A Bit of Physics

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CEA, LIST

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Lagrangian Mechanics

Conservative forces

A force is **conservative** when the work

$$\int_{t_1}^{t_2} F(q(t)) \cdot \dot{q}(t)$$

only depends on the endpoints $q(t_1)$ and $q(t_2)$.

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Remark

This is not true for friction for instance since it clearly depends on the path: we neglect heat loss!

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Principle

All forces are conservative.

Remark

When the space is simply connected, this is equivalent to

$$dF \quad = \quad \nabla \times F = 0$$

which is equivalent to

$$F \quad = \quad -\nabla V$$

Newton's law

In the case of a conservative force, Newton's law gives

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which turns out to be equivalent to the fact that the *action*

$$S \quad = \quad \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{x}^2 - V(x)\right) \mathrm{d}t$$

is stationary wrt variations of the path $x(t)$.

The Lagrangian

Principle (Hamilton)

A mechanical system is characterized by a function

$$L(q, \dot{q}, t)$$

called the **Lagrangian** where q is (a vector of) *position*, $\dot{q}$ is (a vector of) *speed* and t is the *time* and the paths it takes follows the **least action principle**: it minimizes the **action**

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, \mathrm{d}t$$

between any two instants t_1 and t_2.

The Lagrangian

More formally, the position is a point q in a manifold M (for instance for the double pendulum in $\mathbb{R}^3$, $M \cong S^2 \times S^2$) and the evolution of the system is given by a path

$$q \quad : \quad [t_1, t_2] \quad \to \quad M$$

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The velocity is $\dot{q}(t) \in T_{q(t)}M$. The Lagrangian is a function

$$L \quad : \quad TM \quad \to \quad \mathbb{R}$$

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Notice that when we write $L(q^i, \dot{q}^i)$, $\dot{q}^i$ is a coordinate not the derivative of something.

The least action principle

Suppose that we perturb the position by taking

$$q + \delta q$$

where δq is a (always small) function such that

$$\delta q(t_1) = \delta q(t_2) = 0$$

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The resulting change in action is

$$\delta S \quad = \quad \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t)\, \mathrm{d}t - \int_{t_1}^{t_2} L(q, \dot{q}, t)\, \mathrm{d}t$$

and the least action principle says

$$\delta S \quad = \quad 0$$

Formalizing the δ

In order to make this formal, we consider a family of paths

$$q_s \quad : \quad [0, T] \to M$$

smoothly indexed by $s \in [-1, 1]$, such that $q_s(0) = a$, $q_s(1) = b$ and $q_0 = q$.

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We write

$$\delta \qquad \text{for} \qquad \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}$$

so that the least action principle is

$$\delta S \quad = \quad 0$$

Euler-Lagrange equation

If we suppose that $q_s = q$ for every s outside a given chart,

$$
\begin{aligned}
0 = \delta S &= \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, \mathrm{d}t \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right) \mathrm{d}t
\end{aligned}
$$

with $(q^i, \dot{q}^i)$ local basis for TM (by abuse of notation!).

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with $(q^i, \dot{q}^i)$ local basis for TM (by abuse of notation!).
Since $\delta \dot{q} = \mathrm{d}\delta q / \mathrm{d}t$, we have

$$0 = \delta S = \left[\frac{\partial L}{\partial \dot{q}^i} \delta q^i \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i \, \mathrm{d}t$$

and this it must be true for all δq:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0$$

called the **Euler-Lagrange equation**.

Momentum and force

The Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} \quad = \quad 0$$

relates

- the **momentum**:

$$p_i \quad = \quad \frac{\partial L}{\partial \dot{q}^i}$$

- the **force**:

$$F_i \quad = \quad \frac{\partial L}{\partial q^i}$$

Momentum and force

The Euler-Lagrange equation

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relates

- the **momentum**:

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

- the **force**:

$$F_i = \frac{\partial L}{\partial q^i}$$

In other words, it states

$$F_i = \dot{p}_i$$

In the case of a particle

We have

$$L = T - V$$

where

- $T = \frac{1}{2}mv^2$ is the **Kinetic energy**
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In the E-L equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

- $F = \partial L/\partial q$ is the **force**
- $p = \partial L/\partial \dot{q}$ is the **momentum** $p = mv$

in other words, we have recovered **Newton's law**

$$F(q(t)) = ma(t)$$

How do we know that?

Principle (Galileo's relativity)

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$$L = L(v^2)$$

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By elaborating on these ideas, we find L proportional to v^2:

$$L = \frac{1}{2}mv^2$$

Conservation of energy

We have (with Einstein summation convention)

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t}$$

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By homogeneity of time, $\partial L/\partial t = 0$ and since, by E-L we have $\partial L/\partial q_i = (\mathrm{d}/\,\mathrm{d}t)(\partial L/\partial \dot{q}_i)$

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \dot{q}_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}_i \frac{\partial L}{\partial q_i}\right)\end{aligned}$$

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Therefore **energy is conserved**:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right) = 0$$

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(for a particle, $E = mv^2 - \left(\frac{1}{2}mv^2 - V\right) = \frac{1}{2}mv^2 + V$).

Conservation of momentum

Similarly the **momentum is conserved** by invariance of space.

Noether's theorem

"If a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in time."

Noether's theorem

Consider a smooth map $\mathbb{R} \times \Gamma \to \Gamma$, called *family of symmetries*,

$$(s, q) \mapsto q_s$$

with $q_0 = q$

Noether's theorem

Consider a smooth map $\mathbb{R} \times \Gamma \to \Gamma$, called *family of symmetries*,

$$(s, q) \mapsto q_s$$

(e.g. $q_s(t) = q(s + t)$ or $q_s(t) = q(t) + sv$, etc.)

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Consider a smooth map $\mathbb{R} \times \Gamma \to \Gamma$, called *family of symmetries*,

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$$\delta L \quad = \quad \frac{\mathrm{d}\ell}{\mathrm{d}t}$$

i.e. for every path q,

$$\frac{\mathrm{d}}{\mathrm{d}s} L(q_s(t), \dot{q}_s(t)) \Big|_{s=0} \quad = \quad \frac{\mathrm{d}}{\mathrm{d}t} \ell(q_s(t), \dot{q}_s(t))$$

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then

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(p_i \delta q^i - \ell \right) \quad = \quad 0$$

Noether's theorem

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_i \delta q^i - \ell\right) = 0$$

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(p_i \delta q^i - \ell\right) &= \dot{p}_i \delta q_i + p_i \delta \dot{q}_i - \frac{\mathrm{d}\ell}{\mathrm{d}t} \\
&= \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i} - \delta L \\
&= \delta L - \delta L \\
&= 0
\end{aligned}$$

□

Applications of Noether's theorem

Conservation of energy

Consider

$$q_s(t) \quad = \quad q(t+s)$$

We have

$$\delta L = \left.\frac{\mathrm{d}L(q_s)}{\mathrm{d}s}\right|_{s=0} = \frac{\mathrm{d}L}{\mathrm{d}t} = \dot{L}$$

and taking $\ell = L$, we deduce that the **energy**

$$E \quad = \quad p_i \dot{q}^i - L$$

is conserved.

Applications of Noether's theorem

Conservation of momentum
Consider

$$q_s(t) \quad = \quad q_s(t) + sv$$

For a free particle, we have $L = \frac{1}{2}m\dot{q}^2$, and

$$\delta L \quad = \quad 0$$

because $\delta\dot{q} = 0$ and L only depends on $\dot{q}$ (not on q).
Taking $\ell = 0$, we deduce that the **momentum**

$$p_i \delta q^i \quad = \quad m\dot{q}_i v^i \quad = \quad m\dot{q} \cdot v$$

is conserved.

(notice that this "momentum" is not the same as before, even though it has the same value on usual examples)

Applications of Noether's theorem

Conservation of angular momentum

Consider for $X \in \mathfrak{so}(n)$ an antisymmetric matrix (so that $\mathrm{e}^{sX} \in SO(n)$),

$$q_s(t) \quad = \quad \mathrm{e}^{sX}\, q(t)$$

We have

$$\delta L \quad = \quad \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta q^i$$

In the case of a free particle $\frac{\partial L}{\partial q^i} = 0$, $\frac{\partial L}{\partial \dot{q}^i} = m\dot{q}_i$, and

$$\delta \dot{q}^i = \left.\frac{\mathrm{d}\dot{q}^i}{\mathrm{d}s}\right|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\left.\left(\mathrm{e}^{sX}\, q\right)\right|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}t} Xq = X\dot{q}$$

i.e.

$$\delta L = m\dot{q} \cdot (X\dot{q}) = 0$$

by anisymmetry of X. The **angular momentum**

Hamiltonian Mechanics

The Hamiltonian

Instead of starting from the Lagrangian $L(q, \dot{q})$

$$L \quad : \quad TM \quad \to \quad \mathbb{R}$$

we can characterize the system from the energy

$$H(q, p) \quad = \quad p_i q^i - L(q, \dot{q})$$

called **Hamiltonian** and seen as

$$H \quad : \quad T^* M \quad \to \quad \mathbb{R}$$

since

$$p_i \quad = \quad \frac{\partial L}{\partial q^i}$$

Changing coordinates

We have a map

$$\lambda \quad : \quad TM \quad \to \quad T^*M$$
$$(q, \dot{q}) \quad \mapsto \quad (q, p)$$

where

$$p_i \quad = \quad \frac{dL}{dq^i}$$

Changing coordinates

We have a map

$$\begin{array}{cccc} \lambda & : & TM & \rightarrow & T^*M \\ & & (q, \dot{q}) & \mapsto & (q, p) \end{array}$$

where

$$p_i = \frac{\mathrm{d}L}{\mathrm{d}q^i}$$

which can be described in a coordinate-free way.

Regular Lagrangians

L is **regular** if it induces a diffeomorphism

$$\lambda \quad : \quad TM \quad \to \quad X \subseteq T^*M$$

to the **phase space** X. It is **strongly regular** when $X = T^*Q$.

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to the **phase space** X. It is **strongly regular** when $X = T^*Q$.

When $\lambda : TM \to X \subseteq T^*M$ is an isomorphism, we can see

$$\dot{q}^i : TM \to \mathbb{R} \qquad \text{as} \qquad \dot{q}^i \circ \lambda : X \to \mathbb{R}$$

which we both write $\dot{q}^i$. An in particular, we can see $p_i = \frac{\partial L}{\partial q^i}$ as $X \to \mathbb{R}$ instead of $M \to \mathbb{R}$.

Hamilton's equations

We have

$$\mathrm{d}L \quad = \quad \frac{\partial L}{\partial q^i}\,\mathrm{d}q^i + \frac{\partial L}{\partial \dot{q}^i}\,\mathrm{d}\dot{q}^i \quad = \quad \dot{p}_i\,\mathrm{d}q^i + p_i\,\mathrm{d}\dot{q}^i$$

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and

$$\begin{aligned}\mathrm{d}H \quad &= \quad \mathrm{d}(p_i\dot{q}^i - L) \quad = \quad \dot{q}^i\,\mathrm{d}p_i + p_i\,\mathrm{d}\dot{q}^i - \left(\dot{p}_i\,\mathrm{d}q^i + p_i\,\mathrm{d}\dot{q}^i\right) \\ &= \quad q^i\,\mathrm{d}p_i - \dot{p}_i\,\mathrm{d}q^i\end{aligned}$$

Hamilton's equations

We have

$$\mathrm{d}L = \frac{\partial L}{\partial q^i}\,\mathrm{d}q^i + \frac{\partial L}{\partial \dot{q}^i}\,\mathrm{d}\dot{q}^i = \dot{p}_i\,\mathrm{d}q^i + p_i\,\mathrm{d}\dot{q}^i$$

and

$$\begin{aligned}\mathrm{d}H &= \mathrm{d}(p_i\dot{q}^i - L) = \dot{q}^i\,\mathrm{d}p_i + p_i\,\mathrm{d}\dot{q}^i - \left(\dot{p}_i\,\mathrm{d}q^i + p_i\,\mathrm{d}\dot{q}^i\right)\\ &= q^i\,\mathrm{d}p_i - \dot{p}_i\,\mathrm{d}q^i\end{aligned}$$

And therefore

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

The principle of least action

Notice that the action can be defined as

$$S = \int_{t_1}^{t_2} \left(p_i \dot{q}^i - H \right) \mathrm{d}t$$

and the principle of least action holds iff Hamilton's equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

hold.

Given a function $f(q,p,t)$ on the manifold, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f &= \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial p}\frac{\partial H}{\partial p} + \frac{\partial f}{\partial q}\frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \\ &= \{f, H\} + \frac{\partial f}{\partial t}\end{aligned}$$

The Poisson bracket

Given a function $f(q,p,t)$ on the manifold, we have

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where the **Poisson bracket** is defined by

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial f}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial f}{\partial q^i}$$

The Poisson bracket

Given a function $f(q, p, t)$ on the manifold, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} f &= \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t} \\
&= \frac{\partial f}{\partial p}\frac{\partial H}{\partial p} + \frac{\partial f}{\partial q}\frac{\partial H}{\partial q} + \frac{\partial f}{\partial t} \\
&= \{f, H\} + \frac{\partial f}{\partial t}
\end{aligned}$$

where the **Poisson bracket** is defined by

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial f}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial f}{\partial q^i}$$

In particular, an invariant $f(q, p)$ satisfies $\{f, H\} = 0$.

The Poisson bracket

Notice that we have

$$\dot{q} = \frac{\partial H}{\partial p} = \{q, H\} \qquad \dot{p} = \frac{\partial H}{\partial q} = \{p, H\}$$

The Poisson bracket

Notice that we have

$$\dot{q} = \frac{\partial H}{\partial p} = \{q, H\} \qquad \dot{p} = \frac{\partial H}{\partial q} = \{p, H\}$$

And also

$$\left\{q^i, q^j\right\} = 0 \qquad \{p_i, p_j\} = 0 \qquad \left\{q^i, p_j\right\} = \delta_{ij}$$

Symplectic manifolds

The phase space can be more generally modeled as:

Definition

A **symplectic manifold** M is a manifold equipped with a 2-form ω which is

- closed:

$$\mathrm{d}\omega \quad = \quad 0$$

- non-degenerate: for every $p \in M$ and $v \in TM$,

$$\omega_p(v, -) \quad : \quad TM \quad \to \quad \mathbb{R}$$

  is not 0 (everywhere)

Hamiltonian on a simplectic manifold

Since ω is non-degenerate, it provides a vector bundle isomorphism

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determines a vector field $X_H \in \Gamma TM$ such that

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The **Poisson bracket** is then defined by

$$\{f, g\} \quad = \quad \omega(X_g, X_f) \quad = \quad \mathrm{d}g(X_f)$$

Hamiltonian on a simplectic manifold

For instance, given M of dimension $2n$ with canonical coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$, the simplectic form is

$$\omega \quad = \quad \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i$$

and we have

$$X_H \quad = \quad \left(\frac{\partial H}{\partial p_i}, \frac{\partial H}{\partial q^i}\right)$$

Special Relativity

The principle of relativity

Principle (Einstein)

The speed c of light is the same in two referentials moving at constant speed.

What can we draw from this?

Suppose that a particle moves at speed c from (x_1, y_1, z_1) to (x_2, y_2, z_2) between instants t_1 and t_2. We have

$$-c^2(t_2 - t_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = 0$$

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But also

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$$-c^2(t'_2 - t'_1)^2 + (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 = 0$$

This suggests to introduce a metric of the form

$$\begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

on a **spacetime** manifold, which should be invariant of the referential.

Lorentz transformations

Suppose that we have a referential R' moving at speed v along x axis wrt R. Classically, we have

$$t' = t \qquad x' = x - vt \qquad y' = y \qquad z' = z$$

This is not consistent with relativity principle:

$$x^2 + y^2 + z^2 = ct \qquad \text{vs} \qquad (x - vt)^2 + y^2 + z^2 = ct$$

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And actually, now we have **Lorentz transformations**

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad y' = y \qquad z' = z$$

Deriving Lorentz transformations

Suppose that light is moving along y axis in R.

- in R:

$$c = \frac{y}{t}$$

- in R':

$$c = \frac{\sqrt{y^2 + v^2 t^2}}{t'}$$

and therefore

$$t' = t\frac{\sqrt{y^2 + v^2 t^2}}{y}$$

The proper distance

One thing that one can notice about the metric defined by

$$s = \frac{1}{c}\sqrt{-c^2(t_2 - t_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

is that the distance between two events is invariant under Lorentz transformations!

(which is not the case of distances, or time differences)

Moving clocks

From the fact that s is invariant it is easy to show that during a time $\mathrm{d}t$ in rest frame, in a frame moving at speed v a clock will have advanced from $\mathrm{d}t'$ such that

$$\mathrm{d}t' \quad = \quad \frac{\mathrm{d}s}{c} \quad = \quad \mathrm{d}t\sqrt{1-\frac{v^2}{c^2}}$$

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Moving clocks go more slowly!

The relativistic Lagrangian

For a free particle, the action must be of the form

$$S = -\alpha \int_a^b \mathrm{d}s$$

with $\alpha \geq 0$.

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Therefore

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Therefore

$$S \quad = \quad -\int_{t_1}^{t_2} \alpha c \sqrt{1 - \frac{v^2}{c^2}}$$

Imposing $\lim_{c \to \infty} L = \frac{1}{2} m v^2$ implies $\alpha = mc$, i.e.

$$L \quad = \quad -mc^2 \sqrt{1 - \frac{v^2}{c^2}}$$

Momentum and energy

The **relativistic momentum** of a free particle is

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

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Notice that we recover the classical notion of energy when $v \ll c$:

$$E \approx mc^2 + \frac{mv^2}{2} + \dots$$

Hamiltonian

From preceding formulas we have

$$\frac{E^2}{c^2} = p^2 + m^2 c^2$$

and therefore

$$H = c\sqrt{p^2 + m^2 c^2}$$

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$$H = c\sqrt{p^2 + m^2c^2}$$

In particular, when $v \ll c$,

$$H \approx mc^2 + \frac{p^2}{2m} + \dots$$

Electromagnetics

The electric force

The electric force from a charge q' on a charge q distant from $\vec{r}$

$$q' \xrightarrow{\vec{r}} q$$

is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^2}\frac{\vec{r}}{r}$$

where

- q and q' are the charges (in Coulomb)
- r is the distance (in meters)
- F is the force in (in Newtons)
- ε_0 is the permittivity of free space (in $C^2 m^{-2} N^{-1}$)

Electric field

This can be reformulated by saying that a charge q is subject to a force

$$\vec{F} = q\vec{E}$$

and generates an electric field

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{\vec{r}}{r}$$

The nabla symbol

In the following, we are going to make use of the nabla operator

$$\nabla \quad = \quad \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Divergence

Definition

The **divergence** of a vector field $\vec{F}$ measures its flux

$$\nabla \cdot \vec{F} \quad = \quad \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 \quad = \quad \lim_{V \to \{*\}} \iint_{S(V)} \frac{\vec{F} \cdot \vec{n}}{|V|} \, \mathrm{d}S$$

Curl

Definition

The **curl** measures rotation

$$\begin{aligned}\nabla \times \vec{F} &= (\partial_2 F_3 - \partial_3 F_2, \partial_1 F_3 - \partial_3 F_1, \partial_1 F_2 - \partial_2 F_1) \\ &= \lim_{A \to \{*\}} \oint_A \left(\frac{\vec{F} \cdot \mathrm{d}\vec{r}_i}{|A|} \right)\end{aligned}$$

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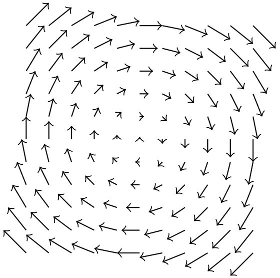
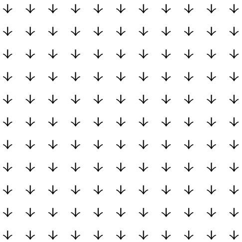
Example

$$\nabla \times (y \, \mathrm{d}x - x \, \mathrm{d}y) = -2 \, \mathrm{d}z$$





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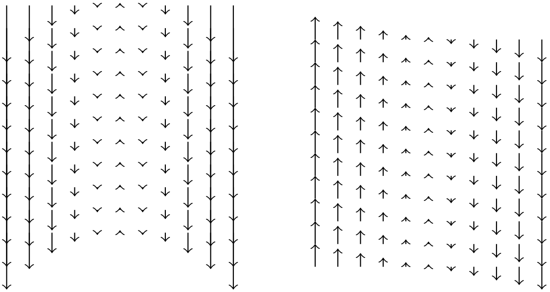
$$\nabla \times -x^2\,\mathrm{d}y = -2x\,\mathrm{d}z$$



Maxwell equations

$$
\begin{aligned}
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\
\nabla \cdot \vec{E} &= \rho \\
\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial r} &= \vec{j}
\end{aligned}
$$

where:

- $\vec{E}$ is the **electric field**
- $\vec{B}$ is the **magnetic field**
- ρ is the **charge density**
- $\vec{j}$ is the **electric current density**

Quantum Mechanics

Complex vector spaces

We will consider vector spaces over the field $\mathbb{C}$.

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We write $\overline{-}$ for the functor **Vect** $\to$ **Vect** such that a linear $f : \overline{V} \multimap W$ is an **antilinear** $f : V \multimap W$, i.e.

$$f(\lambda v) \quad = \quad \overline{\lambda} f(v)$$

$\overline{V}$ is the same as V excepting that λv in $\overline{V}$ is $\overline{\lambda} v$ in V.

Hilbert spaces

Definition

A **Hilbert space** H is a complex (or real) inner product space:

$$\langle - | - \rangle \quad : \quad \overline{H} \otimes H \quad \multimap \quad \mathbb{C}$$

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- $\langle x | \lambda y \rangle = \lambda \langle x | y \rangle$
- $\langle x | y_1 + y_2 \rangle = \langle x | y_1 \rangle + \langle x | y_2 \rangle$
- $\langle x | y \rangle = \overline{\langle y | x \rangle}$
- $\langle x | x \rangle \geq 0$ with equality precisely when $x = 0$

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Remark

- Inner prod is antilinear wrt first argument: $\langle \lambda x | y \rangle = \overline{\lambda} \langle x | y \rangle$
- $\langle x | x \rangle$ is real

Examples

The famous examples

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- ℓ^2: the sequences $(z_i)_{i\in\mathbb{N}}$ such that

$$\sum_{i\in\mathbb{N}} |z_i|^2 \quad < \quad \infty$$

with

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- $L^2(X,\mu)$: given a measure space (X,M,μ) where M is a σ-algebra of subsets of X, the space of functions $f : X \to \mathbb{C}$ such that

  $$\int_X |f|^2 \, \mathrm{d}\mu \quad < \quad \infty$$

  with

  $$\langle g|f\rangle \quad = \quad \int_X \overline{g(t)} f(t) \, \mathrm{d}t$$

A category

The most general notion of morphism we consider are continuous linear functions between Hilbert spaces.

The category of Hilbert spaces is denoted

$$\mathbf{Hilb}$$

and the full subcategory of finite dimensional spaces

$$\mathbf{FdHilb}$$

Riesz representation theorem

Theorem

Given a Hilbert space H

$$\overline{H} \quad \cong \quad \mathbf{Hilb}(H, \mathbb{C})$$

Proof.

- To $v \in \overline{H}$, we associate $\langle v|-\rangle : H \multimap \mathbb{C}$.
- To $f : H \to \mathcal{C}$, $\ker f$ is one-dimensional. Take $z \in \ker f$ such that $\|z\| = 1$. Then $x = \overline{f(z)}\ z$ suits. □

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Remark

We also have $H \cong \mathbf{Hilb}(\overline{H}, \mathbb{C})$.

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We also have $H \cong \mathbf{Hilb}(\overline{H}, \mathbb{C})$.

Notation

We define the functor

$$-^\dagger \quad : \quad \mathbf{Hilb} \quad \to \quad \mathbf{Hilb}^{\mathrm{op}}$$

by

$$H^\dagger \quad = \quad \mathbf{Hilb}(\overline{H}, \mathbb{C})$$

Notations

- We write

$$|v\rangle$$

for a vector

$$v \quad : \quad 1 \quad \multimap \quad H$$

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$$v^\dagger \quad : \quad H \quad \multimap \quad 1$$
- Wunderbar, this justifies the notation
$$\langle w|v\rangle \quad = \quad \langle w| \circ |v\rangle \quad : \quad 1 \quad \multimap \quad 1$$

Orthonormal basis

A finite basis $|1\rangle, |2\rangle, \ldots$ is **orthonormal** when

$$\langle i|j\rangle = \delta_{ij}$$

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Proposition (Graham-Schmidt)

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Proposition (Graham-Schmidt)

A finite basis can be transformed into an orthonormal one.

In such a basis, for a vector $v = (v_1, \ldots, v_n)$, we have $v_i = \langle i|v\rangle$:

$$|v\rangle \quad = \quad \sum_i |i\rangle \langle i|v\rangle$$

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In such a basis, we write for an element $v \in H$:

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So that

$$\langle w|v\rangle \quad = \quad \sum_i \sum_j \overline{w_i} v_i$$

As vectors

We can see those as vectors

$$|v\rangle = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad \langle v| = \begin{pmatrix} \overline{v_1} & \dots & \overline{v_n} \end{pmatrix}$$

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and we have

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Operators

An **operator** is a morphism

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Notice that

$$\langle \lambda v| = \overline{\lambda} \, \langle v| \qquad\qquad \langle Av| = \langle v| \, A^\dagger$$

Self-adjoint operators

Definition

An operator A is **self-adjoint** (or **hermitian**) when

$$A^\dagger \quad = \quad A$$

and **skew-adjoint** (or **anti-hermitian**)when

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Proposition

We can generalize the decomposition of real / imaginary:

$$A = \frac{A + A^\dagger}{2} + \frac{A - A^\dagger}{2}$$

Spectral theorem

Lemma

The eigenvalues of a self-adjoint operator are real.

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Theorem

Given a self-adjoint operator on a finite-dimensional space, there exists an orthonormal basis in which it is diagonal.

Spectral theorem

Lemma

The eigenvalues of a self-adjoint operator are real.

Theorem

Given a compact self-adjoint operator A, there exists an orthonormal basis constituted of eigenvectors of A.

Definition

A is **compact** if the image of a bounded set is relatively compact (its closure is compact).

Unitary operators

Definition

An operator

$$A \quad : \quad H \quad \multimap \quad H'$$

is **unitary** when

$$A^\dagger A \quad = \quad \mathrm{id}_H$$

and

$$AA^\dagger \quad = \quad \mathrm{id}_{H'}$$

Definition

A **dagger category** is a category equipped with a functor

$$-^{\dagger} \quad : \quad \mathcal{C}^{\mathsf{op}} \to \mathcal{C}$$

such that

- $\mathsf{id}_A^{\dagger} = \mathsf{id}_A$ (the functor is identity-on-objects)
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On says that

- an invertible morphism $f : A \to B$ is **unitary** when $f^\dagger = f^{-1}$
- an endomorphism $f : A \to A$ is **self-adjoint** when $f^\dagger = f$

Dagger monoidal categories

Definition

A **dagger symmetric monoidal category** is a symmetric monoidal category equipped with a dagger such that

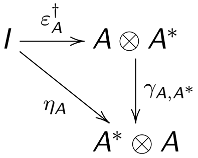
- the dagger functor is strictly monoidal
- the components of the structural natural transformations $\alpha, \lambda, \rho, \sigma$ are unitary, e.g.

$$\alpha^{\dagger}_{A,B,C} \quad = \quad \alpha^{-1}_{A,B,C} \quad : \quad A \otimes (B \otimes C) \quad \to \quad (A \otimes B) \otimes C$$

Dagger compact categories

Definition

A **dagger compact category** is a dagger symmetric monoidal category which is compact closed, such that



In infinite dimensions

Warning: the few next slides are sloppy (maybe someday I'll dig into measures and distributions).

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We can consider the space of functions $[0, 1] \to \mathbb{C}$ equipped with

$$\langle f|g\rangle \quad = \quad \int_0^1 \overline{f(x)}g(x)\,\mathrm{d}x$$

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An orthonormal basis for those is Dirac's "functions" δ_y such that

- $\delta_y(x) = 0$ when $x \neq y$
- $\int_0^1 \delta_y(x)\, \mathrm{d}x = 1$

with which

$$\langle x|f\rangle \quad = \quad \langle \delta_x|f\rangle \quad = \quad f(x)$$

About Dirac's functions

We can think of δ as

$$\delta_y(x) \quad = \quad \lim_{\Delta \to 0} \frac{1}{\sqrt{\pi \Delta^2}} \exp\left(-\frac{x-y}{\Delta^2}\right)$$

or using Fourier transforms

$$\delta_y(x) \quad = \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(x-y)t} \, \mathrm{d}t$$

The derivation operator

Consider the operator D such that

$$D\,|f\rangle \quad = \quad |f'\rangle$$

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with

$$\langle x|\,D\,|y\rangle \quad = \quad \delta'_y(x)$$

i.e.

$$\int \delta'_y(x) f(x)\,\mathrm{d}x \quad = \quad f'(x)$$

A self-adjoint derivation operator

Notice that D looks skew-adjoint:

$$D^{\dagger}_{xy} = \langle y|\, D \,|x\rangle = \delta'_x(y) = -\delta'_y(x) = -\langle x|\, D \,|y\rangle = D^{\dagger}_{xy}$$

we thus would get a self-adjoint operator

$$K \quad = \quad -\mathrm{i}\, D$$

A self-adjoint derivation operator

Notice that D looks skew-adjoint:

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But this is not enough: we also want

$$\begin{aligned}
\langle g|\, K \,|f\rangle &= \overline{\langle f|\, K \,|g\rangle} \\
\int\int \langle g|x\rangle \langle x|\, K \,|y\rangle \langle y|f\rangle \,\mathrm{d}x\,\mathrm{d}y &= \overline{\int\int \langle f|x\rangle \langle x|\, K \,|y\rangle \langle y|g\rangle \,\mathrm{d}x\,\mathrm{d}y} \\
\int \overline{g(x)}\,(-\mathrm{i}\, f'(x))\,\mathrm{d}x &= \int f(x)\left(\mathrm{i}\,\overline{g'(x)}\right)\mathrm{d}x \\
-\mathrm{i}\,\overline{g(x)}f(x)\Big|_0^1 &= 0
\end{aligned}$$

(using integration by parts)

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