A NON-STANDARD SEMANTICS FOR KAHN NETWORKS IN CONTINUOUS TIME

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20TH CONFERENCE ON COMPUTER SCIENCE LOGIC

13 SEPTEMBER 2011

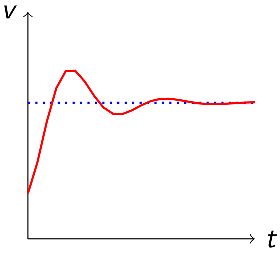
HYBRID SYSTEMS

A model for systems operating in **continuous time** $t \in \mathbb{R}^+$.

A controller to **cruise control** a car:

$$I(t) \quad = \quad K_p e(t)$$

- error: $e(t) = v_{\mathrm{desired}} - v_{\mathrm{actual}}$
- intensity of the engine: I
- parameters of the control: K_p



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A PID-controller to **cruise control** a car:

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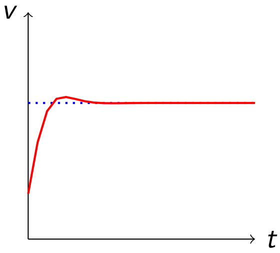

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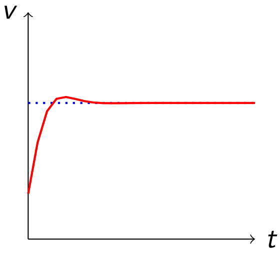

We also want to have discontinuities!

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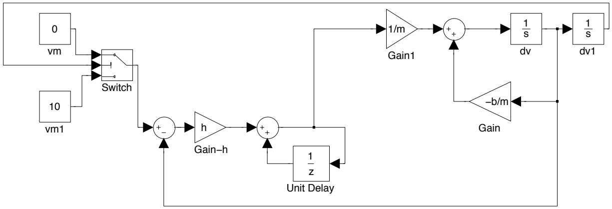

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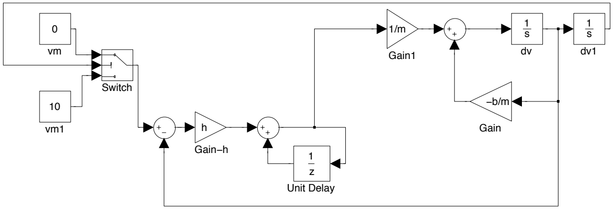

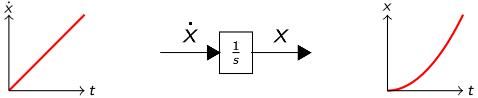

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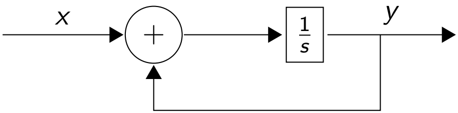

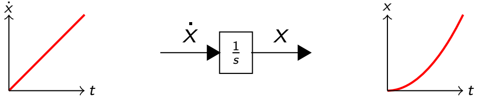

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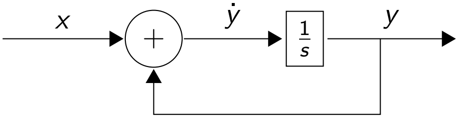

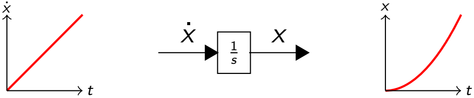

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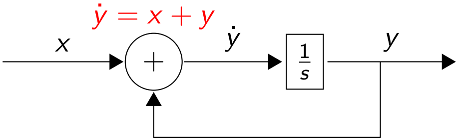

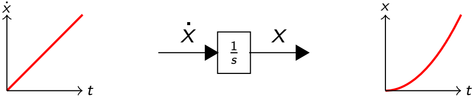

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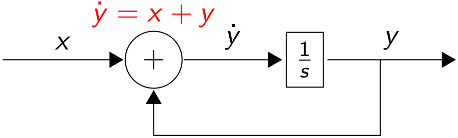

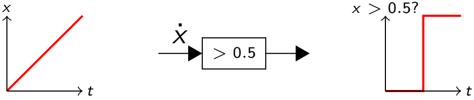

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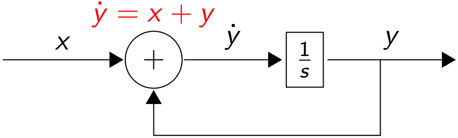

How can we define a semantics for those systems?

The streams $f : \mathbb{R}^+ \to \mathbb{R}$ on the wires could

- be integrable / derivable
- have discontinuities (zero-crossings)
- exhibit complex behaviors such as Zeno
- be approximated...

Let's take inspiration
from discrete time semantics.

KAHN PROCESS NETWORKS

A semantics for ***distributed asynchronous*** *computations*:
processes exchanging *sequences of data* on channels.

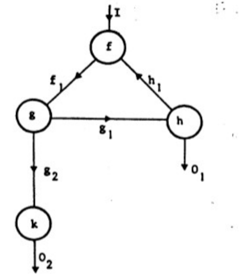

Fig.3. A parallel program schema.

G. Kahn. The semantics of a simple language for parallel programming. *Information processing*, 74:471–475, 1974.

$$1, 2, 3, \ldots \longrightarrow \times 2 \longrightarrow 2, 4, 6, \ldots$$

KAHN NETWORKS IN CONTINUOUS TIME

It is difficult to define a semantics for hybrid systems.
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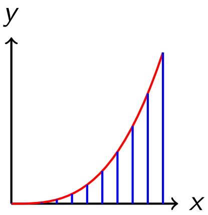
The sampling principle

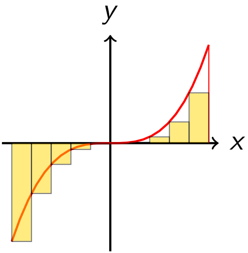
We can to consider a continuous stream $t \mapsto x_t$ (with $t \in \mathbb{R}^+$)
as a discrete stream x_i (with $i \in \mathbb{N}$)
where the data x_i occurs at time $i\varepsilon$, with ε infinitesimal.



A continuous stream sampled at every ε seconds.

NON-STANDARD ANALYSIS

In the 60s, Robinson introduced an extension ${}^*\mathbb{R}$ of $\mathbb{R}$
(the *hyperreals*)
in which one can formally consider **infinitesimals**.



$$f'(x) = \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

$$\int_{t=0}^{T} f(t)\mathrm{d}t = \sum_{0 \leq i \leq T/\varepsilon} f(i)\varepsilon$$

with ε infinitesimal

THE PLAN

1. Define Kahn networks and their semantics
   - formalization of Kahn networks
   - Kahn networks form a free fixpoint category

2. A non-standard semantics for Kahn networks
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Related works:

1. Semantics of Kahn networks
   - Kahn
   - categorical structure:
     Hildebrandt, Panangaden, Winskel, Stark, ...

2. Using non-standard analysis to model hybrid systems
   - Bliudze, Krob
   - Benveniste, Caillaud, Pouzet

KAHN
NETS

EXAMPLES OF KAHN NETWORKS

- Prepend a 0:

$$1, 2, 3, \dots \longrightarrow \boxed{\zeta} \longrightarrow 0, 1, 2, 3, \dots$$

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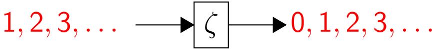
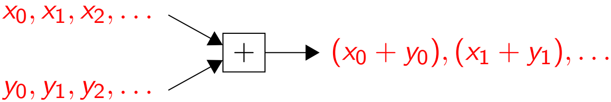


- Add two discrete streams:



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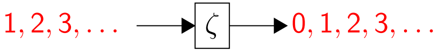

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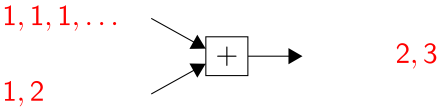

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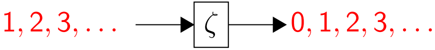

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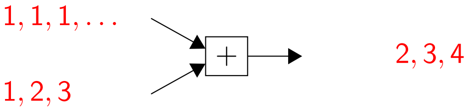

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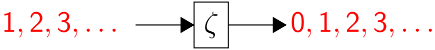

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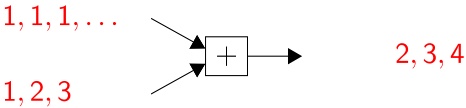
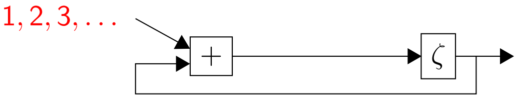


- A net composed of **generators**:



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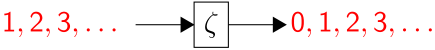

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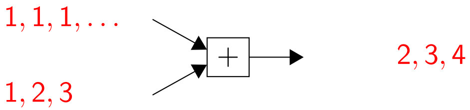

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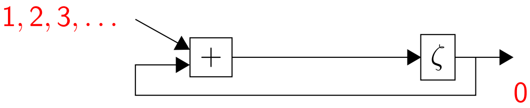

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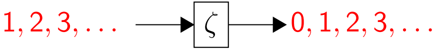

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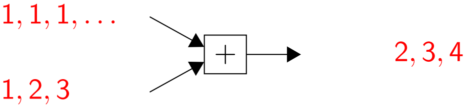

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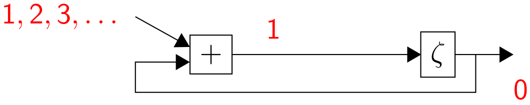

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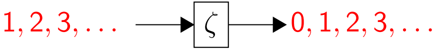

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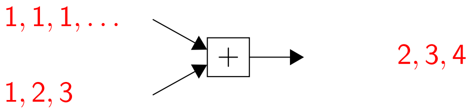

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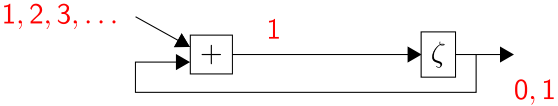

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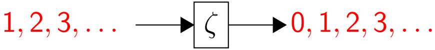

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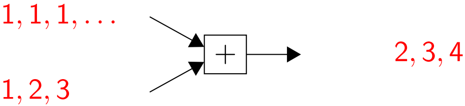

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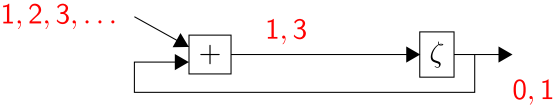

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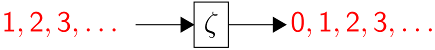

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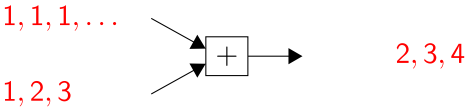

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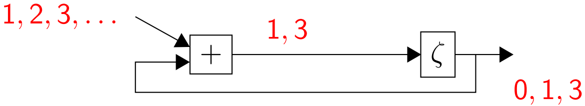

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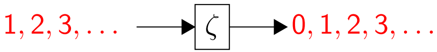

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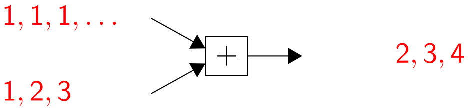

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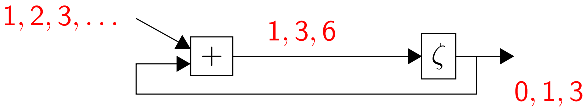

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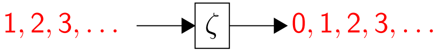

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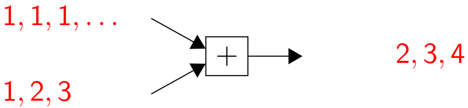

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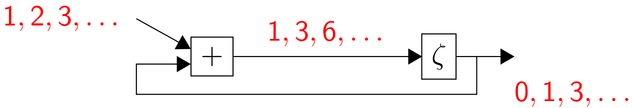

SEMANTICS OF KAHN NETWORKS

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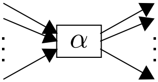
To each generator $\alpha : m \to n$



we associate a *Scott-continuous function* $K^m \to K^n$.

SEMANTICS OF KAHN NETWORKS

The semantics of a composed net is given by associating a set of equations to the network

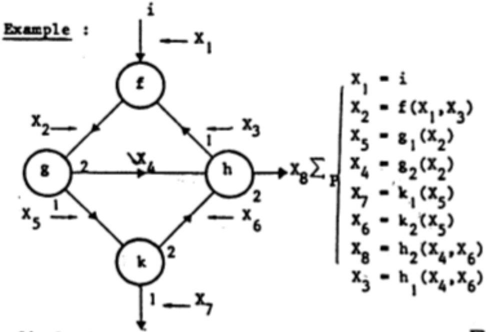


fig.7. The program P and the associated system Σ_P.

and taking the unique *minimal solution* (which exists).

A semantics of what?

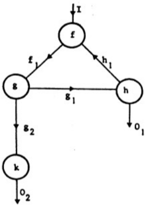


Fig.3. A parallel program schema.

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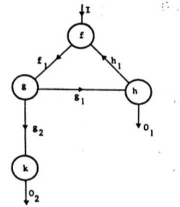

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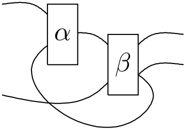
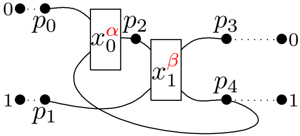


is formalized by



A UNIVERSAL DESCRIPTION OF NETS

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Any interpretation of the generators in a fixpoint category canonically induces an interpretation of all Kahn nets.

- We can give semantics of KN in other fixpoint categories.
- Avoids some technical details (solving systems of equations).

A SEMANTICS OF HYBRID SYSTEMS?

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- They can be seen as partial functions $\ell : \mathbb{N} \to \mathbb{R}$ whose domain of definition is an initial segment of $\mathbb{N}$:

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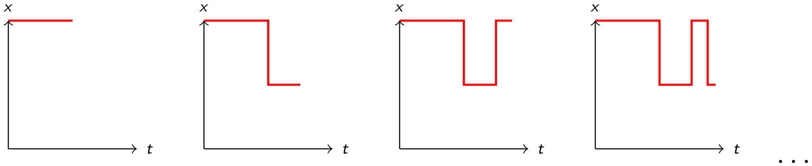

f_0	f_1	f_2	f_3	. . .
$[0, 1/2]$	$[0, 3/4]$	$[0, 7/8]$	$[0, 15/16]$	. . .

. . . because of the Zeno effect.

NON-STANDARD ANALYSIS
A CRASH COURSE

NON-STANDARD ANALYSIS

In order to give a meaning to the infinitesimals, we replace reals by **hyperreals** which are sequences $(x_i)_{i \in \mathbb{N}}$ of reals.

- A real x is seen as the constant sequence (x).
- An infinitesimal number is a sequence converging towards 0.
- An "infinite" number is a sequence converging towards $\pm\infty$.
- The usual operations are extended pointwise on sequences: $(x_i) \times (y_i) = (x_i \times y_i)$

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- The usual operations are extended pointwise on sequences: $(x_i) \times (y_i) = (x_i \times y_i)$
- What is the inverse of $(0, 1, 0, 1, 0, 1, \ldots)$?
- In order to recover usual properties one has to consider *equivalence classes* of sequences.

HYPERREALS

We will define a collection $\mathcal{F}$ of subsets (called **large**) of $\mathbb{N}$ and consider the equivalence relation such that

$$(x_i) \equiv (y_i) \qquad \text{whenever} \qquad \{i \in \mathbb{N} \mid x_i = y_i\} \in \mathcal{F}$$

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The set $\mathcal{F}$ should satisfy properties:

- two sequences equal at every index excepting a finite number should be equal: $\mathcal{F}$ should contain all cofinite sets
- two sequences are either equal (equivalent) or different:

$$\forall U \subseteq \mathbb{N}, \qquad U \in \mathcal{F} \quad \text{or} \quad \mathbb{N} \setminus U \in \mathcal{F}$$

- ...

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The set $\mathcal{F}$ should be a *non-principal ultrafilter* on $\mathbb{N}$.

Definition

An **ultrafilter** $\mathcal{F}$ on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ such that

1. intersection: $\forall U, V \in \mathcal{F}, \qquad U \cap V \in \mathcal{F}$
2. supersets: $\forall U \in \mathcal{F}, \forall V \subseteq \mathbb{N}, \qquad U \subseteq V \Rightarrow V \in \mathcal{F}$
3. proper: $\emptyset \notin \mathcal{F}$
4. complement: $\forall U \subseteq \mathbb{N}, \qquad U \in \mathcal{F} \quad \text{or} \quad \mathbb{N} \setminus U \in \mathcal{F}$

(with AC such an $\mathcal{F}$ exists).

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Definition

- The field of **hyperreals** ${}^*\mathbb{R}$ is $\mathbb{R}^{\mathbb{N}}/\equiv$.
- The ring of **hyperintegers** ${}^*\mathbb{N}$ is $\mathbb{N}^{\mathbb{N}}/\equiv$.

INFINITESIMAL AND UNLIMITED

Definition

An hyperreal $x \in {}^*\mathbb{R}$ is

- **infinitesimal**: if $x \neq 0$ and $\forall r \in \mathbb{R}, |x| < r$
- **unlimited**: if $\forall r \in \mathbb{R}, |x| > r$

Example

- infinitesimal: $x = \langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$
- unlimited: $x = \langle 1, 2, 3, 4, \ldots \rangle$

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Intuition

- ${}^*\mathbb{R}$ is $\mathbb{R}$ completed with infinitesimals and unlimited.
- ${}^*\mathbb{N}$ is $\mathbb{N}$ completed with unlimited

NON-STANDARD ANALYSIS

Continuity, derivation, integration, etc.
have the “expected” formulations
in this framework.

NON-STANDARD SEMANTICS OF KAHN NETWORKS

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- The elements of the Kahn cpo are the streams:

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$$f(0), f(\varepsilon), f(2\varepsilon), f(3\varepsilon), \ldots$$

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We have to take in account infinitesimal variations!

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But for every $n \in \mathbb{N}$, $n\varepsilon$ is an infinitesimal!

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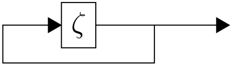
We consider hypersequences of hyperreals.

FIXPOINTS IN ${}^*\mathbb{R}^{\leq {}^*\mathbb{N}}$

The semantics of the following net should be the constant stream f such that $\forall n \in {}^*\mathbb{N}, f(n\varepsilon) = 0$:



However, if we compute its semantics using the fixpoint construction we get the stream f such that

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The value $f(n\varepsilon)$ is not defined for unlimited hyperintegers $n \in {}^*\mathbb{N}$!

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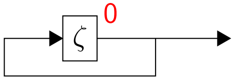

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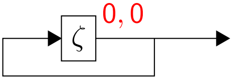

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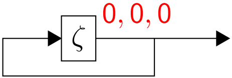

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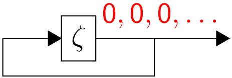

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INTERNAL THINGS

From $(D_i \subseteq \mathbb{R})_{i\in\mathbb{N}}$ we can define a subset $D \subseteq {}^*\mathbb{R}$:

- $D_0 = \{x_0, y_0, z_0, \ldots\}$
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- $\ldots$

$D = \{\langle z_0, x_1, y_2, \ldots\rangle, \ldots\}$

Definition

An **internal set** $D \subseteq {}^*\mathbb{R}$ is a set such that there exists a family $(D_i \subseteq \mathbb{R})_{i\in\mathbb{N}}$ for which

$$D = \langle D_i \rangle = \{\langle x_i \rangle \mid \forall i \in \mathbb{N}, x_i \in D_i\}$$

Internal functions $f = \langle f_i \rangle$, *internal relations*, etc. are defined similarly.

THE TRANSFER PRINCIPLE

Proposition

The transfer principle:

a first-order formula is satisfied for $\mathbb{R}$

iff

it is satisfied for ${}^*\mathbb{R}$,

if we suppose that all the sets, etc in the formula to be <u>internal</u>

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Lemma

Internal induction principle: *if* D *is an* *internal* *subset of* $^*\mathbb{N}$ *s.t.*

- $0 \in D$
- $\forall n \in D, n + 1 \in D$

then $D = {}^*\mathbb{N}$.

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Remark

An internal cpo $(D, \leq)$ is not necessarily a cpo!

THE INFINITESIMAL-TIME DOMAIN

Definition

- The category **ICPO**:
  internal cpo and internal Scott-continuous functions
- The **infinitesimal-time domain** $IT \in \mathbf{ICPO}$:
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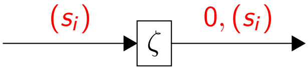
Proposition

The category **ICPO** *is a fixpoint category.*

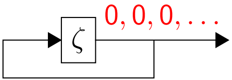
A Σ-object in this category canonically induces
a semantics of Kahn networks

EXAMPLE – THE CONSTANT STREAM

If we interpret



then the net



is interpreted as the constant stream $s : {}^*\mathbb{R}^{{}^*\mathbb{N}}$ such that

$$\forall i \in {}^*\mathbb{N}, \qquad s_i = 0$$

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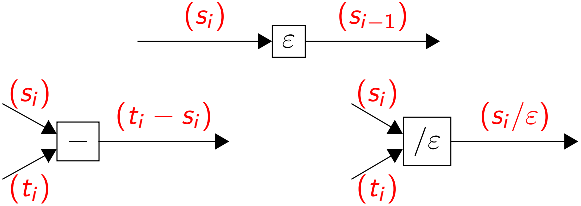
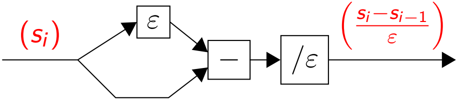


(with ε infinitesimal) then the net



is interpreted as the function

$$\begin{array}{rccc} D & : & {}^*\mathbb{R}^{{}^*\mathbb{N}} & \rightarrow & {}^*\mathbb{R}^{{}^*\mathbb{N}} \\ & & (s_i) & \mapsto & \left(\frac{s_i - s_{i-1}}{\varepsilon}\right) \end{array}$$

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Definition

In order to compare

- $CT = \mathbb{R}^{\leq \mathbb{R}^+}$: the continuous time model
- $IT = {}^*\mathbb{R}^{\leq {}^*\mathbb{N}}$: the infinitesimal time model

we introduce

sampling	**standardisation**
$\begin{array}{rcccc} S & : & CT & \to & IT \\ & & s & \mapsto & (s(i\varepsilon))_{i \in {}^*\mathbb{N}} \end{array}$	$\begin{array}{rcccc} T & : & IT & \to & CT \\ & & (s_i)_{i \in {}^*\mathbb{N}} & \mapsto & t \mapsto \mathrm{st}(s_{\lfloor t/\varepsilon \rfloor}) \end{array}$

Proposition

For any continuously differentiable $s \in CT$, $T(D(S(s))) = s'$.

NON-STANDARD SEMANTICS GOING FURTHER

In this way, we give a non-standard **semantics** for hybrid systems:

- we can interpret all the common building blocks (derivation, integration, zero-crossing, etc.)
- we relate it to the "continuous time model" through S and T

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  - we should have $T \circ S(s) = s$
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- NSA enables us to use discrete techniques for continuous: continuous-time bisiumlations, game semantics, etc.?

THANKS!

Any questions?