathbf{Cat}_\omega$$

reworked by Ara, Lafont and Métayer in 2023 in

Orientals as free algebras

The images of objects n of Δ can be pictured as



O_0 $\qquad O_1$ $\qquad O_2$ $\qquad \dots$

Thomason equivalences

In 1987, Street has defined a functor

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which induces a **nerve** functor

$$N : \mathbf{Cat}_\omega \to \hat{\Delta}$$

with

$$(NC)_n = \mathbf{Cat}_\omega(O_n, C)$$

The **Thomason equivalences** are induced on $\mathbf{Cat}_\omega$ by the one of $\hat{\Delta}$.

Ara's counter-example

Consider the polygraph P with

- one 0-generator $\star$
- one 2-generator $\alpha : \mathrm{id}_\star \Rightarrow \mathrm{id}_\star$

Its homology is the one of the chain complex

$$\mathbb{Z} \longleftarrow 0 \longleftarrow \mathbb{Z} \longleftarrow 0 \longleftarrow 0 \longleftarrow \cdots$$

In particular,

$$H_n(P) = 0$$

for $n > 2$.

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$$C_2 = \mathbb{N}$$

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which is the homology of $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$ and we have

$$H_n^{\mathrm{Th}}(C) = \begin{cases} \mathbb{Z} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

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In particular,

$$H_4^{\mathrm{Pol}}(C) = 0 \neq \mathbb{Z} = H_4^{\mathrm{Th}}(C)$$

Part II

Polygraphs in HoTT

Weak polygraphs

The morale is that polygraphs are not right from a topological point of view.

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Or rather working with *strict* ω-categories is a bad idea.

One way out consists in working with polygraphs adapted to *weak* ω*-categories* (Batanin) which is very technical.

Polygraphs for groupoids

Suppose that we are interested in ∞-**groupoids** instead of ∞-categories.

We get everything for “free” in **homotopy type theory**:

- every type is an ∞-groupoid
- polygraphs can be obtained as (some) higher inductive types

Homotopy type theory

Given a type A and two elements $x, y : A$, there is a type $x =^A y$ of **identities** between x and y.

We can think that

- A is a space
- $x, y : A$ are points in A
- $p : x =^A y$ is a path from x to y in A

Let's do a crash course in 2 slides.

Identity types

The identity types are characterized the fact that

- for every $x : A$, there is a constant path $\mathsf{refl}_x : x = x$
- given a predicate $P : \{y : A\} \to (x = y) \to \mathcal{U}$,
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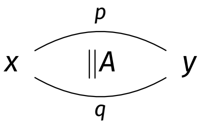
Because of this types act as ∞-groupoids:

- 0-cells are points $x, y : A$
- 1-cells are paths $p : x =^A y$
- 2-cells are paths between paths $\alpha : p =^{x =^A y} q$
- etc.



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Lemma
Given a path $p : x = y$, there is an "inverse" path $\overline{p} : y = x$.

Proof.
In the case where p is refl_x (thus x and y are the same), we can take $\overline{p} := \mathsf{refl}_x$. □

Similarly, we can compose paths, composition is associative up to higher cells, etc.

Equivalences

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- there are homotopies

$$\eta : g \circ f \sim \mathsf{id}_A \qquad\qquad \varepsilon : f \circ g \sim \mathsf{id}_B$$

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In homotopy type theory, **univalence** states that an identity $A = B$ is the same as an equivalence between A and B:

$$(A = B) \xrightarrow{\sim} (A \simeq B)$$

Resolutions in HoTT

Those equivalence play an analogous role as weak equivalence for ω-categories.

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Given a type A of interest, our goal is to construct a **resolution**,
i.e. a polygraph P such that

$$P \simeq A$$

…for a decent notion of “polygraph”.

Inductive types

In good programming languages, we can define inductive types:

```
type Bool : Type =
  false : Bool
  true  : Bool
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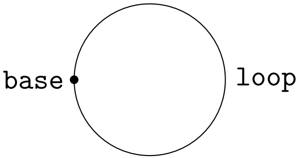
```
type S¹ : Type =
  base : S¹
  loop : base = base
```



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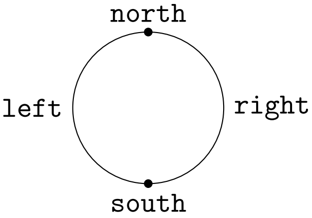
type S¹ : Type =
  north : S¹
  south : S¹
  left  : base = base'
  right : base = base'
```



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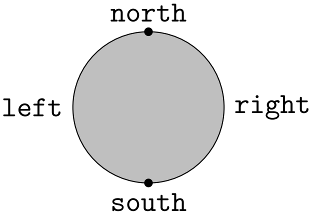
type S² : Type =
  north : S²
  south : S²
  left  : base = base'
  right : base = base'
  back  : left = right
  front : left = right
```



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  succ : Nat → Nat
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In good type theories, we can define **higher inductive types**:

```
type id : A → A → Type =
  refl : (x : A) → id x x
```

Higher inductive types

Since HITs are obtained by successively attaching disks, they play an analogous role of to **polygraphs** or **cellular complexes**.

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At least some of them.

Homotopy levels

A type A can be

-2. contractible:

$$\mathsf{isContr}(A) := \Sigma(x : A).(y : A) \to x =^A y$$

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$$\mathsf{isSet}(A) :\equiv (x\ y : A) \to \mathsf{isProp}(x =^A y)$$

n. a n-type: for every $x, y : A$, $x =^A y$ is an $(n-1)$-type

Propositional truncation

The **propositional truncation** $\|A\|_{-1}$ turns a type A into a proposition in a universal way: for every *proposition* B,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle t}\downarrow & \nearrow_{\tilde{f}} & \\ \|A\|_{-1} & & \end{array}$$

i.e. the map

$$- \circ t : (\|A\|_{-1} \to B) \to (A \to B)$$

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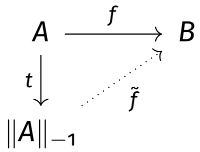

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The ***n*-truncation** $\|A\|_n$ can be defined similarly for n-types.

Propositional truncation

Propositional truncation can be implemented as a HIT:

```
type ‖A‖₋₁ : 𝒰 =
  in   : A → ‖A‖₋₁
  path : (x y : ‖A‖₋₁) → x = y
```

Propositional truncation

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\begin{aligned}
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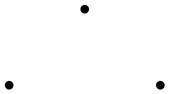
&\quad \texttt{in}\quad\ \ :\ A \to \|A\|_{-1} \\
&\quad \texttt{path}\ :\ (x\,y\ :\ \|A\|_{-1}) \to x = y
\end{aligned}
$$

For instance,



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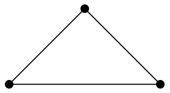
\begin{aligned}
&\texttt{type } \|A\|_{-1} : \mathcal{U} \texttt{ =} \\
&\quad \texttt{in} \quad : A \to \|A\|_{-1} \\
&\quad \texttt{path} : (x\, y : \|A\|_{-1}) \to x = y
\end{aligned}
$$

For instance,



Propositional truncation

Propositional truncation can be implemented as a HIT:

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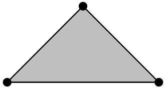
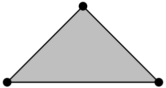


Propositional truncation

Propositional truncation can be implemented as a HIT:

```
type ‖A‖₋₁ : 𝒰 =
  in   : A → ‖A‖₋₁
  path : (x y : ‖A‖₋₁) → x = y
```

For instance,



We can similarly define higher truncations $\|A\|_n$:
it fills in all spheres of dimension $k > n$.

Propositional truncation

Propositional truncation can be implemented as a HIT:

$$\begin{aligned}&\texttt{type}\ \|A\|_{-1} : \mathcal{U}\ \texttt{=}\\ &\quad \texttt{in}\quad : A \to \|A\|_{-1}\\ &\quad \texttt{path} : (x\,y : \|A\|_{-1}) \to x = y\end{aligned}$$

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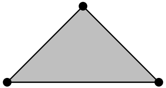

Problem solved?

Propositional truncation

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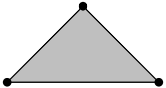

This is a **recursive** HIT, we do not want this as a "polygraph".

Part III

The rewriting approach

Presenting $B\mathbb{Z}_2$

Consider the type $B\mathbb{Z}_2$. All we need to know is that

- it has one connected component
- its fundamental group is $\pi_1(B\mathbb{Z}_2) = \mathbb{Z}_2$
- its higher homotopy groups are trivial: $\pi_n(B\mathbb{Z}_2) = 1$

$$0 \circlearrowright \star \circlearrowleft 1$$

Suppose that we want to construct a presentation of this type by a polygraph.

Presenting $B\mathbb{Z}_2 = 0 \circlearrowleft \star \circlearrowright 1$

We thus define the following HIT:

$$\texttt{type } P^1 : \mathcal{U} \texttt{ =}$$

which looks like

Presenting $B\mathbb{Z}_2 = \; 0 \circlearrowright \; \star \; \circlearrowleft 1$

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$$\texttt{type}\ P^1 : \mathcal{U}\ \texttt{=}$$
$$\star : P^1$$

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$\star$

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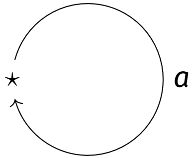
We thus define the following HIT:

$$\texttt{type}\ P^1 : \mathcal{U}\ =$$
$$\star : P^1$$
$$a : \star = \star$$

which looks like

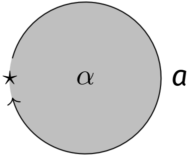


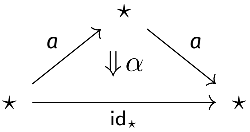
Presenting $B\mathbb{Z}_2 = 0 \circlearrowright \star \circlearrowleft 1$

We thus define the following HIT:

$$
\begin{aligned}
&\texttt{type}\ P^1 : \mathcal{U}\ \texttt{=} \\
&\quad \star : P^1 \\
&\quad a : \star = \star \\
&\quad \alpha : a \cdot a = \mathrm{refl}_\star
\end{aligned}
$$

which looks like





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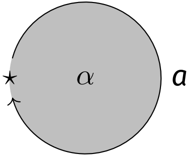

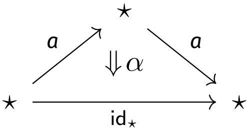

P^1 is a "good approximation" of $B\mathbb{Z}_2$ in the sense that it has one connected component and the right fundamental group, but higher groups are not trivial!

Killing higher groups

One way to obtain a faithful description of $B\mathbb{Z}_2$ consists in considering $\|P^1\|_1$, which amounts to change the definition to

$$
\begin{aligned}
&\texttt{type } Q^1 : \mathcal{U} \texttt{ =} \\
&\quad \star : Q^1 \\
&\quad a : \star = \star \\
&\quad \alpha : a \cdot a = \mathrm{refl}_\star \\
&\quad \texttt{trunc} : (x, y : Q^1)\ (p, q : x = y)\ (\alpha, \beta : p = q) \to \alpha = \beta
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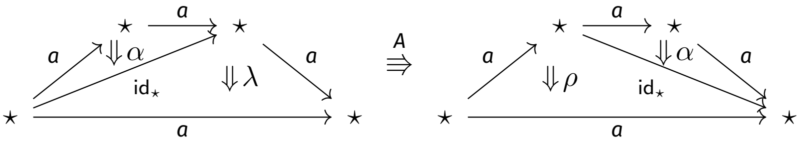
But this is a recursive definition!

Killing $\pi_2(P^1)$

A non-trivial element of $\pi_2(P^1)$ is



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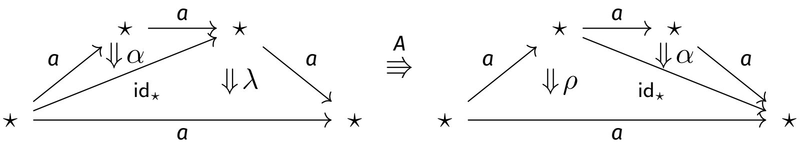

This suggests extending the previous HIT as

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&\quad A : (\alpha \otimes a) \cdot \lambda = (a \otimes \alpha) \cdot \rho
\end{aligned}
$$

Killing $\pi_2(P^1)$

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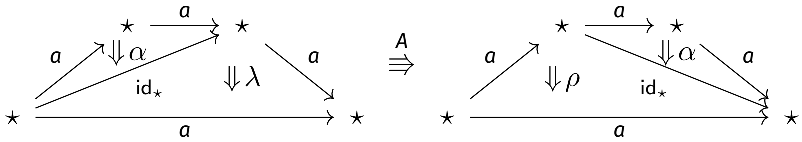


This suggests extending the previous HIT as

```
type P² : U =
```

$\star : P^2$

$a : \star = \star$

$\alpha : a \cdot a = \mathsf{refl}_\star$

$A : (\alpha \otimes a) \cdot \lambda = (a \otimes \alpha) \cdot \rho$

This is enough to have $\pi_2(P^2) = 1$, but how do we show this?
We need to have an induction principle for paths!

Killing $\pi_2(P^1)$

Our aim is to show that

$$\pi_2(P^2) = 1$$

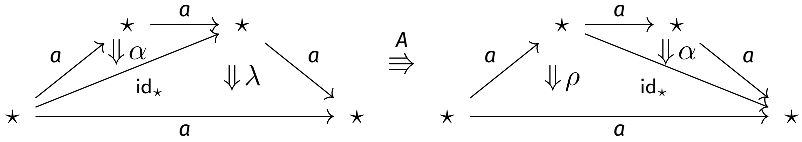
This is equivalent to showing that

- for every paths $p, q : \star = \star$
- for every paths $\alpha, \beta : p = q$

we have that there merely exists a path

$$A : \alpha = \beta$$

as in



Paths in homotopy quotients

Suppose given a type A with a relation $R : A \times A \to \mathcal{U}$, i.e. a **graph**.

Paths in homotopy quotients

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The **homotopy quotient** A/R is

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\begin{aligned}
&\texttt{type}\ A/R : \mathcal{U}\ \texttt{=} \\
&\quad \iota : A \to A/R \\
&\quad \texttt{path} : (x\ y\ :\ A) \to R\ x\ y \to x = y
\end{aligned}
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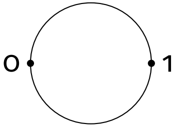
For instance, with $A = \{0, 1\}$ and $R\, x\, y := (x \neq y)$, we have

$$A/R = S^1 =$$



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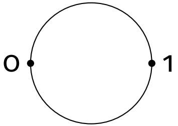
$A/R = S^1 =$ 

Given $x, y : A$, we want to have a description of the type $\iota\ x = \iota\ y$ in A/R.

Paths in homotopy quotients

Given A and $R : A \times A \to \mathcal{U}$, we define the **free groupoid** type

```
type FG : A → A → U =
```

Altenkirch, Kraus, von Raummer have shown

Theorem
Given $x, y : A$, *we have* $(\iota\, x = \iota\, y) = FG\; x\; y$.

Paths in homotopy quotients

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$$\texttt{type}\ FG\ :\ A \to A \to \mathcal{U}\ \texttt{=}$$
$$\texttt{[]} : (x : A) \to FG\ x\ x$$
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```
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   [] : (x : A) → FG x x
   _::_ : (f : R x y) → (p : FG y z) → FG x z
   _::'_ : (f : R y x) → (p : FG y z) → FG x z
```

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Theorem
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Paths in homotopy quotients

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&\quad \_\texttt{::'}\_ : (f\ :\ R\ y\ x) \to (p\ :\ FG\ y\ z) \to FG\ x\ z\\
&\quad \lambda : (f\ :\ R\ x\ y) \to (p\ :\ FG\ y\ z) \to f\texttt{::'}f\texttt{::}p = p\\
&\quad \rho : (f\ :\ R\ x\ y) \to (p\ :\ FG\ y\ z) \to f\texttt{::'}f\texttt{::}p = p
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&\quad \texttt{coh} : \ldots
\end{aligned}
$$

Altenkirch, Kraus, von Raummer have shown

Theorem
Given $x, y : A$, we have $(\iota\ x = \iota\ y) = FG\ x\ y$.

Paths in homotopy quotients

If we want to show a *property* on paths, for instance

$$\textit{for every paths } p, q : x = y \textit{ there merely exists an equality } p = q$$

it is enough to reason on the low-dimensional structure, i.e. zig-zags:

$$\begin{aligned}
&\texttt{type}\ FG\ :\ A \to A \to \mathcal{U}\ \texttt{=} \\
&\quad \texttt{[]} : (x : A) \to FG\ x\ x \\
&\quad _\texttt{::}_ : (f\ :\ R\ x\ y) \to (p\ :\ FG\ y\ z) \to FG\ x\ z \\
&\quad _\texttt{::'}_ : (f\ :\ R\ y\ x) \to (p\ :\ FG\ y\ z) \to FG\ x\ z
\end{aligned}$$

and we can reason by induction.

Paths in homotopy quotients

When reasoning with “lists” (or “zig-zags”) in the type

$$\texttt{type}\ FG : A \to A \to \mathcal{U}\ \texttt{=}$$
$$\texttt{[]} : (x : A) \to FG\ x\ x$$
$$\_::\_ : (f : R\ x\ y) \to (p : FG\ y\ z) \to FG\ x\ z$$
$$\_::'\_ : (f : R\ y\ x) \to (p : FG\ y\ z) \to FG\ x\ z$$

there is one problem with the base case: the elements of $FG\ x\ x$ of length 0 is equivalent to $x = x$, i.e. there can be more than simply [].

Things work out if we suppose that A is a set.

A coherent presentation

If we go back to the type

$$
\begin{aligned}
&\texttt{type } P^2 : \mathcal{U} \texttt{ =} \\
&\quad \star : P^2 \\
&\quad a : \star = \star \\
&\quad \alpha : a \cdot a = \mathsf{refl}_\star \\
&\quad A : (\alpha \otimes a) \cdot \lambda = (a \otimes \alpha) \cdot \rho
\end{aligned}
$$

This implies that

- any path $\star = \star$ has a representative as a list over $\{a, \overline{a}\}$

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This implies that

- any path $\star = \star$ has a representative as a list over $\{a, \overline{a}\}$
- identities between two such lists are generated by

$$laal' = ll' \qquad l\overline{a}al' = ll' \qquad la\overline{a}l' = ll'$$

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$$
la a l' = l l' \qquad l\overline{a} a l' = l l' \qquad l a \overline{a} l' = l l'
$$

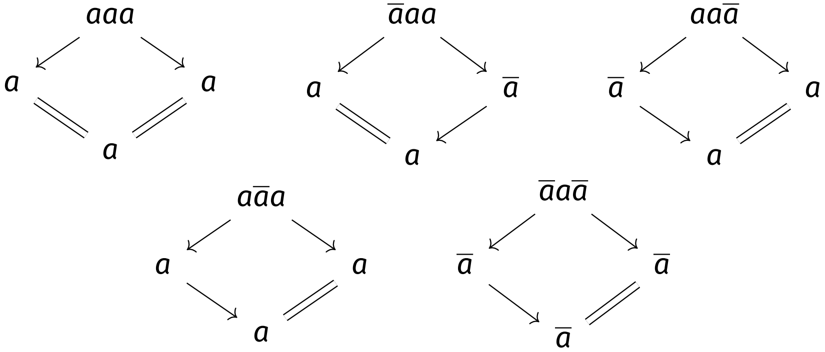
- identities between identities are generated by A and `coh`

A coherent presentation

It can be noted that the string rewriting system over $\{a, \overline{a}\}$ with rules

$$aa \to 1 \qquad \overline{a}a \to 1 \qquad a\overline{a} \to 1 \qquad \overline{a} \to a$$

is convergent



and those coherence correspond to identities between identities.

A coherent presentation

By the **Squier theorem** (with polygraphs implemented in type theory!),

any two zig-zags can thus be filled by identities between identities and we deduce (Kraus, von Raummer) that in the type

$$
\begin{aligned}
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&\quad \star : P^2\\
&\quad a : \star = \star\\
&\quad \alpha : a \cdot a = \mathrm{refl}_\star\\
&\quad \beta : \overline{a} = a\\
&\quad A : (\alpha \otimes a) \cdot \lambda = (a \otimes \alpha) \cdot \rho\\
&\quad B : \ldots\\
&\quad C : \ldots
\end{aligned}
$$

we have $\pi_2(P^2) = 1$.

A coherent presentation

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\\
&\quad A : (\alpha \otimes a) \cdot \lambda = (a \otimes \alpha) \cdot \rho \\
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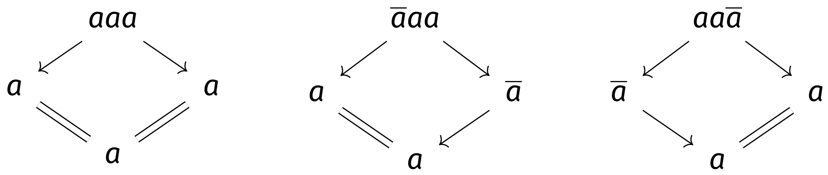
we have $\pi_2(P^2) = 1$.

The Squier theorem

$$aa \to 1 \qquad \bar{a}a \to 1 \qquad a\bar{a} \to 1 \qquad \bar{a} \to a$$

We can show the following:

1. the local confluence diagrams commute modulo equality



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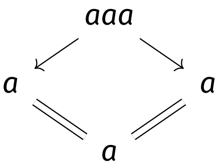
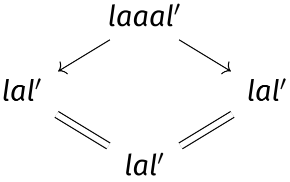
2. the local confluence diagrams can be extended under context



implies



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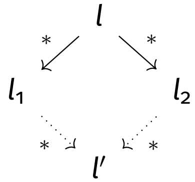
2. the local confluence diagrams can be extended under context
3. the rewriting system is terminating, and thus we have confluence

(modulo equality!)



The Squier theorem

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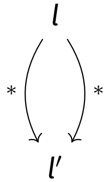
2. the local confluence diagrams can be extended under context
3. the rewriting system is terminating, and thus we have confluence
4. any two parallel zig-zags are equal



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2. the local confluence diagrams can be extended under context
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4. any two parallel zig-zags are equal

We thus have

$$\pi_2(P^2) = 1$$

What we have so far

We have constructed a type P^2 which has the same π_0, π_1 and π_2 as $B\mathbb{Z}_2$.

What we have so far

We have constructed a type P^2 which has the same π_0, π_1 and π_2 as $B\mathbb{Z}_2$.

Extending P^2 into P^3 which also has the same π_3 could certainly be done, but this would require tremendous amounts of work!

What we have so far

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Also, because of the previous remark, we are forced to reason with *set-theoretic* polygraphs, where we have *sets* of cells.

Part IV

The Milnor construction

Some other methods allow us to construct P^∞ such that

$$P^\infty = B\mathbb{Z}_2$$

Projective spaces

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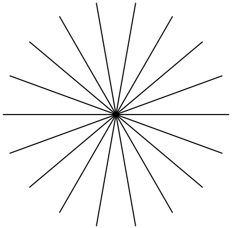
There is a wonderful construction based on the **join** construction, due to Milnor, Rijke, Finster, Joyal, …

…and the construction of orientals by Ara, Lafont and Métayer in *Orientals as free algebras* is closely related to the join construction.

Projective spaces

The **projective space P^n** is the space of lines in $\mathbb{R}^{n+1}$:



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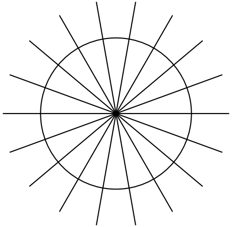
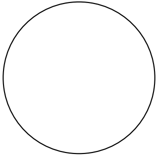


Projective spaces

The **projective space** P^n is the quotient of S^n under the antipodal action:



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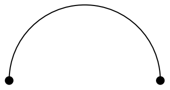
The **projective space** P^n is a disk D^n with antipodal points identified in ∂D^n:



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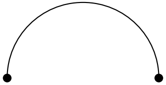

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$$\begin{array}{ccc} S^0 & \longrightarrow & D^1 \\ {\scriptstyle p^0}\downarrow & & \downarrow \\ P^0 & \longrightarrow & P^1 \end{array}$$

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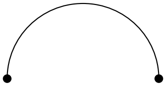

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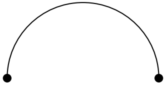

We thus have a pushout

$$\begin{array}{ccc} S^n & \longrightarrow & 1 \\ {\scriptstyle p^n}\big\downarrow & \ulcorner & \big\downarrow \\ P^n & \longrightarrow & P^{n+1} \end{array}$$

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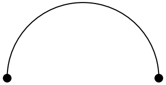

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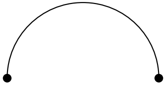

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Moreover, there are exactly two points y such that $p^n(y) = x$, i.e. $\mathrm{fib}_{p^n}(x) = S^0$.

The join construction

Given two types A and B, their **coproduct** $A \sqcup B$ is

$$\begin{array}{ccc} & & B \\ & & \downarrow \iota_2 \\ A & \xrightarrow{\iota_1} & A \sqcup B \end{array}$$

In type theory this can be defined as

$$\begin{aligned} &\texttt{type } A * B : \mathcal{U} \ \texttt{=} \\ &\quad \iota_1 : A \to A \sqcup B \\ &\quad \iota_2 : B \to A \sqcup B \end{aligned}$$

The join construction

Given two types A and B, their **join** $A * B$ is the homotopy pushout

$$\begin{array}{ccc} A \times B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & & \vdots\,{\scriptstyle \iota_2} \\ A & \underset{\iota_1}{\cdots\cdots\rightarrow} & A * B \end{array}$$

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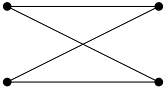

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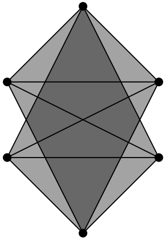

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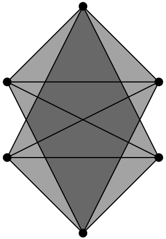

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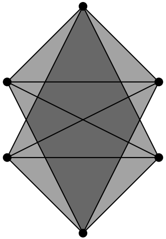

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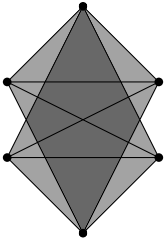

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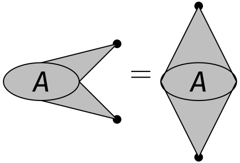

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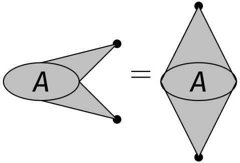

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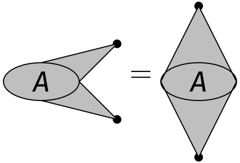

so that

$$(S^0)^{*n} = \Sigma^n S^0 = S^n$$

The inductive limit is

$$(S^0)^{*\infty} = S^\infty$$

which is known to be contractible.

Connecteness

A type A is n-**connected** when $\|A\|_n = 1$.

Proposition
If A is m-connected and B is n-connected then $A * B$ is $(m+n+1)$-connected.

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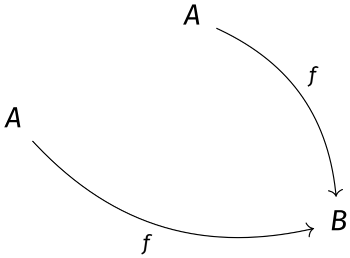
This construction was known as the **Milnor construction** (1956).

The join of maps

Given a map $f : A \to B$, consider the following construction



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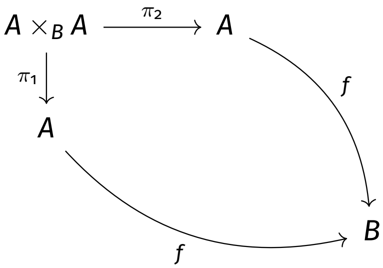

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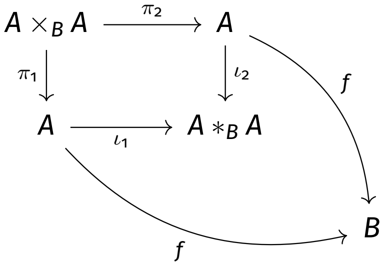

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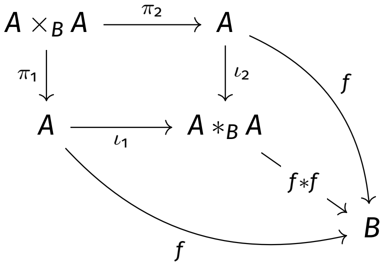

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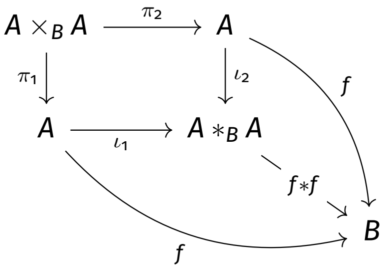

For $b : B$, we have

$$\mathrm{fib}_{f*f}(b) = \mathrm{fib}_f(b) * \mathrm{fib}_f(b)$$

with $\mathrm{fib}_f(b) = \Sigma(a : A).f\ a = b$.

The join of maps

If we compute iterated joins $f^{*n} = f * f * \ldots$, the fibers get more and more connected and at the colimit we have

$$\mathrm{fib}_{f^{*\infty}}(b) = (\mathrm{fib}_f(b))^{*\infty} = \mathbf{1}$$

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excepting when $\mathrm{fib}_f(b) = 0$ where we get 0.

In other words, $f^{*\infty}$ is the canonical inclusion

$$\mathrm{im}\, f := \Sigma(b : B).\| \mathrm{fib}_f(b) \|_{-1} \hookrightarrow B$$

Plan for the construction of $B\mathbb{Z}_2$

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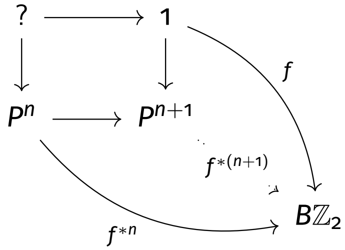
This recipe works for any BG (excepting the last point)!

Iterated joins

We have $f^{*(n+1)} = f^{*n} * f$:



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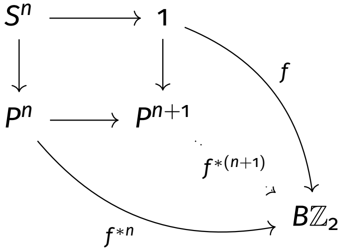

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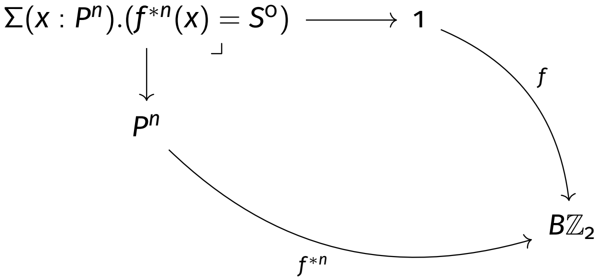

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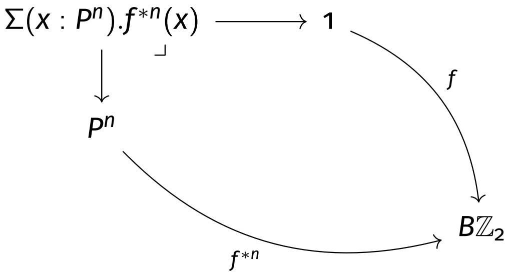

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$$
\begin{array}{ccc}
\Sigma(x : P^n).f^{*n}(x) & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
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\end{array}
$$

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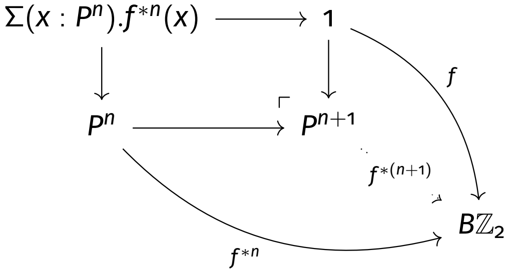

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$$
\begin{array}{ccc}
\Sigma(z : \Sigma(x : P^n).f^{*n}(x)).f(\star) & \longrightarrow & \Sigma(x : 1).f(\star) \\
\downarrow & & \downarrow \\
\Sigma(x : P^n).f^{*n} & \longrightarrow & \ulcorner\ \Sigma(x : P^{n+1})
\end{array}
$$

Iterated joins: inductive case

$$
\begin{array}{ccc}
(\Sigma(x : P^n).f^{*n}(x)) \times S^0 & \longrightarrow & S^0 \\
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We can therefore define $B\mathbb{Z}_2$ as the connected component of $\mathbb{B}$ in the universe:

$$B\mathbb{Z}_2 = \Sigma(X : \mathcal{U}).\|X = \mathbb{B}\|_{-1}$$

which satisfies

$$\Omega B\mathbb{Z}_2 = \mathbb{Z}_2$$

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More generally, we have, for $A : B\mathbb{Z}_2$

$$(\mathbb{B} = A) \quad = \quad A$$

and thus, for $A : X \to B\mathbb{Z}_2$,

$$\Sigma(x : X).(\mathbb{B} = A(x)) \quad = \quad \Sigma(x : X).A(x)$$

A fiber sequence

Writing

$$p : S^n \to P^n$$

it can be shown that for $x : P^n$, we have

$$\mathrm{fib}_p(x) = \Sigma(y : S^n).(p(y) = x) = S^0$$

i.e. we have a **fiber sequence**

$$S^0 \hookrightarrow S^n \xrightarrow{p} P^n$$

A long exact sequence

By general arguments, such a fiber sequence induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_3(S^0) \longrightarrow \pi_3(S^n) \longrightarrow \pi_3(P^n)$$

$$\longrightarrow \pi_2(S^0) \longrightarrow \pi_2(S^n) \longrightarrow \pi_2(P^n)$$

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A long exact sequence

By general arguments, such a fiber sequence induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow 0 \longrightarrow \pi_3(S^n) \longrightarrow \pi_3(P^n) \longrightarrow 0 \longrightarrow \pi_2(S^n) \longrightarrow \pi_2(P^n) \longrightarrow 0 \longrightarrow \pi_1(S^n) \longrightarrow \pi_1(P^n)$$

The P^n we have defined thus has the right homotopy groups!

Part V

Lens spaces

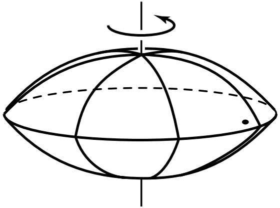
This generalizes to cyclic groups $\mathbb{Z}_m$!

(which requires a bit more than replacing 2 by m)

(work in progress with Émile Oleon)

Lens spaces

There is a geometric construction for $B\mathbb{Z}_m$ called **infinite lens spaces**



We can implement it in HoTT.

We first need to define $B\mathbb{Z}_m$.

Equality between endomorphisms

We write

$$\mathsf{Fin}\, m = \{0, 1, \ldots, m-1\}$$

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$$\begin{array}{ccc} \mathsf{Fin}\, m & \xrightarrow{f} & \mathsf{Fin}\, m \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ \mathsf{Fin}\, m & \xrightarrow[f]{} & \mathsf{Fin}\, m \end{array}$$

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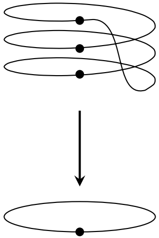
$$f(k) = f(s^k(0)) = s^k(f(0))$$

i.e. $(s = s) = \mathsf{Fin}\, m$.

Equality between endomorphisms

The picture to have in mind is



Cyclic groups

We define the type of **endomorphisms**

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$$\sigma = (\mathsf{Fin}\, m, s)$$

We define

$$B\mathbb{Z}_m = \mathcal{U}^{\circlearrowleft}_{S} = \Sigma(X : \mathcal{U}^{\circlearrowleft}).\|X = \sigma\|_{-1}$$

Lens spaces

We have a map

$$f : 1 \to B\mathbb{Z}_m$$

given by S and we can define

$$B\mathbb{Z}_m = \operatorname{im} f = \partial^-(f^{*\infty})$$

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We also have a map

$$f : S^1 \to B\mathbb{Z}_m$$

and we can define

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Lens spaces

To be more precise there are multiple maps

$$f^k : S^1 \to B\mathbb{Z}_m$$

$$\star \mapsto \sigma$$

$$\texttt{loop} \mapsto s^k : \sigma \simeq \sigma$$

Given $k_1, \ldots, k_n$ all relatively prime to m, we can define

$$L(k_1, \ldots, k_n) = \partial^-(f^{k_1} * \ldots * f^{k_n})$$

which correspond to the well-known **lens spaces**.

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which correspond to the well-known **lens spaces**.

By default,

$$L^n = L(1, \ldots, 1)$$

and

$$L^\infty = \text{colim}_n\, L^n$$

Lens spaces

It can be shown that we have a pushout

$$\begin{array}{ccc} S^{2n+1} & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ L^n & \longrightarrow & L^{n+1} \end{array}$$

from which we can deduce that we have a cellular decomposition
with one cell in each dimension:
at each step we are adding a cell in dimension $2n$ and one in dimension $2n+1$.

Lens spaces: traditional definition

We can see S^{2n-1} as a subset of $\mathcal{C}^n$:

$$S^{2n-1} = \{(z_1, \ldots, z_n) \mid |z_1|^2 + \ldots + |z_n|^2 = 1\}$$

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There is a free action ζ of $\mathbb{Z}_m$ on S^{2n-1} given by

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Lens spaces: traditional definition

We can see S^{2n-1} as a subset of $\mathcal{C}^n$:

$$S^{2n-1} = (S^1)^{*n}$$

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$$g : B\mathbb{Z}_m \to \mathcal{U} \qquad g(x) = \mathrm{fib}_{f^{*n}}(x) \qquad g(\star) = S^{2n-1}$$

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i.e. an action of $B\mathbb{Z}_m$ on S^{2n-1} and we have

$$S^{2n-1}/B\mathbb{Z}_m = \Sigma(x : B\mathbb{Z}_m).g(x) = \Sigma(x : B\mathbb{Z}_m).\,\mathrm{fib}_{f^{*n}}(x) = L^n$$

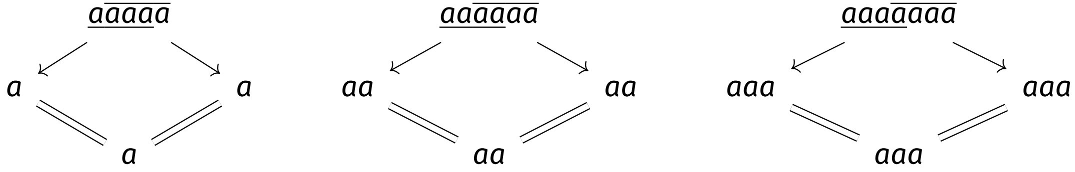
which corresponds to the usual definition of lens spaces!

Compared to rewriting

We have a presentation

$$\mathbb{Z}_m = \langle a \mid a^m = 1 \rangle$$

If we compute the critical branchings for $\mathbb{Z}_4$, we get 3 of them:



which would inevitably lead to a larger presentation…

Lots remains to be done!

Closing words

Thank you François!