Towards Efficient Computation of Trace Spaces of Concurrent Programs

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CEA, LIST

Goal

When verifying a concurrent program,
there is a priori a large number of possible interleavings to check
(exponential in the number of processes)

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Joint work with M. Raussen, L. Fajstrup, É. Goubault and
E. Haucourt.

Programs generate trace spaces

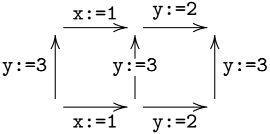
Consider the program

`x:=1;y:=2 | y:=3`

It can be scheduled in three different ways:

`y:=3;x:=1;y:=2` `x:=1;y:=3;y:=2` `x:=1;y:=2;y:=3`

Giving rise to the following graph of traces:



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$(x, y) = (1, 2)$	$(x, y) = (1, 2)$	$(x, y) = (1, 3)$

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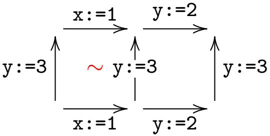

homotopy: commutation / filled square

Mutexes

Concurrent access to shared variables should be protected using **mutexes** $a, b, \ldots$:

- P_a: lock the mutex a
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$$P_b\,;\texttt{x:=1};V_b\,;P_a\,;\texttt{y:=2};V_a \mid P_a\,;\texttt{y:=3};V_a$$

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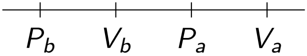
$$P_b.V_b.P_a.V_a \quad | \quad P_a.V_a$$

Geometric semantics

A program will be interpreted as a **directed space**:

- $P_b.V_b.P_a.V_a$



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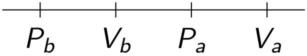

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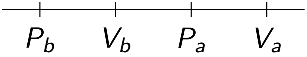

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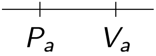

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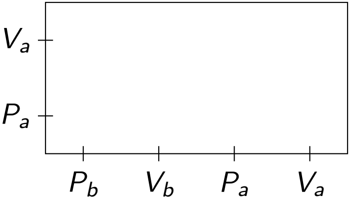

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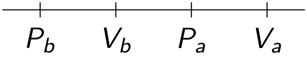

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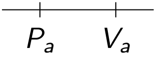

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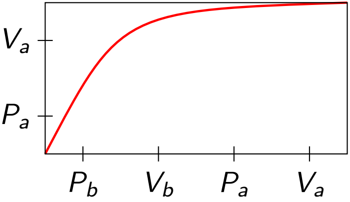

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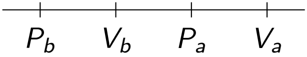

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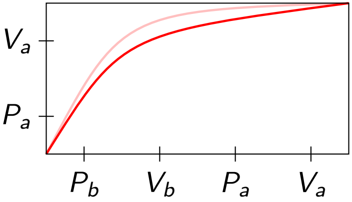
Homotopy



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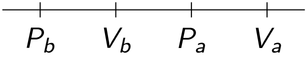

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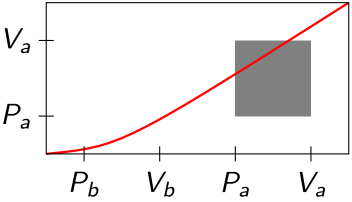
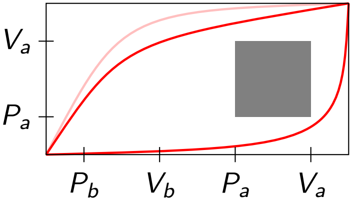


$P_b.V_b.P_a.P_a.V_a.V_a$ Forbidden region

Schedulings

A **scheduling** is the homotopy class of a path.



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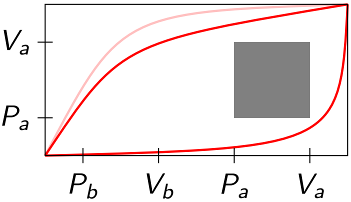

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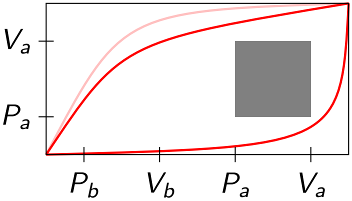
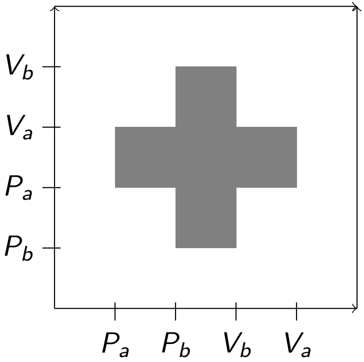


We want to compute *a path in every scheduling*

We do this by testing possible ways to go around forbidden regions:

The Swiss flag

$$P_a.P_b.V_b.V_a \quad | \quad P_b.P_a.V_a.V_b$$



A forbidden region

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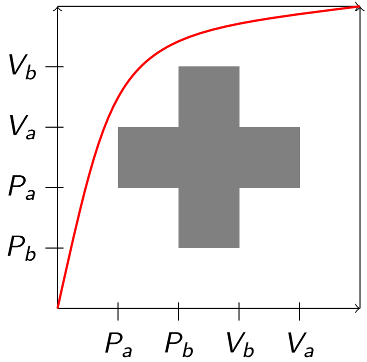

A trace: $P_b.P_a.V_a.P_a.V_b.P_b.V_b.V_a$

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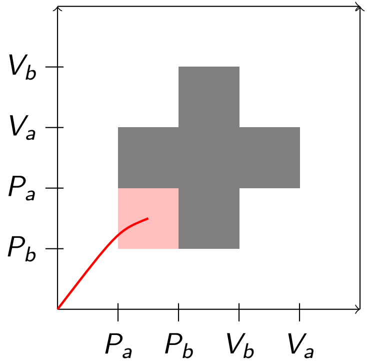

A deadlock: $P_b.P_a$

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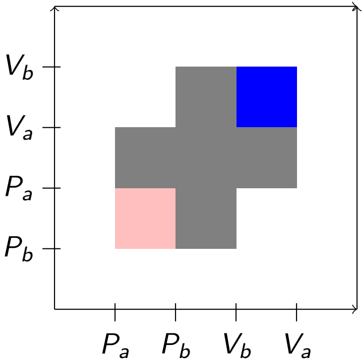
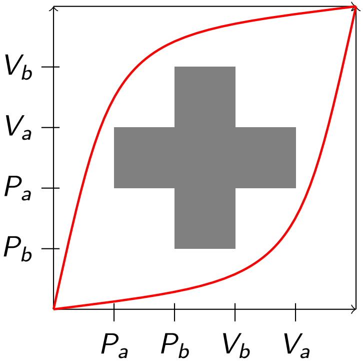
$$P_a.P_b.V_b.V_a \quad | \quad P_b.P_a.V_a.V_b$$



An unreachable region

The Swiss flag

$$P_a.P_b.V_b.V_a \quad | \quad P_b.P_a.V_a.V_b$$



Here we are interested in maximal paths modulo homotopy

Plan

1. Trace semantics of programs
2. Geometric semantics of programs
3. Computation of the trace space

Resources

We suppose fixed a set $\mathcal{R}$ of **resources** *a* with capacity $\kappa_a \in \mathbb{N}$.

The execution of programs are such that

1. a resource *a* cannot be locked (V_a) more than κ_a times
2. a resource *a* cannot be freed if it has not been locked

Example

A mutex is a resource of capacity 1.

Programs

We consider programs of the form:

$$p \quad ::= \quad \mathbf{1} \quad | \quad P_a \quad | \quad V_a \quad | \quad p.p \quad | \quad p|p \quad | \quad p+p \quad | \quad p^*$$

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Programs

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$$p \quad ::= \quad \mathbf{1} \;\mid\; P_a \;\mid\; V_a \;\mid\; p.p \;\mid\; p|p$$

We omit non-deterministic choice, loops, thread creation an join:

$$\begin{array}{lcll} A & ::= & P_a \mid V_a & \textit{actions} \\ t & ::= & A.t \mid \mathbf{1} & \textit{threads} \\ p & ::= & t|t|\ldots|t & \textit{programs} \end{array}$$

Trace semantics

The trace semantics of a program will be an **asynchronous graph**:

- a graph $G = (V, E)$ labeled by actions
- with an *independence relation* I

$$\begin{tikzcd}
y_1 \arrow[r, "B"] & z \\
x \arrow[u, "A"] \arrow[r, "B"'] & y_2 \arrow[u, "A"']
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relating paths of length 2

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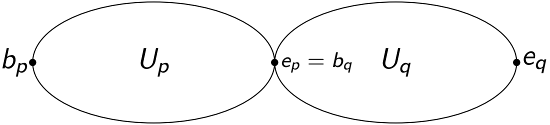
relating paths of length 2

Homotopy is the smallest congruence on paths containing I.

Trace semantics

To every program p we associate (U_p, b_p, e_p) defined by:

- $U_{\mathbf{1}}$: terminal graph
- U_{P_a}: $b_{P_a} \xrightarrow{P_a} e_{P_a}$ $\qquad$ U_{V_a}: $b_{P_a} \xrightarrow{V_a} e_{V_a}$
- $U_{p.q}$:



- $U_{p|q}$ is the "cartesian product" of U_p and U_q:

$$(x, y) \xrightarrow{A} (x', y) \qquad \text{when} \quad x \xrightarrow{A} x' \ \in U_p$$

$$(x, y') \xrightarrow{B} (x, y') \qquad \text{when} \quad y \xrightarrow{B} y' \ \in U_q$$

$$\begin{tikzcd}
(y, x') \arrow[r, "B"] & (y, y') \\
(x, x') \arrow[u, "A"] \arrow[r, "B"'] \arrow[ur, phantom, "\sim"] & (x, y') \arrow[u, "A"']
\end{tikzcd}$$

Trace semantics

Example:

$$P_b.V_b.P_a.V_a \quad | \quad P_a.V_a$$

$$\begin{tikzcd}
{} \arrow[r, "P_b"] & {} \arrow[r, "V_b"] & {} \arrow[r, "P_a"] & {} \arrow[r, "V_a"] & e_p \\
{} \arrow[u, "V_a"] \arrow[r, "P_b"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"] \arrow[r, "V_b"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"] \arrow[r, "P_a"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"] \arrow[r, "V_a"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"'] \\
b_p \arrow[u, "P_a"] \arrow[r, "P_b"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"] \arrow[r, "V_b"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"] \arrow[r, "P_a"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"] \arrow[r, "V_a"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"']
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\end{tikzcd}$$

The **resource function** r_a associates to every vertex x:

number of releases of a - number locks of a

Trace semantics

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Ex: $r_a(x) = -1$, $r_b(x) = 0$

Trace semantics

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The **resource function** r_a associates to every vertex x:

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Ex: $r_a(y) = -2$, $r_b(y) = 0$

Trace semantics

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The **resource function** r_a associates to every vertex x:

number of releases of a - number locks of a

Ex: $r_a(y) = -2 < -1 = \kappa_a$

Trace semantics

Trace semantics T_p:
U_p where we remove vertices x which do not satisfy

$$0 \leqslant r_a(x) + \kappa_a \leqslant \kappa_a$$

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\end{tikzcd}$$

Geometric semantics

The trace semantics is difficult to use to build intuitions. . .

In a similar way, one can define a **geometric semantics** where programs are interpreted by *directed spaces*.

Geometric semantics

A **path** in a topological space X is a continuous map $I = [0, 1] \to X$.

Definition

A **d-space** (X, dX) consists of

- a topological space X
- a set dX of paths in X, called *directed paths*, such that
  - *constant paths*: every constant path is directed,
  - *reparametrization*: dX is closed under precomposition with increasing maps $I \to I$, which are called *reparametrizations*,
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Example

$(X, \leqslant)$ space with a partial order, $dX = \{\text{increasing maps } I \to X\}$

$\vec{I}$: d-space induced by $[0,1]$

Geometric semantics

A **path** in a topological space X is a continuous map $I = [0, 1] \to X$.

Definition

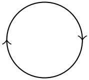
A **d-space** (X, dX) consists of

- a topological space X
- a set dX of paths in X, called *directed paths*, such that
  - *constant paths*: every constant path is directed,
  - *reparametrization*: dX is closed under precomposition with increasing maps $I \to I$, which are called *reparametrizations*,
  - *concatenation*: dX is closed under concatenation.

Example

$S^1 = \{\mathrm{e}^{\mathrm{i}\,\theta}\} 0 \leqslant \theta < 2\pi$

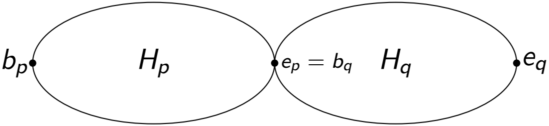
dS^1: $p(t) = \mathrm{e}^{\mathrm{i}\,f(t)}$ for some increasing function $f : I \to \mathbb{R}$



Geometric semantics

To each program p we associate a d-space (H_p, b_p, e_p):

- $H_{\mathbf{1}}$: $\bullet$
- $H_{P_a} = \vec{I}$ $\qquad$ $H_{V_a} = \vec{I}$
- $H_{p.q}$:



- $H_{p|q}$: $H_p \times H_q$, $b_{p|q} = (b_p, b_q)$, $e_{p|q} = (e_p, e_q)$

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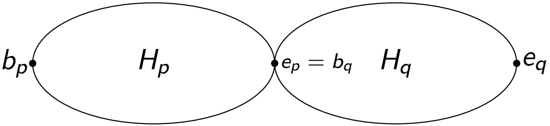

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Resource function: $r_a(x) \in \mathbb{N}$ for each $a \in \mathcal{R}$ and point x

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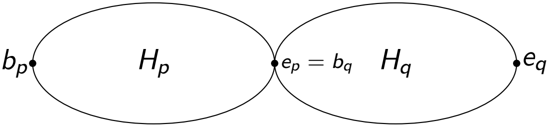

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$F_p = \{x \in H_p \;/\; \exists a \in \mathcal{R}, \quad r_a(x) + \kappa_a < 0 \quad \text{or} \quad r_a(x) > 0\}$

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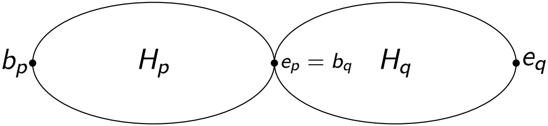

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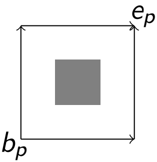
Geometric semantics: $G_p = H_p \setminus F_p$

Examples of geometric semantics

$P_a.V_a|P_a.V_a$



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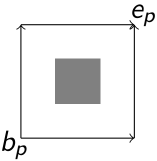

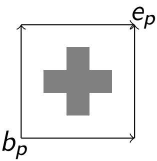

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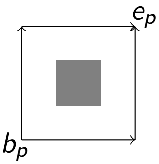

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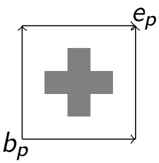
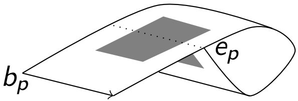
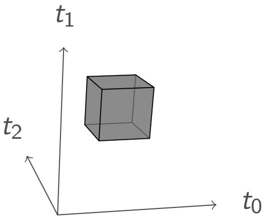


$P_a.(V_a.P_a)^*|P_a.V_a$

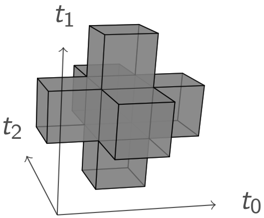


Examples of geometric semantics

$P_a.V_a|P_a.V_a|P_a.V_a$
$(\kappa_a = 2)$



$P_a.V_a|P_a.V_a|P_a.V_a$
$(\kappa_a = 1)$



Geometric realization

The two semantics are "essentially the same": the geometric semantics is the **geometric realization** of a *cubical set*

$$G_p \quad = \quad \int^{n \in \square} T_p(n) \cdot \vec{I}^n$$

Proposition

Given a program p, with T_p as trace semantics and G_p as geometric semantics,

- *every path $\pi : b \to e$ in T_p induces a path $\overline{\pi} : b \to e$ in G_p,*
- *$\pi \sim \rho$ in T_p implies $\overline{\pi} \sim \overline{\rho}$ in G_p*
- *every path ρ of G_p is homotopic to a path $\overline{\pi}$ (π path in G_p)*

Computing the trace space

Goal

Given a program p, we describe an algorithm to compute a trace in each equivalence class of traces $\pi : b_p \to e_p$ up to homotopy in G_p.

The proposition before ensures that it is the same to compute this in the trace semantics or in the geometric semantics.

The algorithm

Suppose given a program

$$p \quad = \quad p_0|p_1|\dots|p_{n-1}$$

with n *threads*.

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$$p \quad = \quad p_0 | p_1 | \ldots | p_{n-1}$$

with n *threads*.

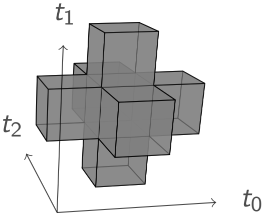
Under mild assumptions, the geometric semantics is of the form

$$G_p \quad = \quad \vec{I}^n \setminus \bigcup_{i=0}^{l-1} R^i$$

where

$$R^i = \prod_{j=0}^{n-1}]x^i_j, y^i_j[$$

are l *open rectangles*.



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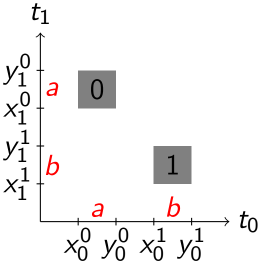
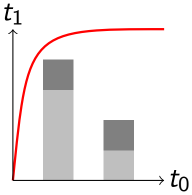
Example

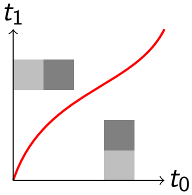
$$P_a.V_a.P_b.V_b|P_b.V_b.P_a.V_a$$

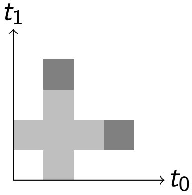


The algorithm

The main idea of the algorithm is to extend the forbidden cubes downwards in various directions and look whether there is a path from b to e in the resulting space.







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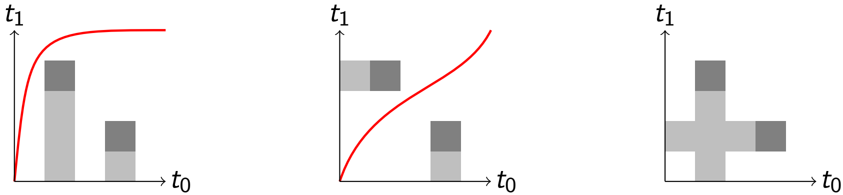

By combining those information, we will be able to compute traces modulo homotopy.

The directions in which to extend the holes will be coded by boolean matrices M.

The index poset

$\mathcal{M}_{l,n}$: boolean matrices with l rows and n columns.

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X_M:

space obtained by *extending*
for every (i,j) such that $M(i,j)=1$
the forbidden cube i downwards
in every direction other than j

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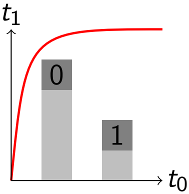
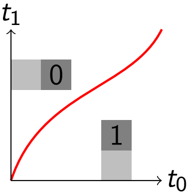


$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$



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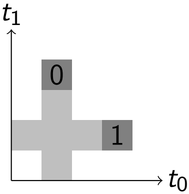

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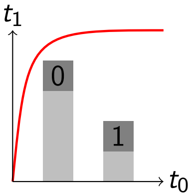

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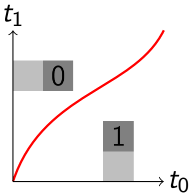

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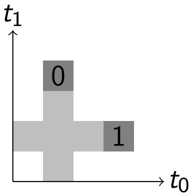


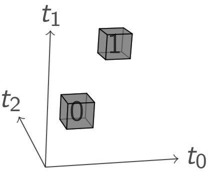
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Psi : \mathcal{M}_{l,n} \to \{0,1\}$:

- $\Psi(M)=0$ if there is a path $b \to e$: M is alive
- $\Psi(M)=1$ if there is no path $b \to e$: M is dead

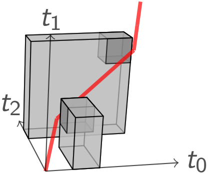
The index poset

$$P_a.V_a.P_b.V_b \quad \Big| \quad P_a.V_a.P_b.V_b \quad \Big| \quad P_a.V_a.P_b.V_b$$



$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

alive



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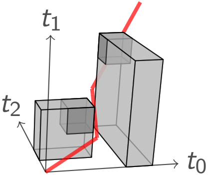

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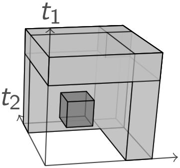
alive



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dead

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- $\mathcal{M}_{l,n}$ is equipped with the pointwise ordering
- Ψ is increasing: more $1 \Rightarrow$ more obstructions
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$D(X) \quad \rightsquigarrow \quad \mathcal{C}(X) \quad \rightsquigarrow$ homotopy classes of traces

The dead poset

Proposition

A matrix $M \in \mathcal{M}_{l,n}^{C}$ *is in* $D(X)$ *iff it satisfies*

$$\forall (i,j) \in [0:l[\times [0:n[, \qquad M(i,j) = 1 \quad \Rightarrow \quad x_j^i < \min_{i' \in R(M)} y_j^{i'}$$

where $R(M)$*: indexes of non-null rows of* M*.*

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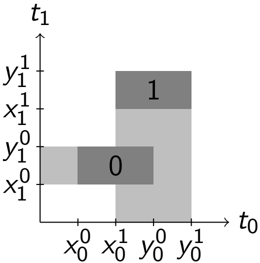
where $R(M)$: indexes of non-null rows of M.

Example

M is dead:



$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1^0 &= 1 < 2 = \min(y_1^0, y_1^1) \\ x_0^1 &= 2 < 3 = \min(y_0^0, y_0^1) \end{aligned}$$

The index poset

Proposition

A matrix M is in $\mathcal{C}(X)$ iff for every $N \in D(X)$, $N \not\leq M$.

The index poset

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A matrix M is in $\mathcal{C}(X)$ iff for every $N \in D(X)$, $N \nleq M$.

Remark

$N \nleq M$: there exists (i,j) s.t. $N(i,j) = 1$ and $M(i,j) = 0$.

Remark

Since $\mathcal{C}(X)$ is downward closed it will be enough to compute the set $\mathcal{C}_{\max}(X)$ of maximal alive matrices.

Remark

The index poset contains all the geometrical information!

Connected components

$M \wedge N$: pointwise min of M and N

Definition

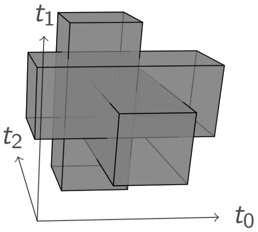
Two matrices M and N are **connected** when $M \wedge N$ does not contain any null row.

Proposition

The connected components of $\mathcal{C}(X)$ are in bijection with homotopy classes of traces $b \to e$ in X.

Dining philosophers

n processes p_k in parallel:



$$p_k = P_{a_k}.P_{a_{k+1}}.V_{a_k}.V_{a_{k+1}}$$

n	sched.	ALCOOL (s)	ALCOOL (MB)	SPIN (s)	SPIN (MB)
8	254	0.1	0.8	0.3	12
9	510	0.8	1.4	1.5	41
10	1022	5	4	8	179
11	2046	32	9	42	816
12	4094	227	26	313	3508
13	8190	1681	58	∞	∞
14	16382	13105	143	∞	∞

How do we extend this methodology to program with loops?

Loops

Given a thread p, we write p^* for its looping: `while(...){`p`}`.

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Notice that the geometric semantics X_{p^*} can be deduced from the semantics of p by glueing copies of X_p in every direction:

$$p_i^* \quad = \quad p_i . p_i . p_i \ldots$$

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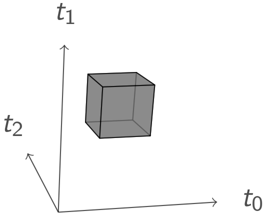
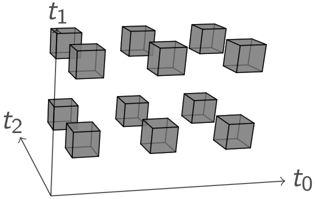
Example

Consider the program $p = q|q|q$ with $q = P_a.V_a$ (and a of arity 3):



X_p



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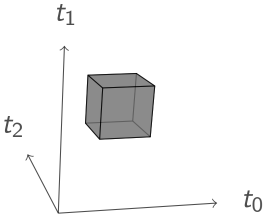
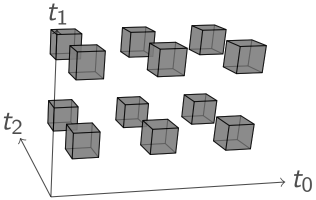


X_p



$X_{p^{(3,2,2)}}$

Finite deloopings:

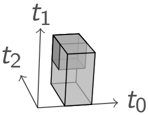
$$X_{p^{(3,2,2)}} \quad = \quad (Y \oplus_1 Y) \oplus_2 (Y \oplus_1 Y) \qquad \text{with} \qquad Y = X_p \oplus_0 X_p \oplus_0 X_p$$

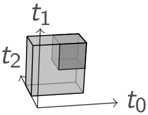
Schedulings

Similarly, given schedulings

$$M = \begin{pmatrix}1 & 0 & 0\end{pmatrix} \qquad \text{and} \qquad N = \begin{pmatrix}0 & 0 & 1\end{pmatrix}$$

of the previous program p





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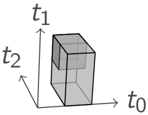

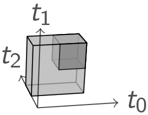

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$$M \oplus_0 N = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

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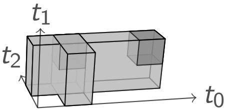

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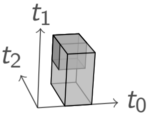

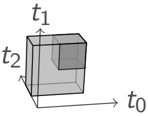

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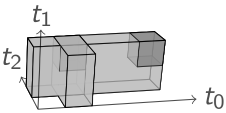
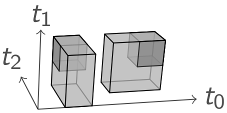


$\neq$



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Shadows

In fact, scheduling drop "*shadows*" on previous schedulings

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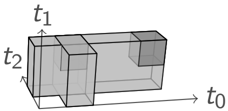

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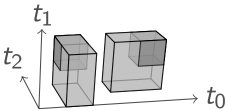

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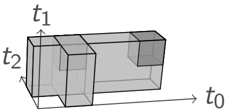

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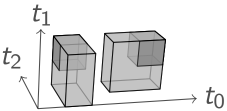
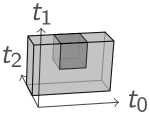


$= X_M \oplus_0 X_N$

Write $X_{M|_j}$ for the **shadow** projected by scheduling M in direction j:

$X_{N|_0} \quad =$



so that

$$X_{M\oplus_j N} \quad = \quad (X_M \cap X_{N|_j}) \otimes_j X_N$$

Alive matrices for programs with loops

Every scheduling M of a delooping of X_p is composed by glueing *submatrices* $(M_{i_1,\dots,i_n})$.

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If X_M contains a deadlock then some subspace $X_{(M_{i_1,\ldots,i_n})}$ contains a deadlock:

Lemma

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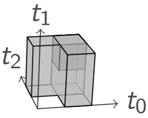
`The converse is not true!`

Shadows can create deadlocks

The following matrices P and Q coding the schedulings



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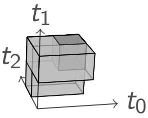
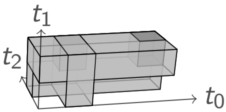


X_Q

of p are alive, however the matrix $P \oplus_0 Q$ is dead:

$X_{P \oplus_0 Q} \quad =$



The shadow automaton

We construct an automaton which describes all the schedulings possible in the future (which won't create deadlocks by their shadow): given a scheduling M and a direction j, it describes all the matrices N such that $M \oplus_j N$ is alive.

The shadow automaton

Definition

The **shadow automaton** of a program p is a non-deterministic automaton whose

- states are shadows
- transitions $N \xrightarrow{j,M} N'$ are labeled by a direction j (with $0 \leqslant j < n$) and a scheduling M

defined as the smallest automaton

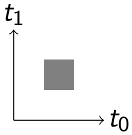
- containing the empty scheduling $\emptyset$
- and such that for every state N', for every direction j and for every scheduling M such that the scheduling $M \cup N'$ is alive, and M is maximal with this property, there is a transition $N \xrightarrow{j,M} N'$ with $N = (M \cup N')|_j$.

All the states of the automaton are both initial and final.

The shadow automaton

For instance consider the program $p = P_a.V_a|P_a.V_a$

$X_p \quad =$



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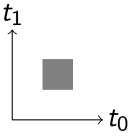

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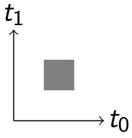
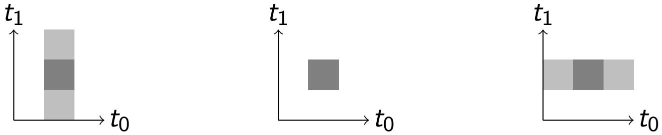


There are two maximal schedulings

which can drop three possible shadows



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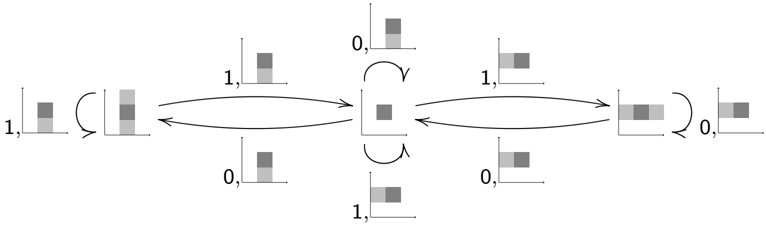
The shadow automaton of p is



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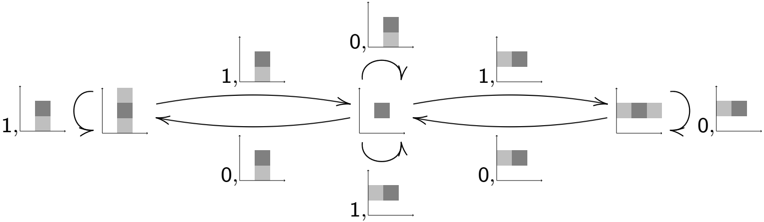
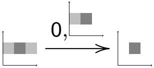
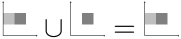


For instance, the transition  is computed as follows:

- consider the shadow $M =$ 
- compute its shadow in direction 0: 

The shadow automaton

Theorem

Given a program p to any total path in a delooping of p is represented by a path in the shadow automaton of p such that

- *every path in the automaton comes from a total path in* $X_{p^?}$
- *if two total paths in* $X_{p^?}$ *correspond to the same path in the automaton then they are homotopic*

Paths in the shadow automaton describe homotopy classes in deloopings of p.

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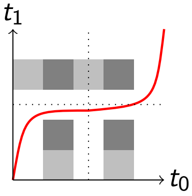
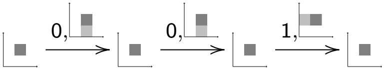


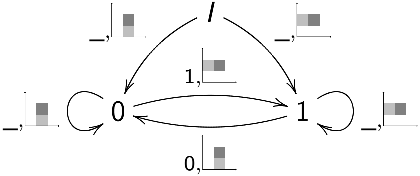
is represented by



Reducing the size of the automaton

The shadow automaton is too big:

- we can determinize it:



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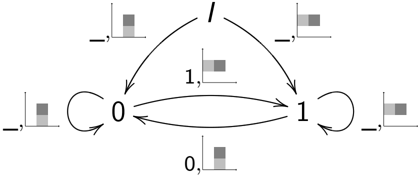

- two distinct paths in the automaton can represent the same homotopy class of paths: we can quotient paths under connexity.

An application to static analysis

The program

$$p^* \quad = \quad \Big(P_a.a := a - 1.V_a\Big)^*\Big|\Big(P_a.\Big(a := \frac{a}{2}\Big).V_a\Big)^*$$

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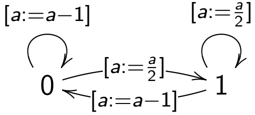

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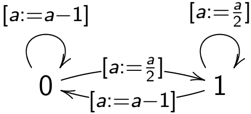

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$$\begin{cases} A_0 & = & I \cup (A_0 - 1) \cup (A_1 - 1) \\ A_1 & = & I \cup \frac{A_1}{2} \cup \frac{A_0}{2} \end{cases}$$

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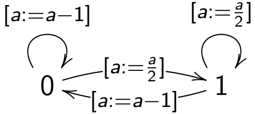

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which admits a least fixed point

$$A_0^\infty \quad = \quad A_1^\infty \quad = \quad]-\infty, 1]$$

An example: the two-phase protocol

We consider n programs locking l resources:

$$p_{n,l} = q|q|\ldots|q \qquad \text{with} \qquad q = P_{a_1}.\ldots.P_{a_l}.V_{a_1}.\ldots.V_{a_l}$$

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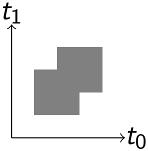

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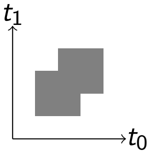

We get the following results compared to spin:

n	l	s	t	s'	t'	s''	t''	s_{SPIN}	t_{SPIN}
2	1	3	8	3	10	1	1	58	65
2	2	3	8	3	10	1	1	112	129
2	3	3	8	3	10	1	1	180	209
3	1	19	90	4	24	1	1	171	218
3	2	19	90	4	24	1	1	441	602
3	3	19	90	4	24	1	1	817	1128

About geometric models

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$\Rightarrow$ Yes: computers are much better at manipulating booleans than complex algebraic structures

Future works

We compute **one execution trace in each homotopy class**.

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What remains to do:

- use these trace to do static analysis
  (e.g. abstract interpretation)
- speed improvements
- implementation improvements (e.g. GPU)
- lots of work remain to be done on the theoretical side