A NON-STANDARD SEMANTICS FOR KAHN NETWORKS IN CONTINUOUS TIME

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A model for systems operating in \textit{continuous time} $t \in \mathbb{R}^+$. 

A controller to \textbf{cruise control} a car:

$$I(t) = K_p e(t)$$

- error: $e(t) = v_{\text{desired}} - v_{\text{actual}}$
- intensity of the engine: $I$
- parameters of the control: $K_p$
HYBRID SYSTEMS

A model for systems operating in continuous time \( t \in \mathbb{R}^+ \).

A PID-controller to cruise control a car:

\[
I(t) = K_p e(t) + K_i \int_0^t e(t)\,dt + K_d \frac{de}{dt}
\]

- error: \( e(t) = v_{\text{desired}} - v_{\text{actual}} \)
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A PID-controller to **cruise control** a car:

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- **intensity of the engine**: $I$
- **parameters of the control**: $K_p$, $K_i$, $K_d$

We also want to have discontinuities!
A model for systems operating in continuous time $t \in \mathbb{R}^+$. 
A model for systems operating in **continuous time** $t \in \mathbb{R}^+$. 

**Diagram Description:**

- **Equations:**
  - $\dot{x}(t)$
  - $\frac{1}{s}$
  - $x(t)$

- **System Diagram:**
  - **Switch**
  - **Gain** $\frac{1}{m}$
  - **Unit Delay**
  - **Gain** $h$
  - **Gain** $b/m$

- **Variables:**
  - $0$
  - $vm$
  - $vm1$
  - $10$

**Mathematical Derivation:**

The system described by the hybrid model operates in continuous time $t \in \mathbb{R}^+$ and involves components such as switches, gains, and unit delays to model the system dynamics.
A model for systems operating in **continuous time** $t \in \mathbb{R}^+$. 

\[
\dot{y} = x + y \\
\dot{x} = 1/s \\
x(t) \rightarrow y(t)
\]
HYBRID SYSTEMS

A model for systems operating in \textit{continuous time} \( t \in \mathbb{R}^+ \).

\[
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\[
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A model for systems operating in **continuous time** $t \in \mathbb{R}^+$. 

$$\dot{y} = x + y$$
HYBRID SYSTEMS

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\[
\dot{y} = x + y \\
\dot{x} > 0.5 \quad x > 0.5?
\]
A model for systems operating in continuous time $t \in \mathbb{R}^+$. 

$\dot{y} = x + y$

How can we define a semantics for those systems?

The streams $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ on the wires could

- be integrable / derivable
- have discontinuities (zero-crossings)
- exhibit complex behaviors such as Zeno
- be approximated...
Let’s take inspiration from discrete time semantics.
KAHN PROCESS NETWORKS

A semantics for *distributed asynchronous computations*: processes exchanging *sequences of data* on channels.


\[1, 2, 3, \ldots \rightarrow \times 2 \rightarrow 2, 4, 6, \ldots\]
It is difficult to define a semantics for hybrid systems. Can we adapt the works on Kahn networks?
KAHN NETWORKS IN CONTINUOUS TIME

It is difficult to define a semantics for hybrid systems. Can we adapt the works on Kahn networks?

The sampling principle

We can consider a continuous stream \( t \mapsto x_t \) (with \( t \in \mathbb{R}^+ \)) as a discrete stream \( x_i \) (with \( i \in \mathbb{N} \)) where the data \( x_i \) occurs at time \( i \varepsilon \), with \( \varepsilon \) infinitesimal.

A continuous stream sampled at every \( \varepsilon \) seconds.
In the 60s, Robinson introduced an extension $^\ast \mathbb{R}$ of $\mathbb{R}$ (the hyperreals) in which one can formally consider infinitesimals.

$$f'(x) = \frac{f(x + \varepsilon) - f(x)}{\varepsilon}$$

$$\int_{t=0}^{T} f(t) \, dt = \sum_{0 \leq i \leq T/\varepsilon} f(i) \varepsilon$$

with $\varepsilon$ infinitesimal
THE PLAN

1. Define Kahn networks and their semantics
   ▶ formalization of Kahn networks
   ▶ Kahn networks form a free fixpoint category

2. A non-standard semantics for Kahn networks
   ▶ non-standard semantics using internal cpo
THE PLAN

1. Define Kahn networks and their semantics
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Related works:

1. Semantics of Kahn networks
   ▶ Kahn
   ▶ categorical structure:
     Hildebrandt, Panangaden, Winskel, Stark, . . .

2. Using non-standard analysis to model hybrid systems
   ▶ Bliudze, Krob
   ▶ Benveniste, Caillaud, Pouzet
EXAMPLES OF KAHN NETWORKS

- Prepend a 0:

\[ 1, 2, 3, \ldots \xrightarrow{\zeta} 0, 1, 2, 3, \ldots \]
EXAMPLES OF KAHN NETWORKS

▶ Prepend a 0:

\[ 1, 2, 3, \ldots \rightarrow \zeta \rightarrow 0, 1, 2, 3, \ldots \]

▶ Add two discrete streams:

\[ x_0, x_1, x_2, \ldots \]
\[ y_0, y_1, y_2, \ldots \]
\[ (x_0 + y_0), (x_1 + y_1), \ldots \]
EXAMPLES OF KAHN NETWORKS

- Prepend a 0:
  1, 2, 3, ... → \( \zeta \) → 0, 1, 2, 3, ...

- Add two discrete streams:
  1, 1, 1, ... + 1, 2 → 2, 3
EXAMPLES OF KAHN NETWORKS

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  \[1, 2, 3 \quad + \quad 1, 2, 3\]
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\[1, 1, 1, \ldots \quad + \quad 1, 2, 3 \rightarrow 2, 3, 4\]

- A net composed of generators:

\[1, 2, 3, \ldots \quad + \quad \zeta \rightarrow \zeta \]
EXAMPLES OF KAHN NETWORKS

▶ Prepend a 0:

1, 2, 3, ... \rightarrow \zeta \rightarrow 0, 1, 2, 3, ...

▶ Add two discrete streams:

1, 1, 1, ...

1, 2, 3

\rightarrow + \rightarrow 2, 3, 4

▶ A net composed of generators:

1, 2, 3, ...

\rightarrow + \rightarrow \zeta \rightarrow 0
EXAMPLES OF KAHN NETWORKS

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  \[ 1, 1, 1, \ldots \]
  \[ 1, 2, 3 \]
  \[ \xrightarrow{\text{+}} \]
  \[ 2, 3, 4 \]

- A net composed of *generators*:
  \[ 1, 2, 3, \ldots \]
  \[ \xrightarrow{\text{+}} 1 \]
  \[ \xrightarrow{\text{+}} \zeta \]
  \[ 0 \]
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- **A net composed of **generators**:\n  \[ 1, 2, 3, \ldots \]
  \[ + \rightarrow 1, 3 \]
  \[ \zeta \rightarrow 0, 1, 3 \]
EXAMPLES OF KAHN NETWORKS

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  \[1, 2, 3, \ldots \overset{\zeta}{\longrightarrow} 0, 1, 2, 3, \ldots\]

- Add two discrete streams:
  \[1, 1, 1, \ldots \overset{+}{\longrightarrow} 2, 3, 4\]
  \[1, 2, 3 \overset{+}{\longrightarrow} 2, 3, 4\]

- A net composed of **generators**:
  \[1, 2, 3, \ldots \overset{+}{\longrightarrow} 1, 3, 6 \overset{\zeta}{\longrightarrow} 0, 1, 3\]
EXAMPLES OF KAHN NETWORKS

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Definition

The **Kahn domain** \( (K, \sqsubseteq) \) is the complete partial order whose elements are the finite or infinite lists of elements in \( \mathbb{R} \), ordered by prefix.
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To each generator \(\alpha : m \to n\)

we associate a **Scott-continuous function** \(K^m \to K^n\).
The semantics of a composed net is given by associating a set of equations to the network and taking the unique minimal solution (which exists).
A semantics of what?

Fig. 3. A parallel program schema.
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A UNIVERSAL DESCRIPTION OF NETS

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**Definition**

A **fixpoint category** is a category with

- cartesian products
- a trace.

Theorem

The category $\text{Net}_\Sigma$ is the free fixpoint category containing a $\Sigma$-object (i.e. an interpretation of the generators).

Any interpretation of the generators in a fixpoint category canonically induces an interpretation of all Kahn nets.

We can give semantics of KN in other fixpoint categories.

Avoids some technical details (solving systems of equations).
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- Avoids some technical details (solving systems of equations).
The Kahn model

- The elements of the Kahn cpo are discrete streams, i.e. lists $\mathcal{L}$ of reals.
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- They can be seen as partial functions $\ell : \mathbb{N} \rightarrow \mathbb{R}$ whose domain of definition is an initial segment of $\mathbb{N}$:
A SEMANTICS OF HYBRID SYSTEMS?

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$$\mathbb{R}^{\leq \mathbb{N}}$$

The intuitive continuous-time model

- We want to model hybrid systems where streams are now partial functions $f : \mathbb{R}^+ \to \mathbb{R}$:

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A SEMANTICS OF HYBRID SYSTEMS?

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These do not form a cpo!

\[ 0, 1/2 \] \[ 0, 3/4 \] \[ 0, 7/8 \] \[ 0, 15/16 \] ...

...because of the Zeno effect.
In order to give a meaning to the infinitesimals, we replace reals by *hyperreals* which are sequences \((x_i)_{i \in \mathbb{N}}\) of reals.

- A real \(x\) is seen as the constant sequence \((x)\).
- An infinitesimal number is a sequence converging towards 0.
- An “infinite” number is a sequence converging towards \(\pm \infty\).
- The usual operations are extended pointwise on sequences: 
  \[(x_i) \times (y_i) = (x_i \times y_i)\]
In order to give a meaning to the infinitesimals, we replace reals by **hyperreals** which are sequences \((x_i)_{i \in \mathbb{N}}\) of reals.

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- An “infinite” number is a sequence converging towards \(\pm \infty\).
- The usual operations are extended pointwise on sequences: \((x_i) \times (y_i) = (x_i \times y_i)\)
- What is the inverse of \((0, 1, 0, 1, 0, 1, \ldots)\)?
- In order to recover usual properties one has to consider *equivalence classes* of sequences.
We will define a collection $\mathcal{F}$ of subsets (called **large**) of $\mathbb{N}$ and consider the equivalence relation such that

$$(x_i) \equiv (y_i) \quad \text{whenever} \quad \{i \in \mathbb{N} \mid x_i = y_i\} \in \mathcal{F}$$
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The set $\mathcal{F}$ should satisfy properties:

- two sequences equal at every index excepting a finite number should be equal: $\mathcal{F}$ should contain all cofinite sets
- two sequences are either equal (equivalent) or different:

  $$\forall U \subseteq \mathbb{N}, \quad U \in \mathcal{F} \quad \text{or} \quad \mathbb{N} \setminus U \in \mathcal{F}$$

- ...
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$$(x_i) \equiv (y_i) \text{ } \text{ whenever } \{i \in \mathbb{N} \mid x_i = y_i\} \in \mathcal{F}$$

The set $\mathcal{F}$ should be a non-principal ultrafilter on $\mathbb{N}$.

**Definition**

An ultrafilter $\mathcal{F}$ on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ such that

1. intersection: $\forall U, V \in \mathcal{F}, \quad U \cap V \in \mathcal{F}$
2. superset: $\forall U \in \mathcal{F}, \forall V \subseteq \mathbb{N}, \quad U \subseteq V \Rightarrow V \in \mathcal{F}$
3. proper: $\emptyset \notin \mathcal{F}$
4. complement: $\forall U \subseteq \mathbb{N}, \quad U \in \mathcal{F} \text{ or } \mathbb{N} \setminus U \in \mathcal{F}$

(with AC such an $\mathcal{F}$ exists).
We will define a collection $F$ of subsets (called large) of $\mathbb{N}$ and consider the equivalence relation such that

$$(x_i) \equiv (y_i) \quad \text{whenever} \quad \{i \in \mathbb{N} \mid x_i = y_i\} \in F$$

**Definition**

- The field of **hyperreals** $\mathbb{R}^*$ is $\mathbb{R}^\mathbb{N} / \equiv$.
- The ring of **hyperintegers** $\mathbb{N}^*$ is $\mathbb{N}^\mathbb{N} / \equiv$. 
INFINITESIMAL AND UNLIMITED

Definition
An hyperreal $x \in \mathbb{R}$ is

- **infinitesimal**: if $x \neq 0$ and $\forall r \in \mathbb{R}, |x| < r$
- **unlimited**: if $\forall r \in \mathbb{R}, |x| > r$

Example

- infinitesimal: $x = \langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$
- unlimited: $x = \langle 1, 2, 3, 4, \ldots \rangle$
INFINITESIMAL AND UNLIMITED

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Example

- **infinitesimal**: \( x = \langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle \)
- **unlimited**: \( x = \langle 1, 2, 3, 4, \ldots \rangle \)

Intuition

- \( \mathbb{R}^* \) is \( \mathbb{R} \) completed with infinitesimals and unlimited.
- \( \mathbb{N}^* \) is \( \mathbb{N} \) completed with unlimited
Continuity, derivation, integration, etc. have the “expected” formulations in this framework.
NON-STANDARD SEMANTICS OF KAHN NETWORKS
The elements of the Kahn cpo are the streams:

\[ \mathbb{R}^\leq \mathbb{N} \]
TOWARDS A NON-STANDARD SEMANTICS

- The elements of the Kahn cpo are the streams:

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\mathbb{R} \leq \mathbb{N}
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- The “continuous time semantics” failed:

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We will see those functions as sequences

\[ f(0), f(\epsilon), f(2\epsilon), f(3\epsilon), \ldots \]

for some infinitesimal \( \epsilon \)

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We have to take into account infinitesimal variations!
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  for some infinitesimal \( \varepsilon \)

  \[ \ast \mathbb{R}^{\leq} \mathbb{N} \]

But for every \( n \in \mathbb{N} \), \( n\varepsilon \) is an infinitesimal!
The elements of the Kahn cpo are the streams:
\[
\mathbb{R} \leq \mathbb{N}
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The “continuous time semantics” failed:
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We will see those functions as sequences
\[
f(0), f(\varepsilon), f(2\varepsilon), f(3\varepsilon), \ldots
\]
for some infinitesimal \(\varepsilon\)

\[
*\mathbb{R} \leq *\mathbb{N}
\]

We consider hypersequences of hyperreals.
The semantics of the following net should be the constant stream $f$ such that $\forall n \in \ast \mathbb{N}, f(n\varepsilon) = 0$:

However, if we compute its semantics using the fixpoint construction we get the stream $f$ such that

$$\forall n \in \mathbb{N}, f(n\varepsilon) = 0$$

The value $f(n\varepsilon)$ is not defined for unlimited hyperintegers $n \in \ast \mathbb{N}$!
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ζ
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$$\varepsilon 0,0,0$$

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The value $f(n\varepsilon)$ is not defined for unlimited hyperintegers $n \in \ast \mathbb{N}$!
INTERNAL THINGS

From \((D_i \subseteq \mathbb{R})_{i \in \mathbb{N}}\) we can define a subset \(D \subseteq \ast \mathbb{R}\):

- \(D_0 = \{x_0, y_0, z_0, \ldots\}\)
- \(D_1 = \{x_1, y_1, z_1, \ldots\}\)
- \(D_2 = \{x_2, y_2, z_2, \ldots\}\)
- \(\ldots\)
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- \(\ldots\)

\[D = \{\langle z_0, x_1, y_2, \ldots \rangle, \ldots \}\]
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- \(D_2 = \{x_2, y_2, z_2, \ldots\}\)
- \(\ldots\)

\[D = \{\langle z_0, x_1, y_2, \ldots \rangle, \ldots \}\]

**Definition**

An **internal set** \(D \subseteq \ast \mathbb{R}\) is a set such that there exists a family \((D_i \subseteq \mathbb{R})_{i \in \mathbb{N}}\) for which

\[D = \langle D_i \rangle = \{\langle x_i \rangle \mid \forall i \in \mathbb{N}, x_i \in D_i\}\]

*Internal functions* \(f = \langle f_i \rangle\), *internal relations*, etc. are defined similarly.
THE TRANSFER PRINCIPLE

Proposition

The transfer principle: a first-order formula is satisfied for $\mathbb{R}$ iff it is satisfied for $^*\mathbb{R}$, if we suppose that all the sets, etc in the formula to be internal.
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The transfer principle:

a first-order formula is satisfied for $\mathbb{R}$

iff

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Lemma

Internal induction principle: if $D$ is an internal subset of $\ast\mathbb{N}$ s.t.

- $0 \in D$
- $\forall n \in D, \ n + 1 \in D$

then $D = \ast\mathbb{N}$. 
This suggests that

we should consider **internal** cpo!
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**Proposition**

An internal Scott-continuous function $f$ between two internal cpo admits a least (internal) fixpoint

$$\text{fix}(f) = \bigvee \{ f^n(\bot) \mid n \in \ast \mathbb{N} \}$$
This suggests that we should consider internal cpo!

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\[
\text{fix}(f) = \bigsqcup \{ f^n(\perp) \mid n \in \ast \mathbb{N} \}
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**Remark**

An internal cpo \((D, \leq)\) is not necessarily a cpo!
THE INFINITESIMAL-TIME DOMAIN

Definition

- The category **ICPO**: internal cpo and internal Scott-continuous functions
- The **infinitesimal-time domain** \( IT \in ICPO \): the internal cpo of internal functions in \( {}^*\mathbb{R} \leq {}^*\mathbb{N} \)
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Definition

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Proposition

*The category ICPO is a fixpoint category.*

A \( \Sigma \)-object in this category canonically induces a semantics of Kahn networks
EXAMPLE – THE CONSTANT STREAM

If we interpret

\[(s_i)\] 0, (s_i)

then the net

\[0, 0, 0, \ldots\]

is interpreted as the constant stream \(s : \ast \mathbb{R}^{\ast \mathbb{N}}\) such that

\[\forall i \in \ast \mathbb{N}, \quad s_i = 0\]
If we interpret

\[
(s_i) (s_{i-1})
\]

(with \(\varepsilon\) infinitesimal) then the net

is interpreted as the function

\[
D : *\mathbb{R}^* N \rightarrow *\mathbb{R}^* N
\]

\[
(s_i) \mapsto \left( \frac{s_i - s_{i-1}}{\varepsilon} \right)
\]
The function \( D : \ast \mathbb{R}^\ast \mathbb{N} \to \ast \mathbb{R}^\ast \mathbb{N} \) acts as a derivation operator for streams.

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EXAMPLE – DERIVATION

The function

\[ D : \ast \mathbb{R}^* \mathbb{N} \rightarrow \ast \mathbb{R}^* \mathbb{N} \]

acts as a derivation operator for streams.

Definition

In order to compare

- \( CT = \mathbb{R}^{\leq} \mathbb{R}^+ \): the continuous time model
- \( IT = * \mathbb{R}^{\leq} * \mathbb{N} \): the infinitesimal time model

we introduce

<table>
<thead>
<tr>
<th>sampling</th>
<th>standardisation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S : CT \rightarrow IT )</td>
<td>( T : IT \rightarrow CT )</td>
</tr>
<tr>
<td>( s \mapsto (s(i\varepsilon))_{i \in \ast \mathbb{N}} )</td>
<td>( (s_i)<em>{i \in \ast \mathbb{N}} \mapsto t \mapsto st(s</em>{\lfloor t/\varepsilon \rfloor}) )</td>
</tr>
</tbody>
</table>

Proposition

For any continuously differentiable \( s \in CT \), \( T(D(S(s))) = s' \).
In this way, we give a non-standard **semantics** for hybrid systems:

- we can interpret all the common building blocks (derivation, integration, zero-crossing, etc.)
- we relate it to the “continuous time model” through $S$ and $T$
- definitions should be independent of the infinitesimal $\varepsilon$ (in particular, Zeno effects are “non-reasonable”)
- we have a built-in notion of approximation
- NSA enables us to use discrete techniques for continuous: continuous-time bisimulations, game semantics, etc.
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NON-STANDARD SEMANTICS
GOING FURTHER

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➤ we can study when solutions exist
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Any questions?