A NON-STANDARD SEMANTICS FOR KAHN NETWORKS IN CONTINUOUS TIME

SAMUEL MIMRAM & ROMAIN BEAUXIS

SÉMINAIRE PPS

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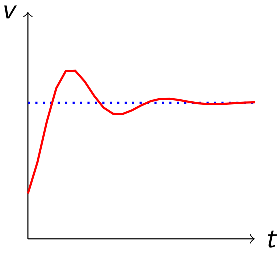
HYBRID SYSTEMS

A model for systems operating in **continuous time** $t \in \mathbb{R}^+$.

A controller to **cruise control** a car:

$$I(t) = K_p e(t)$$

- error: $e(t) = v_{\text{desired}} - v_{\text{actual}}$
- intensity of the engine: I
- parameters of the control: K_p



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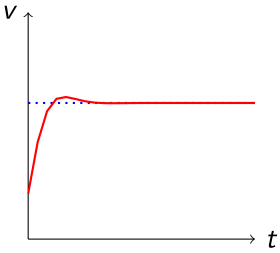

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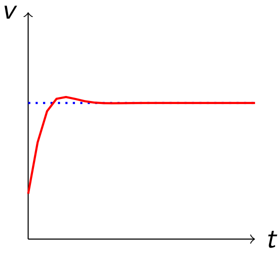

We also want to have discontinuities!

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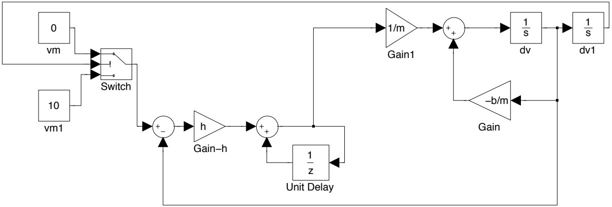

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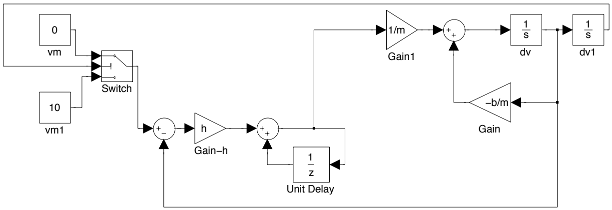

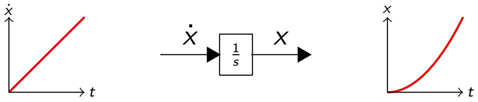

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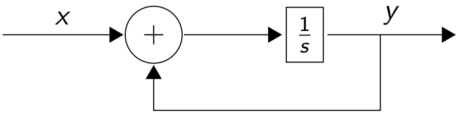

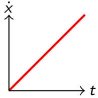

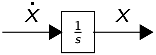

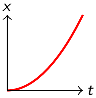

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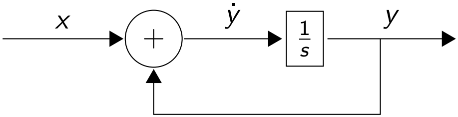

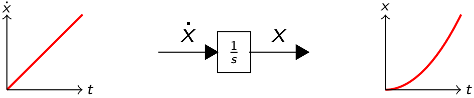

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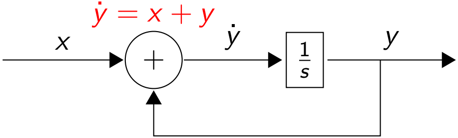

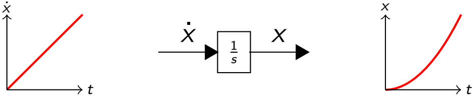

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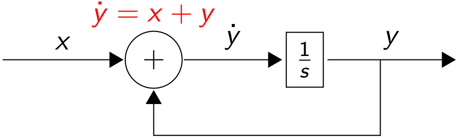

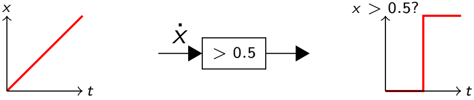

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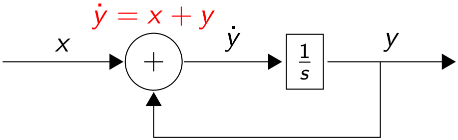

How can we define a semantics for those systems?

The streams $f : \mathbb{R}^+ \to \mathbb{R}$ on the wires could

- be integrable / derivable
- have discontinuities (zero-crossings)
- exhibit complex behaviors such as Zeno
- be approximated...

Let's take inspiration
from discrete time semantics.

KAHN PROCESS NETWORKS

A semantics for ***distributed asynchronous*** *computations*:
processes exchanging *sequences of data* on channels.

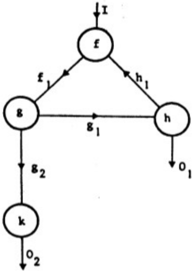

Fig.3. A parallel program schema.

G. Kahn. The semantics of a simple language for parallel programming. *Information processing*, 74:471–475, 1974.

$$1, 2, 3, \ldots \longrightarrow (\times 2) \longrightarrow 2, 4, 6, \ldots$$

KAHN NETWORKS IN CONTINUOUS TIME

It is difficult to define a semantics for hybrid systems.
Can we adapt the works on Kahn networks?

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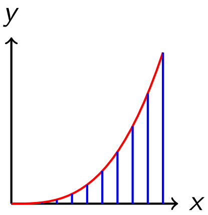
The sampling principle

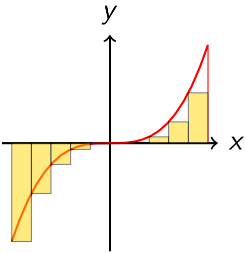
We can to consider a continuous stream $t \mapsto x_t$ (with $t \in \mathbb{R}^+$)
as a discrete stream x_i (with $i \in \mathbb{N}$)
where the data x_i occurs at time $i\varepsilon$, with ε infinitesimal.



A continuous stream sampled at every ε seconds.

NON-STANDARD ANALYSIS

In the 60s, Robinson introduced an extension ${}^*\mathbb{R}$ of $\mathbb{R}$
(the *hyperreals*)
in which one can formally consider **infinitesimals**.



$$f'(x) = \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

$$\int_{t=0}^{T} f(t)\mathrm{d}t = \sum_{0 \le i \le T/\varepsilon} f(i)\varepsilon$$

with ε infinitesimal

THE PLAN

1. Define **Kahn networks** and their semantics
   - formalization of Kahn networks
   - Kahn networks form a free fixpoint category

2. A **non-standard semantics** for Kahn networks
   - non-standard semantics using internal cpo

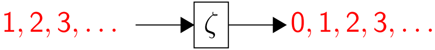
THE PLAN

1. Define **Kahn networks** and their semantics
   - formalization of Kahn networks
   - Kahn networks form a free fixpoint category
   - *related works:*
     - Kahn
     - categorical structure of the models: Hildebrandt, Panangaden, Winskel, Stark, ...
2. A **non-standard semantics** for Kahn networks
   - non-standard semantics using internal cpo
   - *related works:*
     - Bliudze, Krob
     - Benveniste, Caillaud, Pouzet

KAHN
NETS

EXAMPLES OF KAHN NETWORKS

- Prepend a 0:



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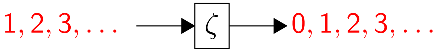

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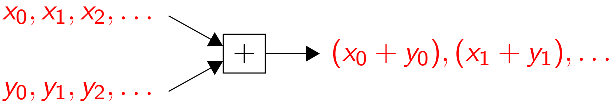

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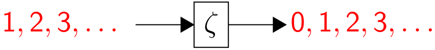

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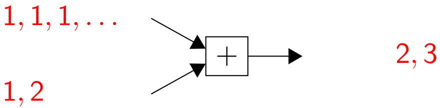

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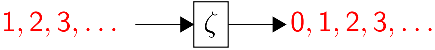

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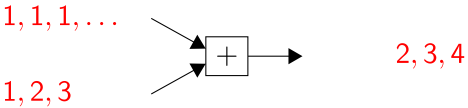

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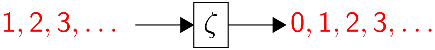

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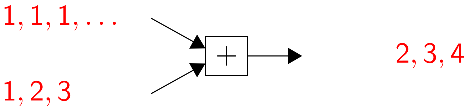

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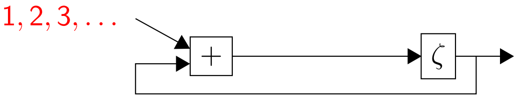

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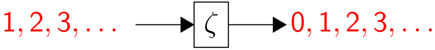

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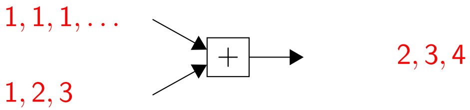

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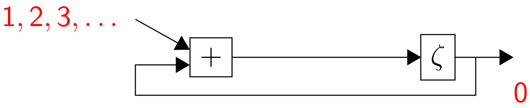

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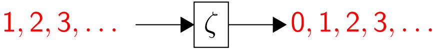

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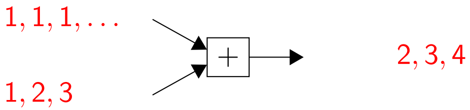

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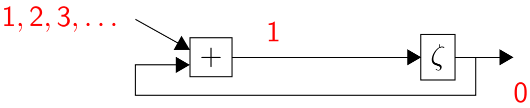

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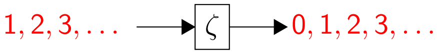

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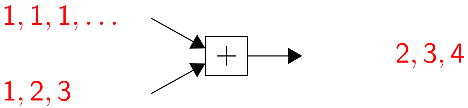

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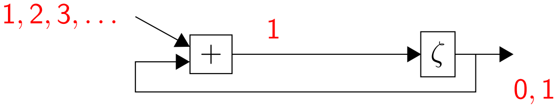

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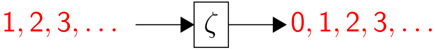

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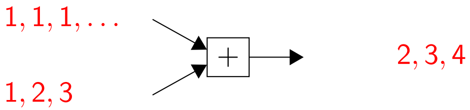

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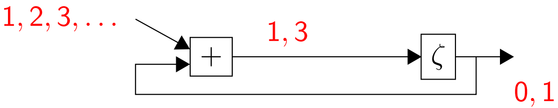

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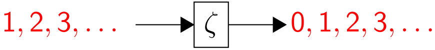

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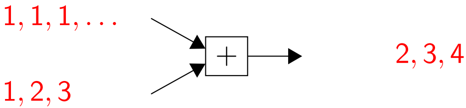

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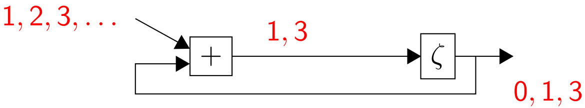

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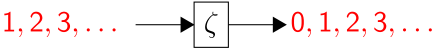

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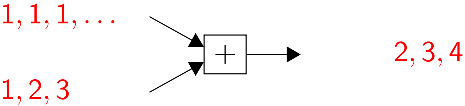

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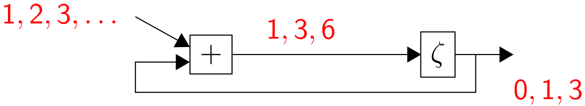

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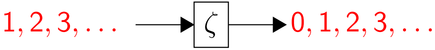

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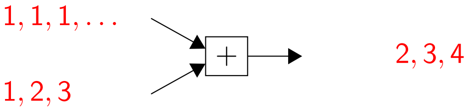

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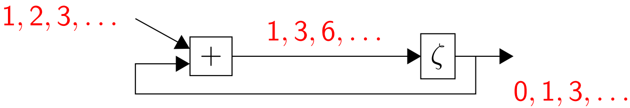

SEMANTICS OF KAHN NETWORKS

Definition

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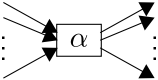
To each generator $\alpha : m \to n$



we associate a *Scott-continuous function* $K^m \to K^n$.

SEMANTICS OF KAHN NETWORKS

The semantics of a composed net is given by associating a set of equations to the network

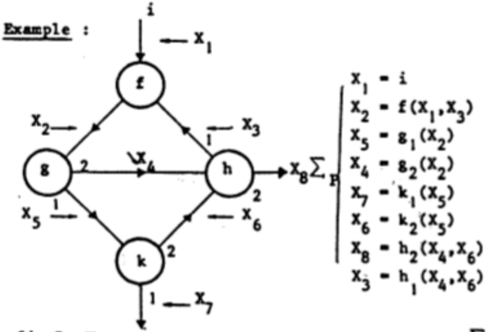


fig.7. The program P and the associated system Σ_P.

and taking the unique *minimal solution* (which exists).

A semantics of what?

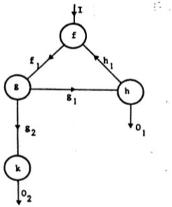


Fig.3. A parallel program schema.

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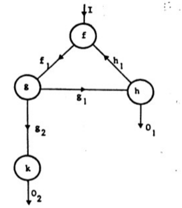

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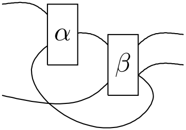
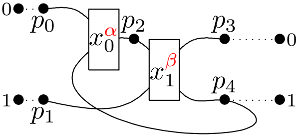


is formalized by



SIGNATURES

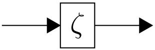
The "basic building blocks" of Kahn networks are the elements of a signature:

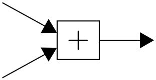
Definition

A **signature** $\Sigma = \{\alpha_i : m_i \to n_i\}$ consists of a set Σ of **symbols** (or **generators**) α_i with **arity** m_i and **coarity** n_i.

Example

For instance,

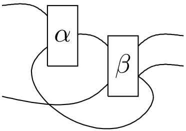




is formalized by the signature

$$\Sigma \quad = \quad \{\zeta : 1 \to 1,\ + : 2 \to 1\}$$

A FORMAL DEFINITION OF NETS



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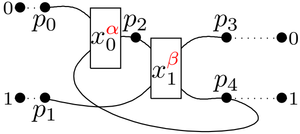
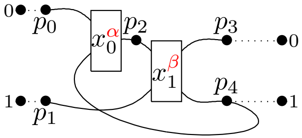


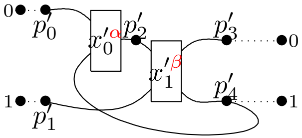
Definition (Net)

A **net** $N = (P, O, \lambda, s, t)$ with m inputs and n outputs consists of

- P: a finite set of *ports*,
- O: a finite set of *operators*,
- $\lambda : O \to \Sigma$: a *labeling function* ,
- $s : S_N \to P$ and $t : T_N \to P$: *source* and inj. *target functions*
  $S_N = \{(x, i) \mid x \in O, i \in \{0 \dots \sigma \circ \lambda(x) - 1\}\} \uplus \{0 \dots n - 1\}$
  $T_N = \{(x, i) \mid x \in O, i \in \{0 \dots \tau \circ \lambda(x) - 1\}\} \uplus \{0 \dots m - 1\}$

α-CONVERSION OF NETS





Definition

Two nets $N = (P, O, \lambda, s, t)$ and $N' = (P', O', \lambda', s', t')$ are α-*convertible* when there exists a pair of bijective functions

$$\varphi_P : P \to P' \qquad \text{and} \qquad \varphi_O : O \to O'$$

such that

- $\forall x \in O, \quad \lambda'(\varphi_O(x)) = \lambda(x)$
- $\forall (x, i) \in S_N, \quad \varphi_P(s(x, i)) = s'(\varphi(x), i)$

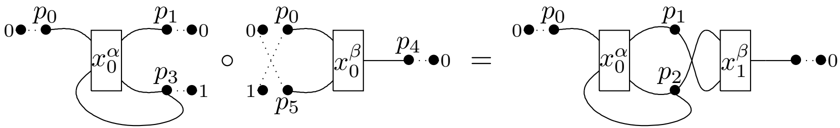
THE CATEGORY OF NETS

We form a category $\mathbf{Net}_\Sigma$ whose

- objects are natural integers $n \in \mathbb{N}$
- morphisms $N : m \to n$ are nets with m inputs and n outputs

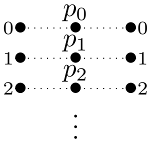
Categorical structure:

- **Composition**: juxtaposing and "linking the wires"



(up to α-conversion, otherwise we get a bicategory)

- **Identities**:



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We form a monoidal category $\mathbf{Net}_\Sigma$ whose

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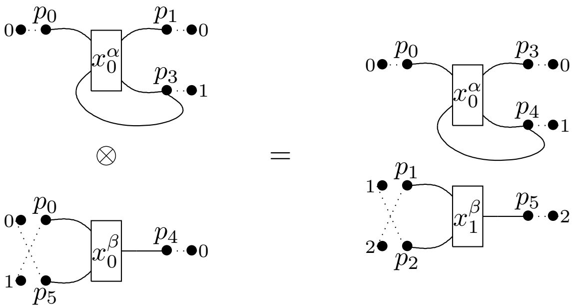
Categorical structure:

- **Tensor product**: from $N : m \to n$ and $N' : m' \to n'$ we can define $N \otimes N' : (m + m') \to (n + n')$



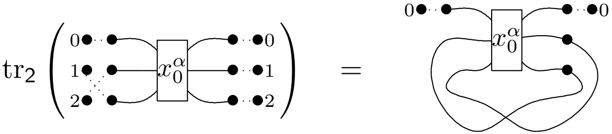
THE CATEGORY OF NETS

We form a traced monoidal category $\mathbf{Net}_\Sigma$ whose

- objects are natural integers $n \in \mathbb{N}$
- morphisms $N : m \to n$ are nets with m inputs and n outputs

Categorical structure:

- **Trace**: from $N : (m+k) \to (n+k)$
  we can define $\mathrm{tr}_k(N) : m \to n$
  satisfying suitable axioms



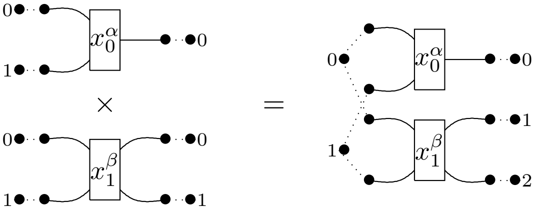
THE CATEGORY OF NETS

We form a traced cartesian(?) category $\mathbf{Net}_\Sigma$ whose

- objects are natural integers $n \in \mathbb{N}$
- morphisms $N : m \to n$ are nets with m inputs and n outputs

Categorical structure:

- The tensor is *almost* a **cartesian product**:
  from $N_1 : m \to n_1$ and $N_2 : m \to n_2$
  we can define $N_1 \times N_2 : m \to (n_1 + n_2)$

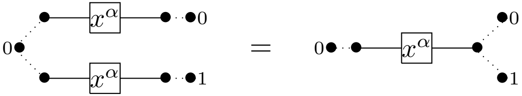


CARTESIAN PRODUCT OF NETS

The "cartesian product" we defined is not one: given $N : m \to n$

it does *not* satisfy

- **sharing**: $N \times N = (\mathrm{id}_n \times \mathrm{id}_n) \circ N$



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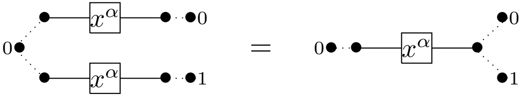

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We consider nets up to the equivalence relation generated by the above equations!

Σ-OBJECTS

Suppose given a signature (Σ, σ, τ).

Definition

A Σ-**object** in a monoidal category $\mathcal{C}$ consists of

- an object A
- for every symbol $\alpha \in \Sigma$, a morphism

$$\alpha \quad : \quad A^{\otimes\sigma(\alpha)} \to A^{\otimes\tau(\alpha)}$$

where

$$A^{\otimes k} \quad = \quad \underbrace{A \otimes A \otimes \ldots \otimes A}_{k \text{ times}}$$

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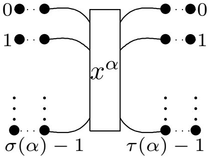
Remark

In the category $\mathbf{Net}_\Sigma$, 1 is a Σ object, where to every symbol $\alpha \in \Sigma$ is associated the morphism



A UNIVERSAL DESCRIPTION OF NETS

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A **fixpoint category** is a category with

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Any interpretation of the generators in a fixpoint category canonically induces an interpretation of all Kahn nets.

- We can give semantics of KN in other fixpoint categories.
- Avoids some technical details (solving systems of equations).

A SEMANTICS OF HYBRID SYSTEMS?

The Kahn model

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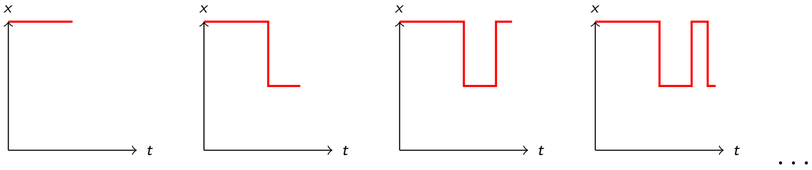

f_0	f_1	f_2	f_3	$\dots$
$[0, 1/2]$	$[0, 3/4]$	$[0, 7/8]$	$[0, 15/16]$	$\dots$

. . . because of the Zeno effect.

NON-STANDARD
ANALYSIS
A CRASH COURSE

NON-STANDARD ANALYSIS

In order to give a meaning to the infinitesimals, we replace reals by **hyperreals** which are sequences $(x_i)_{i \in \mathbb{N}}$ of reals.

- A real x is seen as the constant sequence (x).
- An infinitesimal number is a sequence converging towards 0.
- An "infinite" number is a sequence converging towards $\pm\infty$.
- The usual operations are extended pointwise on sequences:
  $(x_i) \times (y_i) = (x_i \times y_i)$

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- An "infinite" number is a sequence converging towards $\pm\infty$.
- The usual operations are extended pointwise on sequences: $(x_i) \times (y_i) = (x_i \times y_i)$
- What is the inverse of $(0, 1, 0, 1, 0, 1, \ldots)$?
- In order to recover usual properties one has to consider *equivalence classes* of sequences.

HYPERREALS

We will define a collection $\mathcal{F}$ of subsets (called **large**) of $\mathbb{N}$ and consider the equivalence relation such that

$$(x_i) \equiv (y_i) \qquad \text{whenever} \qquad \{i \in \mathbb{N} \mid x_i = y_i\} \in \mathcal{F}$$

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The set $\mathcal{F}$ should satisfy properties:

- two sequences equal at every index excepting a finite number should be equal: $\mathcal{F}$ should contain all cofinite sets
- two sequences are either equal (equivalent) or different:

$$\forall U \subseteq \mathbb{N}, \qquad U \in \mathcal{F} \quad \text{or} \quad \mathbb{N} \setminus U \in \mathcal{F}$$

- ...

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The set $\mathcal{F}$ should be a *non-principal ultrafilter* on $\mathbb{N}$.

Definition

An **ultrafilter** $\mathcal{F}$ on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ such that

1. intersection: $\forall U, V \in \mathcal{F}, \qquad U \cap V \in \mathcal{F}$
2. supersets: $\forall U \in \mathcal{F}, \forall V \subseteq \mathbb{N}, \qquad U \subseteq V \Rightarrow V \in \mathcal{F}$
3. proper: $\emptyset \notin \mathcal{F}$
4. complement: $\forall U \subseteq \mathbb{N}, \qquad U \in \mathcal{F} \quad \text{or} \quad \mathbb{N} \setminus U \in \mathcal{F}$

(with AC such an $\mathcal{F}$ exists).

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We write $\langle x_i \rangle$ for the equivalence class of (x_i).

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Definition

- The field of **hyperreals** $^*\mathbb{R}$ is $\mathbb{R}^{\mathbb{N}}/\equiv$.
- The ring of **hyperintegers** $^*\mathbb{N}$ is $\mathbb{N}^{\mathbb{N}}/\equiv$.

INFINITESIMAL AND UNLIMITED

Definition

An hyperreal $x \in {}^*\mathbb{R}$ is

- **infinitesimal**: if $x \neq 0$ and $\forall r \in \mathbb{R}, |x| < r$
- **unlimited**: if $\forall r \in \mathbb{R}, |x| > r$

Example

- infinitesimal: $x = \langle \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$
- unlimited: $x = \langle 1, 2, 3, 4, \ldots \rangle$

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Intuition

- ${}^*\mathbb{R}$ is $\mathbb{R}$ completed with infinitesimals and unlimited.
- ${}^*\mathbb{N}$ is $\mathbb{N}$ completed with unlimited

NON-STANDARD ANALYSIS

Continuity, derivation, integration, etc.
have the “expected” formulations
in this framework.

THE STANDARD PART

Notation

Given $x, y \in {}^*\mathbb{R}$, we write $x \approx y$ whenever $x - y$ is infinitesimal.

Lemma

*Given a limited $x \in {}^*\mathbb{R}$, there exists a unique real $\mathrm{st}(x)$, the **standard part** of x, such that $x \approx \mathrm{st}(x)$.*

NON-STANDARD ANALYSIS – CONTINUITY

Proposition

*A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at $x \in \mathbb{R}$ iff*

$$\forall y \in {}^*\mathbb{R}, \qquad y \approx x \quad \Rightarrow \quad f(y) \approx f(x)$$

or equivalently, for every infinitesimal ε there exists an infinitesimal δ such that

$$f(x+\varepsilon) \quad = \quad f(x) + \delta$$

NON-STANDARD ANALYSIS – DERIVATIVE

Proposition

A function $f : \mathbb{R} \to \mathbb{R}$ admits $y \in \mathbb{R}$ as ***derivative*** *at $x \in \mathbb{R}$ iff for every infinitesimal ε,*

$$\frac{f(x+\varepsilon) - f(x)}{\varepsilon} \quad \approx \quad y$$

In this case, for any ε infinitesimal,

$$f'(x) \quad = \quad \mathrm{st}\left(\frac{f(x+\varepsilon) - f(x)}{\varepsilon}\right)$$

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Proposition

Integrals *can be reformulated similarly as Riemann sums...*

NON-STANDARD
SEMANTICS
OF KAHN NETWORKS

TOWARDS A NON-STANDARD SEMANTICS

- The elements of the Kahn cpo are the streams:

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- We will see those functions as sequences

$$f(0), f(\varepsilon), f(2\varepsilon), f(3\varepsilon), \ldots$$

  for some infinitesimal ε

$$\mathbb{R}^{\leq {}^*\mathbb{N}}$$

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We have to take in account infinitesimal variations!

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But for every $n \in \mathbb{N}$, $n\varepsilon$ is an infinitesimal!

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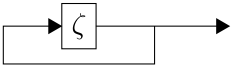
We consider hypersequences of hyperreals.

FIXPOINTS IN $^*\mathbb{R}^{\leq {}^*\mathbb{N}}$

The semantics of the following net should be the constant stream f such that $\forall n \in {}^*\mathbb{N}, f(n\varepsilon) = 0$:



However, if we compute its semantics using the fixpoint construction we get the stream f such that

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The value $f(n\varepsilon)$ is not defined for unlimited hyperintegers $n \in {}^*\mathbb{N}$!

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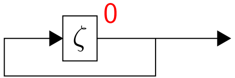

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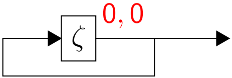

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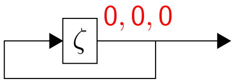

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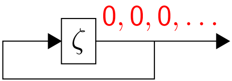

However, if we compute its semantics using the fixpoint construction we get the stream f such that

$$\forall n \in \mathbb{N}, f(n\varepsilon) = 0$$

The value $f(n\varepsilon)$ is not defined for unlimited hyperintegers $n \in {}^*\mathbb{N}$!

INTERNAL THINGS

From $(D_i \subseteq \mathbb{R})_{i \in \mathbb{N}}$ we can define a subset $D \subseteq {}^*\mathbb{R}$:

- $D_0 = \{x_0, y_0, z_0, \ldots\}$
- $D_1 = \{x_1, y_1, z_1, \ldots\}$
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Definition

An **internal set** $D \subseteq {}^*\mathbb{R}$ is a set such that there exists a family $(D_i \subseteq \mathbb{R})_{i \in \mathbb{N}}$ for which

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Internal functions $f = \langle f_i \rangle$, *internal relations*, etc. are defined similarly.

THE TRANSFER PRINCIPLE

Proposition

The transfer principle:

a first-order formula is satisfied for $\mathbb{R}$

iff

it is satisfied for ${}^*\mathbb{R}$,

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Lemma

Internal induction principle: *if* D *is an <u>internal</u> subset of* ${}^*\mathbb{N}$ *s.t.*

- $0 \in D$
- $\forall n \in D, n+1 \in D$

then $D = {}^*\mathbb{N}$.

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An <u>internal</u> Scott-continuous function f between two <u>internal</u> cpo admits a least (internal) fixpoint

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Remark

An internal cpo $(D, \leq)$ is not necessarily a cpo!

THE INFINITESIMAL-TIME DOMAIN

Definition

- The category **ICPO**:
  internal cpo and internal Scott-continuous functions
- The **infinitesimal-time domain** $IT \in$ **ICPO**:
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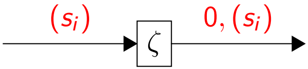
Proposition

The category **ICPO** *is a fixpoint category.*

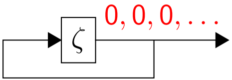
A Σ-object in this category canonically induces
a semantics of Kahn networks

EXAMPLE – THE CONSTANT STREAM

If we interpret



then the net



is interpreted as the constant stream $s : {}^*\mathbb{R}^{{}^*\mathbb{N}}$ such that

$$\forall i \in {}^*\mathbb{N}, \qquad s_i = 0$$

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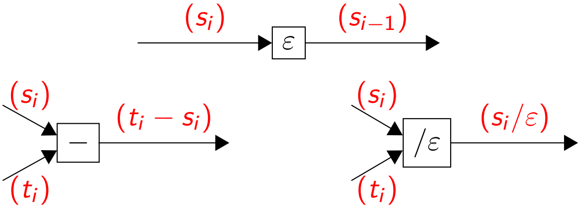
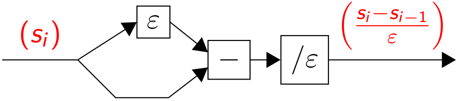


(with ε infinitesimal) then the net



is interpreted as the function

$$\begin{array}{rcccc} D & : & {}^*\mathbb{R}^{{}^*\mathbb{N}} & \to & {}^*\mathbb{R}^{{}^*\mathbb{N}} \\ & & (s_i) & \mapsto & \left(\frac{s_i - s_{i-1}}{\varepsilon}\right) \end{array}$$

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In order to compare

- $CT = \mathbb{R}^{\leq \mathbb{R}^+}$: the continuous time model
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sampling	**standardisation**
$\begin{array}{rcccl} S & : & CT & \to & IT \\ & & s & \mapsto & (s(i\varepsilon))_{i \in {}^*\mathbb{N}} \end{array}$	$\begin{array}{rcccl} T & : & IT & \to & CT \\ & & (s_i)_{i \in {}^*\mathbb{N}} & \mapsto & t \mapsto \mathrm{st}(s_{\lfloor t/\varepsilon \rfloor}) \end{array}$

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Proposition

For any continuously differentiable $s \in CT$, $T(D(S(s))) = s'$.

NON-STANDARD SEMANTICS
GOING FURTHER

In this way, we give a non-standard **semantics** for hybrid systems:

- we can interpret all the common building blocks (derivation, integration, zero-crossing, etc.)
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- we have a built-in notion of approximation
- NSA enables us to use discrete techniques for continuous: continuous-time bisiumlations, game semantics, etc.?

THANKS!

Questions?