Manifolds and Many More

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CEA, LIST

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- Most operations in linear algebra are performed on vectors/linear transformations/etc which are expressed in some *coordinate system*, but nature does not come equipped with those. Can we define things in a way that is independent of a choice of coordinates?
- The general idea is to extend *globally* what we know *locally*.

Differential geometry in $\mathbb{R}^n$

Derivation

Definition

The **derivative** $f'(x_0)$ of a function

$$f \quad : \quad U \subseteq \mathbb{R} \quad \rightarrow \quad \mathbb{R}$$

at $x_0 \in U$ is defined as

$$f'(x_0) \quad = \quad \lim_{\substack{t \rightarrow 0 \\ x_0 + t \in U}} \frac{f(x_0 + t) - f(x_0)}{t}$$

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Or equivalently,

$$f(x_0 + t) \quad = \quad f(x_0) + f'(x_0) \cdot t + |t| \cdot \varepsilon(t)$$

with

$$\varepsilon : \mathbb{R} \to \mathbb{R} \qquad \text{s.t.} \qquad \lim_{t \to 0} \varepsilon(t) = 0$$

Definition

The **differential** $\mathrm{d}f_p$ of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $p \in \mathbb{R}^n$ is a linear map

$$\mathrm{d}f_p \quad : \quad \mathbb{R}^n \quad \multimap \quad \mathbb{R}^m$$

such that, for $v \in \mathbb{R}^n$,

$$f(p+v) \quad = \quad f(p) + \mathrm{d}f_p(v) + \|v\| \cdot \varepsilon(v)$$

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- Does not depend on $\| - \|$: all the norms are equivalent on $\mathbb{R}^n$.
- Uniquely defined: $\mathbb{R}^n$ is complete.

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- Does not depend on $\| - \|$: all the norms are equivalent on $\mathbb{R}^n$.
- Uniquely defined: $\mathbb{R}^n$ is complete.

Notice that differentials might not exist, but I won't bother about definition problems here.

Basic properties

Proposition

- *Given* $f : \mathbb{R}^n \multimap \mathbb{R}^m$ *linear,*

$$f(p+v) = f(p) + f(v) + \|v\| \cdot 0$$

so, $\mathrm{d}f_p = f$.

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- $\mathrm{d}(f^{-1})_{f(p)} = (\mathrm{d}f_p)^{-1}$
- *etc.*

Definition

A **vector space** V over a field $\mathbb{k} = \mathbb{R}$ consists of

- an (additive) abelian group V:

$$(u + v) + w = u + (v + w)$$
$$0 + v = v$$
$$v - v = 0$$
$$v + w = w + v$$

Vector spaces

Definition

A **vector space** V over a field $\mathbb{k} = \mathbb{R}$ consists of

- an (additive) abelian group V:
- an action of $\mathbb{k}$ over V:

$$\alpha(v + w) = \alpha v + \alpha w \qquad (\alpha + \beta)v = \alpha v + \beta v$$
$$\alpha(\beta v) = (\alpha\beta)v \qquad 1v = v$$

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Definition

A **linear map** $f : V \to W$ is a function satisfying

$$f(v + w) = f(v) + f(w) \qquad f(\alpha v) = \alpha f(v)$$

We denote by $V \multimap W$ the set of linear maps between V and W.

Differentiation

Definition
Given

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

its **differential** is

$$\mathrm{d}f \quad : \quad \mathbb{R}^n \to \mathbb{R}^n \multimap \mathbb{R}^m$$

(when it is defined on every point $p \in \mathbb{R}^n$).

Linearity of differentiation

The linear space $V \multimap W$ (pointwisely) inherits a structure of vector space

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$$f + g = v \mapsto f(v) + g(v) \qquad\qquad \alpha f = v \mapsto \alpha f(v)$$

Given $p \in \mathbb{R}^n$, differentiation at p is linear over $\mathbb{R}^n \multimap \mathbb{R}^n$:

$$\mathrm{d}(f+g)_p \quad = \quad \mathrm{d}f_p + \mathrm{d}g_p \qquad\qquad \mathrm{d}(\alpha f)_p \quad = \quad \alpha\,\mathrm{d}f_p$$

Differentials and partial derivation

The **differential** of f is

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- We have considered the **differential** at a point $p \in \mathbb{R}^n$

$$\mathrm{d}f_p \quad = \quad v \mapsto \mathrm{d}f(p, v) \quad : \quad \mathbb{R}^n \multimap \mathbb{R}^m$$

- We can also consider **partial derivative** in direction $v \in \mathbb{R}^n$

$$\partial_v f \quad = \quad \frac{\partial f}{\partial v} \quad = \quad p \mapsto \mathrm{d}f(p, v) \quad : \quad \mathbb{R}^n \to \mathbb{R}^m$$

Differential in a basis

Proposition

Given a basis $(e_i)_{1\leq i\leq n}$ *of* $\mathbb{R}_n$*, we have*

$$\begin{aligned} \mathrm{d}f_p(v) &= \mathrm{d}f_p\left(\sum_i v_i \cdot e_i\right) \\ &= \sum_i v_i \cdot \mathrm{d}f_p(e_i) \\ &= \sum_i v_i \cdot \frac{\partial f}{\partial x^i}(p) \end{aligned}$$

In other words,

$$\mathrm{d}f_p = \sum_i \frac{\partial f}{\partial x^i}(p) \cdot \mathrm{d}x^i$$

with $x^i : \mathbb{R}^n \to \mathbb{R}$ *the canonical i-th projection.*

The chain rule

Given $x : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$, the chain rule says

$$\mathrm{d}(g \circ x)_t \quad = \quad \mathrm{d}g_{x(t)} \circ \mathrm{d}x_t$$

which is a way to write the usual chain rule

$$\frac{\mathrm{d}f}{\mathrm{d}t} \quad = \quad \sum_i \frac{\partial f}{\partial x^i} \frac{\mathrm{d}x^i}{\mathrm{d}t}$$

Manifolds

Manifolds

Definition

An n-dimensional **smooth manifold** consists of

- a topological space X
- an open covering $(U_i)_{i \in I}$ of X: $\bigcup_{i \in I} U_i = X$
- *charts* $\varphi_i : U_i \to V_i \subseteq \mathbb{R}^n$ (invertible and continuous) forming an *atlas*: the *transition functions*

$$\varphi_{ij} \quad = \quad \varphi_j \circ \varphi_i^{-1} : V_i \to V_j$$

  are smooth.

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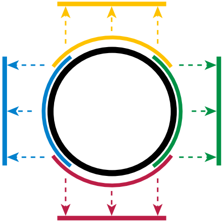
Example

The 1-sphere:
$x^2 + y^2 = 1$



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Example

Every finite-dimensional vector space $V \cong \mathbb{R}^n$.

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  are smooth.

Remark

In the following, we will not bother about definition issues and suppose that $V_i = \mathbb{R}^n$.

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  are smooth.

Remark

There are many possible variations over the definition:

- we can replace "smooth" by other adjective such as "differentiable", "analytic", etc.
- we can replace $\mathbb{R}$ by $\mathbb{C}$ and consider holomorphic transitions

Compatible atlases

Two atlases on X are **compatible** when their union is still an atlas.

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Two atlases on X are **compatible** when their union is still an atlas.

- Compatibility is an equivalence relation.
- The union of an equivalence relation is a **maximal atlas**.
- An atlas is included in a unique maximal atlas.

In theory we can thus use the maximal atlas,
but smaller is simpler in practice.

Morphisms

Definition

Given two m- and n-manifolds $M = (X, U_i, \varphi_i)$ and $N = (Y, V_i, \psi_i)$, a **morphism**

$$f \quad : \quad M \to N$$

is a function $f : X \to Y$ such that for every i, j the function f_{ij} is smooth and satisfies

$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{f_{ij}} & \mathbb{R}^n \\ \varphi_i \uparrow & & \uparrow \psi_j \\ U_i & \xrightarrow[f]{} & V_j \end{array}$$

The category of manifolds

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- We can thus define a **category** of manifolds: **Man**.
- It has coproducts of manifolds of same dimension.
- It has cartesian products (of dim $m + n$)
- It is *not* cartesian closed (the hom-space would be an infinite-dimensional manifold. . .)

Smooth functions

We write

$$M^* \quad = \quad \mathbf{Man}(M, \mathbb{R})$$

for the set of smooth functions from M to $\mathbb{R}$, i.e. functions

$$f \quad : \quad M \to \mathbb{R}$$

such that for every i, $f \circ \varphi_i$ is smooth.

Tangent spaces

Tangent spaces

A **path** is a smooth map $\gamma : (-1, 1) \to M$.

Definition

Fix a chart $\varphi : U \to \mathbb{R}^n$. We define an equivalence relation on paths $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ by

$$\gamma \sim \rho \quad \text{whenever} \quad (\varphi \circ \gamma)'(0) = (\varphi \circ \rho)'(0)$$

The **tangent space** T_pM is the quotient of those paths (*germs*).

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The **tangent space** T_pM is the quotient of those paths (*germs*).

Remark

This equivalence relation is independent of the chart.

Tangent spaces as vector spaces

Proposition

Given a chart (U, φ)*, the map* $T_\varphi : T_pM \to \mathbb{R}^n$ *defined by*

$$T_\varphi(\gamma) \quad = \quad (\varphi \circ \gamma)'(0)$$

is an isomorphism.

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This allows to transfer the structure of vector space of $\mathbb{R}^n$ to T_p, e.g.

$$\gamma + \rho \quad = \quad T_\varphi^{-1}\left((T_\varphi \gamma) + (T_\varphi \rho)\right)$$

(and this does not depend on the choice of the chart).

Actually, since it is enough to "test" linear morphisms coordinatewise, we can define T_pM as follows:

Definition

Fix a chart $\varphi : U \to \mathbb{R}^n$. We consider paths $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ and "copaths" $g : M \to \mathbb{R}$ such that $g(p) = 0$. We define

$$\langle \gamma | g \rangle_p \quad = \quad \frac{\partial (g \circ \gamma)}{\partial t}(0)$$

Two paths γ, ρ are equivalent when

$$\forall g : M \to \mathbb{R}, \qquad \langle \gamma | g \rangle_p = \langle \rho | g \rangle_p$$

T_pM is the set (vector space) of equivalence classes of paths.

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Remark

The dual notion on copaths gives rise to the notion of **cotengent** vector space of **covectors** T_p^*M.

Differentials

Definition

Given a morphism $f : M \to N$, we define its **differential** at $p \in M$

$$\mathrm{d}f_p \quad : \quad T_pM \quad \multimap \quad T_{f(p)}N$$

by

$$\mathrm{d}f_p(\gamma) \quad = \quad f \circ \gamma$$

Smooth functions

Recall that M^* is the set of smooth functions $f : M \to \mathbb{R}$.

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Given two such functions $f, g \in M^*$, we can

- add them: $(f + g)(x) = f(x) + g(x)$
- multiply them by $\alpha \in \mathbb{R}$: $(\alpha f)(x) = \alpha f(x)$
- multiply them: $(f \cdot g)(x) = f(x) \times g(x)$

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The two first equip the space with a structure of vector space, which satisfies

$$(fg)h = f(gh) \qquad fg = gf$$
$$f(g + h) = fg + fh \qquad (f + g)h = fg + gh$$

which is called a (commutative) **algebra**.

Algebras

Definition

An **algebra** is a vector space A together with a *multiplication*

$$- \cdot - \quad : \quad A \otimes A \quad \multimap \quad A$$

such that multiplication is associative

$$\forall a, b, c \in A, \qquad (a \cdot b) \cdot c \quad = \quad a \cdot (b \cdot c)$$

Properties of derivation

Given $f : M \to N$, we have defined $\mathrm{d}f_p : T_p M \multimap T_{f(p)} N$.
Given $v \in T_p M$, we can also define $\partial_v : M^* \to \mathbb{R}$ by

$$\partial_v(f) \quad = \quad \mathrm{d}f_p(v) \quad = \quad (f \circ v)'(0)$$

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Proposition

Suppose given $f, g : M^*$.

- *Differentiation is linear:*

$$\partial_v(f+g) = \partial_v f + \partial_v g \qquad\qquad \partial_v(\alpha f) = \alpha \partial_v f$$

- *It satisfies the **Leibnitz law**: given $v \in T_p M$,*

$$\partial_v(f \cdot g) \quad = \quad \partial_v f \cdot g(p) + f(p) \cdot \partial_v g$$

Tangent space – via derivations

Actually, this can be taken as a definition,
by identifying v with ∂_v!

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Definition

The **tangent space** $T_p M$ is the vector space whose elements are

$$v \quad : \quad M^* \quad \multimap \quad \mathbb{R}$$

such that

$$\begin{aligned} v(f+g) &= v(f)+v(g) \\ v(\alpha g) &= \alpha v(f) \\ v(f \cdot g) &= v(f) \cdot g(p) + f(p) \cdot v(g) \end{aligned}$$

i.e. the space of **derivations** at p of the algebra M^*.

Tangent space – via derivations

With this definition it is easy to show that T_pM is a vector space:

$$(v+w)(f) = v(f) + w(f) \qquad (\alpha v)(f) = \alpha v(f)$$

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$$(v+w)(f) = v(f) + w(f) \qquad (\alpha v)(f) = \alpha v(f)$$

Given a chart $\varphi : U \to \mathbb{R}^n$ with $p \in U$ and a basis (x^i) of $\mathbb{R}^n$, the vectors ∂_i defined by

$$\partial_i(f) \quad = \quad \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}$$

form a basis for this vector space.

Derivation as a functor

We have a functor

$$\mathbf{pMan} \quad \to \quad \mathbf{Vect}$$

which sends

$$(M, x) \qquad \text{to} \qquad T_x M$$

and

$$f : (M, x) \to (N, y) \qquad \text{to} \qquad \mathrm{d}f_x : T_x M \multimap T_y N$$

The chain rule is precisely the axiom of functoriality
wrt composition.

Vector fields

Intuition: a vector field is given by a vector $v_p \in T_pM$ for each point $p \in M$, which varies continuously in p.

We'll use tangent bundles to define them.

Tangent bundle

Definition

The **tangent bundle** is

$$TM \quad = \quad \coprod_{p \in M} T_p M$$

Proposition

If M is an n-manifold, TM is canonically a $2n$-manifold.

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If M is an n-manifold, TM is canonically a $2n$-manifold.

We write $\pi : TM \to M$ for the canonical projection

$$\pi \quad = \quad v \in T_p M \quad \mapsto \quad p$$

Vector fields

Definition

A **vector field** v is a section of the tangent bundle TM, i.e. a map

$$v \quad : \quad M \quad \to \quad TM$$

such that

$$\pi \circ v \quad = \quad \mathrm{id}_M$$

Vectors fields are denoted $\Gamma(TM)$.

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Vectors fields are denoted $\Gamma(TM)$.

This means that $v(p) = (q, v_q \in T_q M)$ such that $q = p$.

Notice that the map v is required to be smooth!

Vector fields – via derivations

Definition

A **vector field** is a function $v : M^* \to M^*$ such that

$$v(f+g) = v(f) + v(g)$$
$$v(\alpha f) = \alpha v(f)$$
$$v(f \cdot g) = v(f) \cdot g + f \cdot v(g)$$

i.e. a **derivation** of the algebra M^*.

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\end{aligned}
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i.e. a **derivation** of the algebra M^*.

Proposition

Vector fields over M form an M^*-module with

$$
\begin{aligned}
(v+w)(f) &= v(f) + w(f) \\
(g \cdot v)(f) &= g \cdot v(f)
\end{aligned}
$$

Pullback and push forward

A morphism $\phi : M \to N$ induces

- a **pullback** function

$$\phi^* \quad : \quad N^* \to M^*$$

defined by

$$\phi^*(f) \quad = \quad f \circ \phi$$

Pullback and push forward

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- a **pullback** function

  $$\phi^* \quad : \quad N^* \to M^*$$

  defined by

  $$\phi^*(f) \quad = \quad f \circ \phi$$

- a **pushforward** function

  $$\phi_* \quad : \quad T_p M \to T_{\phi(p)} N$$

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Remark

Notice that functions are *contravariant* and vectors are *covariant*.

Coordinates

The vector space $\mathbb{R}^n$ is equipped with canonical **coordinate functions**, which are the projections

$$x^i \quad : \quad \mathbb{R}^n \quad \to \quad \mathbb{R}$$

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These induce coordinate functions $\varphi^* x^i : U \to \mathbb{R}^n$, that we (abusively) still denote x^i, called **local coordinates**.

TODO: Change of basis.....

Writing conventions

In the following, we use **Einstein summation convention**: we implicitly sum over repeated indices in a formula, e.g.

$$v = v^i \partial_i$$

(with $v^i = v(x^i)$) means

$$v = \sum_i v^i \partial_i$$

Writing conventions

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(with $v^i = v(x^i)$) means

$$v \quad = \quad \sum_i v^i \partial_i$$

Concerning the indices, we write

- x^i for a contravariant quantities (coordinates, n-forms, etc.)
- ∂_i for a covariant quantities (vectors, etc.)

Notice that

$$v = v^i \partial_i \qquad \text{and} \qquad \omega = \omega_i \, \mathrm{d}x^i$$

Differential
1-forms

Differential

Recall that given $p \in M$ and $f : M \to N$, we have defined

$$\mathrm{d}f_p \quad : \quad T_pM \quad \multimap \quad T_{f(p)}N$$

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and these can be "collected together" into

$$\mathrm{d}f \quad : \quad \Gamma(TM) \quad \to \quad M^*$$

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$$\mathrm{d}f(v)(p) \quad = \quad \mathrm{d}f_p(v_p)$$

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by

$$\mathrm{d}f(v)(p) \quad = \quad \mathrm{d}f_p(v_p)$$

This function can easily be shown to be linear over the module M^*:

$$\mathrm{d}f(v + w) = \mathrm{d}f(v) + \mathrm{d}f(w) \qquad\qquad \mathrm{d}f(\alpha v) = \alpha\,\mathrm{d}f(v)$$

1-forms

Definition

A **differential** 1-**form**

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is a map which is linear over M^*.

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We write $\Omega^1(M)$ for the M^*-*module* of differential forms.

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$$\omega \quad : \quad \Gamma(TM) \multimap M^*$$

is a map which is linear over M^*.

Notation

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Example

Given $f \in M^*$, its **differential** (or **exterior derivative**)

$$\mathrm{d}f \quad = \quad v \mapsto p \mapsto \mathrm{d}f_p(v_p)$$

is a 1-form.

Proposition

*The **exterior derivative*** $\mathrm{d} : M^* \to \Omega^1(M)$ *is*

- *linear:*

$$\mathrm{d}(f+g) = \mathrm{d}f + \mathrm{d}g$$
$$\mathrm{d}(\alpha f) = \alpha\,\mathrm{d}f$$

- *a derivation:*

$$\mathrm{d}(f \cdot g) = \mathrm{d}f \cdot g + f \cdot \mathrm{d}g$$

Proposition

The $\mathrm{d}x^i$ *form a basis of the* M^**-module of* 1*-forms over* $\mathbb{R}^n$*:*

$$df = \sum_i \frac{\partial f}{\partial x^i} \cdot \mathrm{d}x^i$$

(or locally in a manifold).

Cotengent vectors

Definition

A **cotengent vector** at $p \in M$ is an element of $T_p M \multimap \mathbb{R}$.
We thus write $T_p^* M$ for the cotengent vectors at p.

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Proposition

One can form the ***cotengent vector bundle***

$$T^* M \quad = \quad \coprod_{p \in M} T_p^* M$$

and 1*-forms are its sections*

$$\Omega^1(M) \quad \cong \quad \Gamma(T^* M)$$

TODO: cotengent vectors and 1-forms are contravariant
Derivative is natural: given $f \in M^*$ and $\phi : M \to N$,

$$\mathrm{d}(\phi^* f) \quad = \quad \phi^*(\mathrm{d}f)$$

(Co)tangent space as infinitesimals

Given $p \in M$, consider the ideals

$$I_p = \{f \in M^* \mid f(p) = 0\}$$

and

$$I_p^2 = \{\sum_i f_i g_i \mid f_i, g_i \in I\}$$

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Proof.

A derivation D satisfies $D(f) = 0$ for $f \in I_p^2$, i.e. $D : I_p/I_p^2 \to \mathbb{R}$.
Conversely, given $r \in I_p/I_p^2$, $D(f) = r((f - f(x)) + I_p^2)$ is a derivation. □

Towards Algebraic Geometry

Towards algebraic geometry

Given a manifold M, an open set $U \subseteq M$ is also canonically a manifold. We can thus consider the (ring of) smooth functions $U^* = \mathbf{Man}(U, \mathbb{R})$. The collection of all those form a (pre)sheaf:

Definition

A **presheaf** (X, O) is a functor $O : \mathcal{O}(X)^{\mathsf{op}} \to \mathcal{C}$ from the category of open sets in X and reversed inclusions to a category $\mathcal{C}$.

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- Here, we have $X = M$, $O(U) = U^*$ and $\mathcal{C} = \mathbf{Rings}$.
- Given $U \subseteq V$, we have a **restriction** function

  $$O(V) \quad \to \quad O(U)$$

  and we write the image of $f \in V$

  $$f|_U^V \qquad \text{or} \qquad f|_U$$

Definition

A **sheaf** is a presheaf such that, for every open covering (U_i) of any open $U \subseteq X$:

1. *Locality.* If $f, g \in O(U)$ satisfy
$$f|_{U_i} = g|_{U_i}$$
for each U_i then
$$f = g$$
2. *Gluing.* If there exists $f_i \in O(U_i)$ are such that
$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$$
then there exists $f \in O(U)$ such that
$$f_i = f|_{U_i}$$

Sheaves

In the case $\mathcal{C}$ has products, this is equivalent to

Definition

A **sheaf** is a presheaf such that for any covering U_i of U the diagram

$$O(U) \longrightarrow \prod_i O(U_i) \rightrightarrows \prod_{i,j} O(U_i \cap U_j)$$

is an equalizer, where the arrows are products $-|^{U}_{U_i}$, $-|^{U_i}_{U_i \cap U_j}$ and $-|^{U_j}_{U_i \cap U_j}$ respectively.

What can we recover from rings?

Proposition

The points of M are in bijection with the maximal ideals of the algebra M^*.

Proof.

To a point p, one can associate the ideal

$$I_p \quad = \quad \{f \in M^* \mid f(p) = 0\}$$

which is maximal and conversely, every maximal ideal is of this form! □

Germs

Definition

Given a point p and functions $f, g : U \to \mathbb{R}$ with $p \in U$, we define an equivalence relation by

$$f \sim g \qquad \text{when} \qquad f|_V = g|_V$$

for some $V \subseteq U$ with $p \in V$. The equivalence class of a function is its **germ** and the collection of all germs at p is the **stalk** at p.

The tangent space

Definition

The **cotangent space** at p is I_p/I_p^2 where I_p is the maximal ideal of the stalk $O_{M,p}$.

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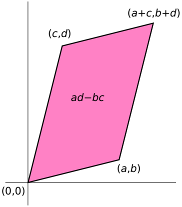
Definition

The **tangent space** is the sheaf of morphisms from O_M into the ring of **dual numbers** $\mathbb{R}[X]/X^2$.

Differential Forms

The area of a parallelogram

What is the area of a parallelogram spanned by vectors u and v?



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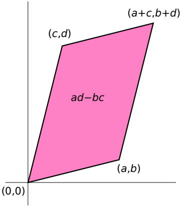

We should have:

- $A(u, v)$ bilinear:

$$A(u_1 + u_2, v) = A(u_1, v) + A(u_2, v) \qquad A(\alpha u, v) = \alpha A(u, v)$$

- $A(u, u) = 0$

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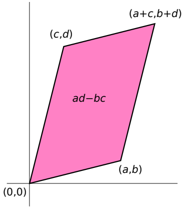

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- $A(u, u) = 0$
- and therefore $A(u, v) = -A(v, u)$

$$A(u+v, u+v) = A(u, u)+A(u, v)+A(v, u)+A(v, v) = A(u, v)+A(v, u)$$

The area of a parallelogram

Up to a multiplicative constant, there is only one alternating linear form:

$$A \quad : \quad V \otimes V \quad \multimap \quad V$$

this is the determinant of 2×2 matrices!

$$A(u, v) \quad = \quad \det([u, v])$$

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So, the area of a parallelogram generated by u and v is

$$\begin{aligned} \det(u, v) &= \det(u_1 x^1 + u_2 x^2, v_1 x^1 + v_2 x^2) \\ &= u_1 v_1 \det(x^1, x^1) + u_1 v_2 \det(x^1, x^2) + u_2 v_1 \det(x^2, x^1) + u_2 v_2 \text{ d} \\ &= (u_1 v_2 - u_2 v_2) \det(x^1, x^2) \end{aligned}$$

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&= (u_1 v_2 - u_2 v_2) \det(x^1, x^2)
\end{aligned}$$

(and this generalizes in higher dimensions)

A basis for areas

A differential 1-form can be seen as a way to measure (infinitesimal) distances:

$$\mathrm{d}x \quad : \quad \Gamma(TM) \quad \multimap \quad M \quad \to \quad \mathbb{R}$$

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In order to measure areas with 2-forms, we should therefore take the pairs $(\mathrm{d}x^i, \mathrm{d}x^j)$ as basis for 2-forms but quotiented by relations imposing that

$$(\mathrm{d}x^i, \mathrm{d}x^j) \quad = \quad -(\mathrm{d}x^j, \mathrm{d}x^i)$$

Change of variables in integration

In dimension 1, the fundamental theorem of calculus gives:

$$\int_{\phi(a)}^{\phi(b)} f(x)\,\mathrm{d}x \quad = \quad \int_a^b f(\phi(t))\phi'(t)\,\mathrm{d}t$$

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More generally, given $U \subseteq \mathbb{R}^n$ open and $\varphi : U \to \mathbb{R}^n$ injective and differentiable with continuous partial derivatives:

$$\int_{\varphi(U)} f\,\mathrm{d}x^1 \ldots \mathrm{d}x^n \quad = \quad \int_U (f \circ \varphi)|\det(D\varphi)|\,\mathrm{d}x^1 \ldots \mathrm{d}x^n$$

where $D\varphi$ is the **Jacobian** of φ: $(D\varphi)_{ij} = \partial_i \varphi_j$.

Division in the ring of dual numbers

The ring of dual numbers is $\mathbb{R}[\varepsilon]/\varepsilon^2$.

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With $c \neq 0$, we have

$$\begin{aligned}
\frac{a+b\varepsilon}{c+d\varepsilon} &= \frac{(a+b\varepsilon)(c-d\varepsilon)}{(c+d\varepsilon)(c-d\varepsilon)} \\
&= \frac{ac+(bc-ad)\varepsilon-db\varepsilon^2}{c^2-d^2\varepsilon^2} \\
&= \frac{a}{c}+\frac{bc-ad}{c^2}\varepsilon
\end{aligned}$$

Exterior algebra

Definition

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$$TV = \bigoplus_{k\in\mathbb{N}} V^{\otimes k}$$

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Definition

The **exterior algebra** ΛV of V is

$$\Lambda V \quad = \quad TV/I$$

where I is the two-sided ideal generated by $x \otimes x$ with $x \in V$.
Its tensor product is written $\wedge$.

Antisymmetry

Proposition

We have $x \wedge x = 0$ *and* $x \wedge y = -y \wedge x$*.*

Proof.

$0 = (x+y)\wedge(x+y) = x\wedge x + x\wedge y + y\wedge x + y\wedge y = x\wedge y + y\wedge x.$ □

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We could have defined $\Lambda V = TV/I$ where I is the two-sided ideal generated by $x \otimes y + y \otimes x$.

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Proposition

Given a basis (e_i) *of* V, *a basis of* ΛV *is* $e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_k}$ *with* $i_1 < i_2 < \ldots < i_k$. *If* $\dim V = n$ *then*

$$\dim \Lambda V = 2^n \qquad \textit{and} \qquad \dim \Lambda^k V = \frac{n!}{k!(n-k)!}$$

Grading

The exterior algebra is naturally graded as a quotient of the tensor algebra by a homogeneous ideal:

$$\Lambda V \quad = \quad \bigoplus_{k \in \mathbb{N}} \Lambda^k V$$

The elements of $\Lambda^k V$ are of the form

$$v_1 \wedge v_2 \wedge \ldots \wedge v_k$$

with $v_i \in V$.

Example

Given $\mathbb{R}^2$ with the canonical orthonormal basis x, y and two vectors v and w, we have

$$\begin{aligned} v \wedge w &= (v_x x + v_y y) \wedge (w_x x + w_y y) = (v_x w_y - v_y w_x)\; x \wedge y \\ &= \det(u, v)\; x \wedge y \end{aligned}$$

where the determinant computes the (signed) area of the parallelogram spanned by v and w.

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where the determinant computes the (signed) area of the parallelogram spanned by v and w.

Example

Similarly in $\mathbb{R}^3$ we have

$$v \wedge w = (v_x w_y - v_y w_x)\ x \wedge y + (v_x w_z - v_z w_x)\ x \wedge z + (v_y w_z - v_z w_y)\ y \wedge z$$

and

$$u \wedge v \wedge w = \det(u, v, w)\ x \wedge y \wedge z$$

The special dimension 3

We have seen that

$$\dim \Lambda^k V = \frac{n!}{k!(n-k)!}$$

When $\dim V = 3$, we have $\dim \Lambda^2 V = \dim V = 3$, so that

$$\Lambda^2 V \quad \cong \quad V$$

but there is no canonical isomorphism, which explains why the "right-hand rule" can be replaced by the "left-hand rule", i.e. there is no particular reason to choose between the two isomorphisms

$$\begin{aligned} x \wedge y &\mapsto z \\ x \wedge z &\mapsto y \\ y \wedge x &\mapsto x \end{aligned} \qquad\qquad \begin{aligned} y \wedge x &\mapsto z \\ z \wedge x &\mapsto y \\ z \wedge y &\mapsto x \end{aligned}$$

Moreover, this does not generalize in other dimensions. . .

p-forms

Definition

p-forms are defined as the exterior algebra of the M^*-module $\Omega^1(M)$:

$$\Omega(M) = \Lambda\Omega^1(M) \qquad \Omega^k(M) = \Lambda^k\Omega^1(M)$$

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A p-form ω can be assimilated to a function

$$\Lambda^k(T_pM) \quad \multimap \quad \mathbb{R}$$

i.e. an alternating multilinear map

$$\omega \quad : \quad T_pM \times \ldots \times T_pM \quad \to \quad \mathbb{R}$$

i.e.

$$\omega(\ldots, x, \ldots, x, \ldots) = 0$$

or

$$\omega(\ldots, x, \ldots, y, \ldots) = -\omega(\ldots, y, \ldots, x, \ldots)$$

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We also have a definition as sections of the exterior power of the cotengent bundle

$$\Omega^k(M) \quad = \quad \Gamma(\Lambda^k T_p^* M)$$

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Remark

$\Omega^0(M) = M^*$.

Remark

A 2-form is the same as an antisymmetric matrix.

Integration

Suppose given $\phi : M \to N$. We can define a **pullback** operation:

- on 0-forms:

$$\phi^* : \Omega^0 N \to \Omega^0 M \qquad \text{by} \qquad \phi^*(f) = f \circ \phi$$

with $f \in \Omega^0 N$

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- on cotangent vectors:

$$\phi^* : T_p^* N \to T_{\phi(p)}^* M \qquad \text{by} \qquad \phi^*(\omega)(v) = \omega(\phi_* v)$$

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  with $\omega \in T_p^* N$ and $v \in T_{\phi(p)} M$

- on 1-forms:

$$\phi^* : \Omega^1 N \to \Omega^1 M \qquad \text{by} \qquad (\phi^* \omega)_p = \phi^*(\omega_{\phi(p)})$$

  with $\omega \in \Omega^1 N$

Pullback

Proposition

Given $\phi : M \to N$ *there exists a unique* ***pullback*** *map*

$$\phi^* \quad : \quad \Omega N \to \Omega M$$

such that ϕ^* *agrees with the previous definition on* $\Omega^0 M$*, on* $\Omega^1 M$ *and such that*

$$\begin{aligned} \phi^*(\alpha\omega) &= \alpha\phi^*\omega \\ \phi^*(\omega + \mu) &= \phi^*\omega + \phi^*\mu \\ \phi^*(\omega \wedge \mu) &= \phi^*\omega \wedge \phi^*\mu \end{aligned}$$

Integration on $\mathbb{R}^n$

Given $\omega \in \Omega^n U$ with $U \subseteq \mathbb{R}^n$ open, we can define

$$\int_U \omega \quad = \quad \int_U \omega \, \mathrm{d}x^1 \ldots \mathrm{d}x^n$$

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$$\int_U \omega \quad = \quad \int_U \omega \, \mathrm{d}x^1 \ldots \mathrm{d}x^n$$

In order to check whether this is independent of the choice of basis, recall that

$$\int_{\phi(U)} f \, \mathrm{d}x^1 \ldots \mathrm{d}x^n \quad = \quad \int_U (f \circ \phi) |\det(D\phi)| \, \mathrm{d}x^1 \ldots \mathrm{d}x^n$$

Integration on $\mathbb{R}^n$

Given $\omega \in \Omega^n U$ with $U \subseteq \mathbb{R}^n$ open, we can define

$$\int_U \omega \quad = \quad \int_U \omega \, \mathrm{d}x^1 \ldots \mathrm{d}x^n$$

In order to check whether this is independent of the choice of basis, recall that

$$\int_{\phi(U)} f \, \mathrm{d}x^1 \ldots \mathrm{d}x^n \quad = \quad \int_U (f \circ \phi) |\det(D\phi)| \, \mathrm{d}x^1 \ldots \mathrm{d}x^n$$

Proposition

Given a diffeomorphism $\phi : U \to V$ between two open subsets of $\mathbb{R}^n$ such that $\det(D\phi)$ is of constant sign δ, then for every n-form $\omega \in \Omega^n V$,

$$\int_U \phi^* \omega = \delta \int_V \omega$$

Orientable manifolds

Given two basis x_i and y_i of T_pM with $y_j = T^i_j x_i$ then

$$y_1 \wedge \ldots \wedge y_n \quad = \quad (\det T)\, x_1 \wedge \ldots \wedge x_n$$

Two such **volume elements** have the same **orientation** when $\det(T) > 0$.

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A **volume form** on an n-manifold M is an n-form which is nowhere zero. A manifold is **orientable** when it admits a volume form.

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Definition

A **volume form** on an n-manifold M is an n-form which is nowhere zero. A manifold is **orientable** when it admits a volume form.

Remark

For instance the Möbius strip is not orientable.

Integration on a manifold

We write $\Omega^n_c M$ for the n-forms with compact support.

We can define integration by "splitting over charts":

Proposition

Given a smooth oriented n-manifold M there exits a unique linear

$$\int_M \quad : \quad \Omega^n_c(M) \quad \multimap \quad \mathbb{R}$$

such that if $\operatorname{supp} \omega \subseteq U$ *with* (U, φ) *positively oriented chart then*

$$\int_M \omega \quad = \quad \int_{\varphi(U)} (\varphi^{-1})^* \omega$$

Integration on a manifold

In the case where ω does not have compact support, we have to suppose that M is *paracompact and Hausdorff*. In this case, it admits partitions of unity:

Definition

A **partition of unity** is a collection of functions $f_i \in M^*$ such that

1. f_i is zero outside U_i
2. for every point $p \in M$, $\sum_i f_i(p) = 1$
3. for every point $p \in M$ there is an open neighborhood on which finitely many f_i are nonzero.

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We thus have

$$\omega = \sum_i f_i \omega$$

Integration on a manifold

Since $\omega = \sum_i f_i \omega$, we define

$$\int_M \omega \quad = \quad \sum_i \int_{U_i} f_i \omega$$

where by definition

$$\int_{U_i} f_i \omega \quad = \quad \int_{\varphi(U_i^*)} (\varphi_i^{-1})^* f_i \omega$$

with (U_i, φ_i) a positively oriented chart.

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with (U_i, φ_i) a positively oriented chart.

Proposition

This does not depend on the choice of the partition of unity.

Derivation

Definition

The **exterior derivative**

$$\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$$

is defined by

1. $\mathrm{d} : \Omega^0(M) \to \Omega^1(M)$ is the usual differential
2. d is linear
   (the $\Omega^k(M)$ are real vector spaces)
3. $\mathrm{d}(\omega \wedge \mu) = \mathrm{d}\omega \wedge \mu + (-1)^k \omega \wedge \mathrm{d}\mu$
   for $\omega \in \Omega^k(M)$ and $\mu \in \Omega(M)$
4. $\mathrm{d}(\mathrm{d}\omega) = 0$ for $\omega \in \Omega(M)$

On $\mathbb{R}^3$

In $\mathbb{R}^3$, given

$$\omega \quad = \quad \omega_x \, \mathrm{d}x + \omega_y \, \mathrm{d}y + \omega_z \, \mathrm{d}z$$

for $\omega\_ \in \Omega^0 M = M^*$, we have

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}(\omega_x \, \mathrm{d}x + \omega_y \, \mathrm{d}y + \omega_z \, \mathrm{d}z) \\
&= \mathrm{d}\omega_x \wedge \mathrm{d}x + \omega_x \wedge \mathrm{d}\,\mathrm{d}x + \mathrm{d}\omega_y \wedge \mathrm{d}y + \omega_y \wedge \mathrm{d}\,\mathrm{d}y + \mathrm{d}\omega_z \wedge \mathrm{d}z + \omega_z \wedge \mathrm{d} \\
&= \mathrm{d}\omega_x \wedge \mathrm{d}x + \mathrm{d}\omega_y \wedge \mathrm{d}y + \mathrm{d}\omega_z \wedge \mathrm{d}z \\
&= (\partial_x \omega_x \, \mathrm{d}x + \partial_y \omega_x \, \mathrm{d}y + \partial_z \omega_x \, \mathrm{d}z) \wedge \mathrm{d}x + \ldots \\
&= \partial_x \omega_x \, \mathrm{d}x \wedge \mathrm{d}x + \partial_y \omega_x \, \mathrm{d}y \wedge \mathrm{d}x + \partial_z \omega_x \, \mathrm{d}z \wedge \mathrm{d}x + \ldots \\
&= -\partial_y \omega_x \, \mathrm{d}x \wedge \mathrm{d}y + \partial_z \omega_x \, \mathrm{d}z \wedge \mathrm{d}x + \ldots \\
&= (\partial_x \omega_y - \partial_y \omega_x) \, \mathrm{d}x \wedge \mathrm{d}y + (\partial_y \omega_z - \partial_z \omega_y) \, \mathrm{d}y \wedge \mathrm{d}z + (\partial_z \omega_x - \partial_x \omega_z) \\
&\approx \nabla \times \omega
\end{aligned}$$

On $\mathbb{R}^3$

In $\mathbb{R}^3$, given

$$\omega = \omega_{xy}\,\mathrm{d}x \wedge \mathrm{d}y + \omega_{yz}\,\mathrm{d}y \wedge \mathrm{d}z + \omega_{zx}\,\mathrm{d}z \wedge \mathrm{d}x$$

we have similarly

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}\omega_{xy} \wedge \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}\omega_{yz} \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}\omega_{zx} \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&= \partial_z\omega_{xy}\,\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y + \partial_x\omega_{yz}\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z + \partial_y\omega_{zx}\,\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&= (\partial_x\omega_{yz} + \partial_y\omega_{zx} + \partial_z\omega_{xy})\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&\approx \nabla \cdot \omega
\end{aligned}$$

On $\mathbb{R}^3$

An easy computation shows that

- $\mathrm{d} : \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ is the **gradient**

$$\nabla \quad = \quad f \quad \mapsto \quad \partial_i f \, \mathrm{d}x^i$$

- $\mathrm{d} : \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$ is the **curl**

$$\nabla \times - \quad = \quad \omega \quad \mapsto \quad \partial_i \omega_j \, \mathrm{d}x^i \wedge \mathrm{d}x^j$$

- $\mathrm{d} : \Omega^2(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$ is the **divergence**

$$\nabla \cdot - \quad = \quad \omega \quad \mapsto \quad (\partial_1 \omega_{23} + \partial_2 \omega_{13} + \partial_3 \omega_{12})$$

  with $\omega = \omega_{12} \;\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \omega_{13} \;\, \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \omega_{23} \;\, \mathrm{d}x^2 \wedge \mathrm{d}x^3$.

In local coordinates

Given a multiset $I = (i_1, \ldots, i_k)$ in $\{1, \ldots, k\}$, with $i_1 \leq \ldots \leq i_k$, the exterior derivative of the k-form

$$\omega \quad = \quad f_I \,\mathrm{d}x^I \quad = \quad f_{i_1,\ldots,i_k} \,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}$$

is

$$\mathrm{d}\omega \quad = \quad \sum_{i=1}^{n} \partial_i f_I \,\mathrm{d}x^i \wedge x^I$$

and this extends to general k-forms

$$\omega \quad = \quad \sum_I f_I \,\mathrm{d}x^I$$

Riemannian manifolds

Riemannian manifolds

Definition

A **Riemannian metric** is a bilinear map

$$g \quad : \quad V \otimes V \multimap \mathbb{R}$$

which is symmetric and *positive-definite*:
$g(v, v) \geq 0$ with equality only if $v = 0$

Riemannian manifolds

Definition

A **semi-Riemannian metric** is a bilinear map

$$g \quad : \quad V \otimes V \multimap \mathbb{R}$$

which is symmetric and *nondegenerate*:
if $g(v, w) = 0$ for every $w \in V$ then $v = 0$.

Riemannian manifolds

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Definition

A **semi-Riemannian manifold** M is such that each $T_p M$ is equipped with such a metric g_p which "varies smoothly with p", i.e. for every vector fields $v, w \in \Gamma(TM)$, the function $p \mapsto g_p(v_p, w_p)$ is a smooth function $M \to \mathbb{R}$.

(i.e. we have a smooth section of the positive definite quadratic forms on the tangent bundle).

Definition

The **length** of a curve $\gamma : [0,1] \to M$ is

$$L(\gamma) \quad = \quad \int_0^1 \|\gamma'(t)\| \, \mathrm{d}t \quad = \quad \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))} \, \mathrm{d}t$$

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Every (connected) Riemannian manifold is thus a metric space with

$$d(x, y) \quad = \quad \inf\{\gamma : [0, 1] \to M \mid \gamma(0) = x, \gamma(1) = y\}$$

Volume form

Locally, the components of the metric are

$$g_{ij} \quad = \quad g(\partial_i, \partial_j)$$

Proposition

Given a Riemanian manifold one can define a volume form by

$$\mathrm{vol} \quad = \quad \sqrt{|\det(g_{ij})|}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$$

This allows us to define, for any $f \in M^*$:

$$\int_M f \quad = \quad \int_M f\, \mathrm{vol}$$

Remark

Since g_p is nondegenerate, $g_p(v_p, -) : T_pM \to T_p^*M$ is a bijection.

This allows one to transfer stuff such as the inner product to 1-forms.

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This allows one to transfer stuff such as the inner product to 1-forms.

Orientation allows us to generalize the right-hand rule as follows:

Definition
The **Hodge star operator** on an oriented n-manifold M

$$\star \quad : \quad \Omega^k(M) \to \Omega^{n-k}(M)$$

is the unique M^*-linear map such that for $\omega, \mu \in \Omega^k(M)$

$$\omega \wedge \star\mu \quad = \quad \langle \omega, \mu \rangle \, \mathrm{vol}$$

Maxwell equations

Maxwell equations

The Maxwell equations

$$
\begin{aligned}
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\
\nabla \cdot \vec{E} &= \rho \\
\nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial r} &= \vec{j}
\end{aligned}
$$

Maxwell equations

The Maxwell equations

$$\begin{aligned} \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 \\ \nabla \cdot \vec{E} &= \rho \\ \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial r} &= \vec{j} \end{aligned}$$

become, with $E \in \Omega^1(M)$ and $B \in \Omega^2(M)$ and $M = \mathbb{R} \times S$,

$$\begin{aligned} \mathrm{d}_S B &= 0 \\ \partial_t B + \mathrm{d}_S E &= 0 \\ \star_S \mathrm{d}_S \star_S E &= \rho \\ -\partial_t E + \star_S d_S \star_S B &= j \end{aligned}$$

Maxwell equations

- $\vec{E}$ is the **electric field**
- $\vec{B}$ is the **magnetic field**
- ρ is the **charge density**
- $\vec{j}$ is the **electric current density**
- $\nabla = (\partial_1, \partial_2, \partial_3)$
- the **divergence** measures flux

$$\nabla \cdot \vec{F} \quad = \quad \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 \quad = \quad \lim_{V \to \{*\}} \iint_{S(V)} \frac{\vec{F} \cdot \vec{n}}{|V|} \, \mathrm{d}S$$

- the **curl** measures rotation

$$\begin{aligned} \nabla \times \vec{F} \quad &= \quad (\partial_2 F_3 - \partial_3 F_2, \partial_1 F_3 - \partial_3 F_1, \partial_1 F_2 - \partial_2 F_1) \\ &= \quad \lim_{A \to \{*\}} \oint_A \left(\frac{\vec{F} \cdot \mathrm{d}\vec{r}_i}{|A|} \right) \end{aligned}$$

Lorenzian metrics

For spacetime $M = \mathbb{R} \times S$, we want a **Lorentzian metric** of signature $(n-1,1)$, i.e. something like

$$(g_{ij}) \quad = \quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A vector v is

- **spacelike** if $v \cdot v > 0$
- **timelike** if $v \cdot v < 0$
- **lightlike** if $v \cdot v = 0$

Maxwell equations

By writing the **electromagnetic field**

$$F \quad = \quad B + E \wedge \mathrm{d}t$$

that is

$$F \quad = \quad \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Maxwell equations

By writing the **electromagnetic field**

$$F = B + E \wedge \mathrm{d}t$$

and

$$J = j - \rho\, \mathrm{d}t$$

we arrive at

$$\begin{aligned} \mathrm{d}F &= 0 \\ \star \mathrm{d} \star F &= J \end{aligned}$$

Integration

Closed an exact forms

- A differential form ω such that $\mathrm{d}\omega = 0$ is **closed**
- A differential form $\omega \in \Omega^{k+1}(M)$ for which there exists $\mu \in \Omega^k(M)$ such that $\mathrm{d}\mu = \omega$ is **exact**

So, $\mathrm{d}^2 = 0$ can be phrased: *exact forms are closed*.

Integrating 1-forms

When is a 1-form exact?

Integrating 1-forms

When is a 1-form exact?

Given $\omega \in \Omega^1(M)$ and a (piecewise) smooth path $\gamma : [0, T] \to S$, we can integrate ω along γ by

$$\int_\gamma \omega \quad = \quad \int_0^T \gamma^*(\omega)(t)\,\mathrm{d}t \quad = \quad \int_0^T \omega_{\gamma(t)}(\gamma'(t))\,\mathrm{d}t$$

Integrating 1-forms

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Given $p \in M$, we can (try to) define $f \in M^* = \Omega^0(M)$

$$\mu(q) \quad = \quad \int_\gamma \omega$$

for some path $\gamma : p \rightsquigarrow q$, so that

$$\mathrm{d}\mu \quad = \quad \omega$$

In order for the definition $\mu(q) = \int_\gamma \omega$ to work we have to suppose that M is *simply connected*!

Proposition

Given a homotopy γ_s between paths γ_0 and γ_1,

$$I_s = \int_0^T \omega_{\gamma_s(t)}(\gamma_s'(t))\,\mathrm{d}t$$

does not depend on s when $\mathrm{d}\omega = 0$.

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does not depend on s when $\mathrm{d}\omega = 0$.

Proof.

Up to splitting γ, we can suppose that we are working in a chart. In local coordinates we have

$$\omega_{\gamma_s(t)}(\gamma_s'(t)) = \omega_i(\gamma_s(t))\partial_t\gamma_s^i(t)$$

Proposition

Given a homotopy γ_s *between paths* γ_0 *and* γ_1*,*

$$I_s = \int_0^T \omega_{\gamma_s(t)}(\gamma_s'(t))\,\mathrm{d}t$$

does not depend on s *when* $\mathrm{d}\omega = 0$*.*

Proof.

$$\begin{aligned}
\partial_s I_s &= \int \partial_s[\omega_i(\gamma_s(t))\partial_t\gamma_s^i(t)]\,\mathrm{d}t \\
&= \int [\partial_s\omega_i(\gamma_s(t))\partial_t\gamma_s^i(t) + \omega_i(\gamma_s(t))\partial_s\partial_t\gamma_s^i(t)]\,\mathrm{d}t \\
&= \int [\partial_s\omega_i(\gamma_s(t))\partial_t\gamma_s^i(t) - \partial_t\omega_i(\gamma_s(t))\partial_s\gamma_s^i(t)]\,\mathrm{d}t \\
&= \int \partial_j\omega_i(\gamma_s(t))[\partial_s\gamma_s^j\partial_t\gamma_s^i - \partial_t\gamma_s^j\partial_s\gamma_s^i]\,\mathrm{d}t \\
&= \int (\mathrm{d}\omega)_{ij}\partial_s\gamma^i\partial_t\gamma^j\,\mathrm{d}t
\end{aligned}$$

□

The Poincaré Lemma

We have just shown that the integral only depends on the endpoints of $\gamma : p \rightsquigarrow q$, which always exists.

Theorem

When M is simply connected, every closed 1-form ω (i.e. $\mathrm{d}\omega = 0$) is exact: $\omega = \mathrm{d}\mu$ with

$$\mu(q) \quad = \quad \int_\gamma \omega$$

for some path $\gamma : p \rightsquigarrow q$ from some fixed point p.

The Poincaré Lemma

For instance, in the Maxwell equations we have

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \quad = \quad 0$$

When the second term vanishes (under magneto-static conditions), we have

$$\mathrm{d}\vec{E} \quad = \quad \nabla \times \vec{E} \quad = \quad 0$$

and therefore there exists a scalar function V such that

$$\vec{E} \quad = \quad -\nabla V$$

called the electric potential.

A counter-example

When M is not simply connected, this fails to be true.

Consider $M = \mathbb{R}^2 \setminus \{(0,0)\}$ and with γ a loop around the unit circle (once) in counterclockwise orientation. Take

$$\omega \quad = \quad \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2 + y^2}$$

This 1-form is closed ($\mathrm{d}\omega = 0$) and

$$\int_\gamma \omega \quad = \quad \pi \quad \neq \quad 0 \quad = \quad \int_{\mathrm{id}} \omega$$

where id is a constant loop.

(In order to show this, use polar coordinates by change of basis.)

Formulation with loops

Definition

A manifold is **contractible** if every loop based at a point p is homotopic to the constant loop at p.

Proposition

A 1-form ω is exact iff $\oint_\gamma \omega = 0$ for every loop γ.

Proof.

Use Green's theorem which states that

$$\int_\gamma \omega \quad = \quad \int_0^\varepsilon \int_0^\varepsilon (\partial_i \omega_j - \partial_j \omega_i) \, \mathrm{d}x^i \, \mathrm{d}x^j \qquad \square$$

Towards Stokes' theorem

Recall

- the fundamental theorem of calculus

$$\int_a^b f'(x)\,\mathrm{d}x \quad = \quad f(b) - f(a)$$

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- Stokes' theorem: given a surface S in $\mathbb{R}^3$ with $\partial S = \gamma$,

$$\int_S (\nabla \times \vec{F}) \cdot \vec{n} \quad = \quad \int_\gamma \vec{F}$$

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- Gauss' theorem: given a volume R in $\mathbb{R}^3$,

$$\int_R \nabla \cdot \vec{F} \quad = \quad \int_{\partial R} \vec{F} \cdot \vec{n}$$

Manifolds with boundaries

Definition

A **half-space** H is

$$H^n \quad = \quad \{\pi(x) \geq 0 \mid x \in \mathbb{R}^n\}$$

where $\pi : \mathbb{R}^n \multimap \mathbb{R}$ is a non-zero linear map (typically the projection on x_n).

Its boundary is

$$\partial H^n \quad = \quad \ker \pi \quad \cong \quad \mathbb{R}^{n-1}$$

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Definition

An n-**manifold with boundary** is a manifold with charts

$$\varphi_i \quad : \quad U_i \to H^n$$

and smooth transition maps.

Manifolds with boundaries

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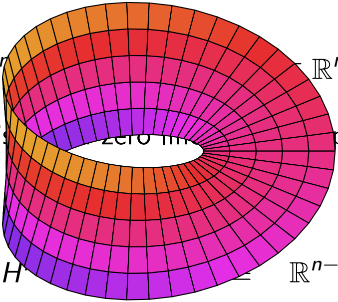

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Definition

An n-**manifold with boundary** is a manifold with charts

$$\varphi_i \quad : \quad U_i \to H^n$$

and smooth transition maps.

Boundary

The **boundary** of such an n-manifold M is

$$\partial M \quad = \quad \{x \in M \mid \exists i, x \in U_i \quad \text{and} \quad \varphi_i(x) \in \partial H^n\}$$

and is canonically an $(n-1)$-manifold.

Stokes' theorem

Theorem

Given a compact oriented n-manifold M with boundary and an $(n-1)$-form ω,

$$\int_M \mathrm{d}\omega \quad = \quad \int_{\partial M} \omega$$

DeRham cohomology

Given a manifold M we have constructed a cochain complex

$$\ldots \xleftarrow{\mathrm{d}^3} \Omega^2 M \xleftarrow{\mathrm{d}^2} \Omega^1 M \xleftarrow{\mathrm{d}^1} \Omega^0 M \xleftarrow{\mathrm{d}^0} 0$$

of vector spaces:

$$\mathrm{d} \circ \mathrm{d} \quad = \quad 0$$

which implies

$$\operatorname{im} \mathrm{d}^k \quad \subseteq \quad \ker \mathrm{d}^{k+1}$$

(exact forms are closed).

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of vector spaces:

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(exact forms are closed).

Definition

We define the **DeRham cohomology groups** by

$$H^k \quad = \quad \ker d^{k+1} / \operatorname{im} d^k$$

We have $H^0 M = \ker \mathrm{d}^1$. Given $f \in H^0 M$, we have locally

$$\mathrm{d}f \quad = \quad \partial_i f \, \mathrm{d}x^i \quad = \quad 0$$

so f is constant on connected components.

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A basis of $H^0 M$ is thus the $f_i \in M^*$, with i indexing connected components of M, such that

$$f_i(p) \quad = \quad \begin{cases} 1 & \text{if } p \text{ in the } i\text{-th connected component} \\ 0 & \text{otherwise} \end{cases}$$

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In other words

$$H^0 M \quad \cong \quad \mathbb{R}^c$$

where c is the number of connected components of M.

Consider $\omega \in \operatorname{im} \mathrm{d}^1 \subseteq \Omega^1 M$: we have

$$\omega \quad = \quad \mathrm{d}f$$

for some $f \in M^*$.

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because ∂S is empty.

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$H^1 M$ counts the "number of holes" in M.

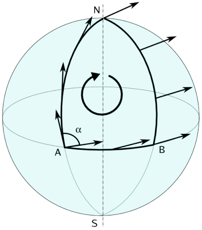
Frobenius Algebras
&
Topological Quantum Field Theories

Connections

Connections

The idea of a connection is to relate nearby tangent spaces in order to

- *parallel transport*:



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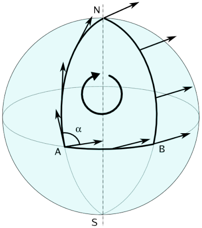

- define the *derivative* of a vector field: the formula

$$D_v w = \lim_{q \to p} \frac{w(q) - w(p)}{\|q - p\|}$$

does not make sense because $w(q) \in T_q M$ and $w(p) \in T_p M$. However, we will manage if we can parallel transport $w(q)$ into $T_p M$.

Bundles

Definition

A **bundle**

$$E \xrightarrow{\pi} M$$

is a manifold E equipped with a projection to M. The **fiber** over $p \in M$ is

$$E_p \quad = \quad \{v \in E \mid \pi(v) = p\}$$

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For instance, the **trivial bundle** with **standard fiber** F is

$$M \times F$$

equipped with the first projection.

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Given a submanifold $U \subseteq M$, we can define the restriction bundle

$$E|_U \quad = \quad \pi^{-1}(U) \quad = \quad \{v \in E \mid \pi(v) \in U\}$$

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A bundle is **locally trivial** with standard fiber F when each point $p \in M$ has a neighborhood U and a bundle isomorphism

$$\varphi \quad : \quad E|_U \quad \to \quad U \times F$$

Vector bundles

Definition

A **vector bundle** is a bundle such that

1. each fiber is a vector space

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A **vector bundle** is a bundle such that

1. each fiber is a vector space
2. and each point $p \in M$ has a neighborhood U and a bundle morphism

$$\varphi \quad : \quad E|_U \quad \rightarrow \quad U \times \mathbb{R}^n$$

such that for every $p \in U$,

$$\varphi(p, -) \quad : \quad E_p \quad \multimap \quad \mathbb{R}^n$$

is a linear isomorphism.

Vector bundles

Common operations on vector spaces extend fiberwise on bundles:

$$(E^*)_p = E_p^* \qquad (E \otimes F)_p = E_p \otimes F_p \qquad \text{etc.}$$

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A vector bundle with a basis is isomorphic to a trivial bundle, so we generally consider basis only locally.

Definition

A **connection** is a bilinear map

$$\nabla \quad : \quad \Gamma TM \otimes \Gamma E \quad \multimap \quad \Gamma E$$

which is

1. M^*-linear in the first variable:

$$\nabla_{fv} w \quad = \quad f \nabla_v w$$

2. Leibniz in the second variable:

$$\nabla_v (fw) \quad = \quad \mathrm{d}f(v) w + v \nabla_v w$$

$\nabla_v w$ is called the **covariant derivative** of w in direction v.

Parallel transport

Definition

A vector field v is **parallel** if $\nabla v = 0$ (i.e. for every w, $\nabla_w v = 0$).

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These often don't exist because parallel transport depend on paths:

Definition

Given a path $\gamma : p \rightsquigarrow q$ and $v_p \in T_p M$, a vector field $v \in \Gamma TM$ is the **parallel transport** of v_p along γ if

1. $v(p) = v_p$
2. $\nabla_{\dot{\gamma}(t)} v(\gamma(t)) = 0$ for every t
   (i.e. v is parallel wrt the pullback connection on the pullback bundle $\gamma^* TM$)

Connections in a basis

If we write locally the *vector potential* A:

$$D_{\partial_k} e_j = A^i_{kj} e_i$$

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If we write locally the *vector potential* A:

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we have, given a section $s \in \Gamma E$,

$$\begin{aligned} D_v s &= D_{v^k \partial_k} s \\ &= v^k D_{\partial_k} s \\ &= v^k D_{\partial_k}(s^i e_i) \\ &= v^k \left(\left(\partial_k s^i \right) e_i + A^j_{ki} s^i e_j \right) \\ &= v^k \left(\partial_k s^i + A^i_{kj} s^j \right) e_i \end{aligned}$$

i.e.

$$(D_k s)^i \quad = \quad \partial_k s^i + A^i_{kj} s^j$$

An End(E)-valued 1-form

A connection is a linear map

$$\nabla \quad : \quad \Gamma TM \otimes \Gamma E \quad \multimap \quad \Gamma E$$

so locally, is described by a section A of

$$T^*U \otimes E^*|_U \otimes E|_U \quad \cong \quad T^*U \otimes (E|_U \multimap E|_U)$$

with coordinates

$$A \quad = \quad A^i_{kj}\, \mathrm{d}x^k \otimes x^j \otimes x_i$$

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called the **vector potential**.

We can thus write

$$(\nabla_v s)^i \quad = \quad v(s^i) + (A(v)s)^i$$

Given a choice of local trivialization of E, the **standard flat connection** is

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The flat connection

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Proposition

Any connection ∇ *can be written as*

$$\nabla \quad = \quad \nabla^0 + A$$

for some potential $A \in \Gamma(T^*M \otimes (E \multimap E))$, *i.e.*

$$\nabla_v s \quad = \quad \nabla^0_v s + A(v)s$$

Torsion and curvature

Definition

The **torsion** of ∇ is

$$T(v, w) \quad = \quad \nabla_v w - \nabla_w v - [v, w]$$

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Definition

The **curvature** of ∇ is

$$R_{u,v}(w) \quad = \quad \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

Levi-Civita connection

Theorem

Given a Riemannian manifold (M, g), there exists a unique connection which is

1. *an isometry:*

$$\nabla g = 0$$

2. *torsion-free: for any $v, w \in \Gamma TM$, $T(v, w) = 0$, i.e.*

$$\nabla_v w - \nabla_w v = [v, w]$$

Differential λ-calculus

Syntax

Terms are built from the syntax

$$t \quad ::= \quad x \quad | \quad tt \quad | \quad \lambda x.t \quad | \quad \alpha t \quad | \quad t + t \quad | \quad 0 \quad | \quad \mathsf{D}t \cdot t$$

with x a variable and $\alpha \in \mathbb{R}$ (or any fixed rig such as $\mathbb{N}$).

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with x a variable and $\alpha \in \mathbb{R}$ (or any fixed rig such as $\mathbb{N}$).

So, we have added linear combination of terms, but more importantly

$$\mathsf{D} t \cdot u$$

which is the derivative of (function) t wrt its argument, and will satisfy

$$\mathsf{D}(\lambda x.t) \cdot u \quad = \quad \lambda x. (\partial_x t \cdot u)$$

Intuitions

The partial derivative $\partial_x t \cdot u$ is the sum of all possible replacement of *one* occurrence of x in u by t:

$$\begin{aligned} \mathrm{D}\,(\lambda x.x(xy)) \cdot u &= \lambda x.\partial_x(x(xy)) \cdot u \\ &= \lambda x.u(xy) + \lambda x.x(uy) \end{aligned}$$

Structural congruence

We consider them up to structural congruence:

- α-conversion
- terms form an $\mathbb{R}$-module:
  $(s + t) + u = s + (t + u)$, $\alpha(\beta t) = (\alpha\beta)t$, ...

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and

$$\begin{aligned}
\left(\sum_k \alpha_k t_k\right) u &= \sum_k \alpha_k t_k u \\
\lambda x. \sum_k \alpha_k t_k &= \sum_k \alpha_k (\lambda x. t_k) \\
\mathrm{D}(st) &= \mathrm{D}(s)t \\
\mathrm{D}(\mathrm{D}t \cdot u) \cdot v &= \mathrm{D}(\mathrm{D}t \cdot v) \cdot u \\
\mathrm{D}\left(\sum_k \alpha_k t_k\right) \cdot \left(\sum_l \beta_l u_l\right) &= \sum_{k,l} \alpha_k \beta_l (\mathrm{D}t_k \cdot u_l)
\end{aligned}$$

Linearity

Notice that we are linear in function only:

$$\left(\sum_k \alpha_k t_k\right) u \quad = \quad \sum_k \alpha_k t_k u$$

Otherwise, we would not be coherent with β-reduction:

$$(\lambda x.xx)(s+t) \quad \longrightarrow \quad (s+t)(s+t) \quad = \quad ss + st + ts + tt$$

vs

$$(\lambda x.xx)s + (\lambda x.xx)t \quad \longrightarrow \quad ss + tt$$

Partial derivative

$$
\begin{array}{rcl}
\partial_x \left(\sum_k \alpha_k t_k \right) \cdot u & = & \sum_k \alpha_k \left(\partial_x t_k \right) \cdot u \\
\partial_x x \cdot u & = & u \\
\partial_x y \cdot u & = & 0 \\
\partial_x (st) \cdot u & = & \left(\partial_x s \cdot u \right) t + \left(\mathrm{D} s \cdot \left(\partial_x t \cdot u \right) \right) t \\
\partial_x (\lambda y . t) \cdot u & = & \lambda y . \left(\partial_x t \cdot u \right) \\
\partial_x \left((\mathrm{D} t \cdot u) \cdot v \right) & = & \mathrm{D} \left(\partial_x t \cdot v \right) + \mathrm{D} t \cdot \left(\partial_x u \cdot v \right)
\end{array}
$$

Reduction

- β-reduction:

$$(\lambda x.s)\,t \quad \longrightarrow \quad s[t/x]$$

- differential reduction:

$$\mathrm{D}\,(\lambda x.t) \cdot u \quad \longrightarrow \quad \lambda x.\,((\partial_x s) \cdot u)$$

i.e. we substitute only one linear occurrence of x (and take the sum over all possibilities)

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Theorem

The reduction is confluent and differential λ-calculus is a conservative extension of λ-calculus (we did not quotient pure λ-terms wrt reduction).

Derivative wrt i-th variable

The original article by Eherhard and Reigner defines D_i, differentiation with the i-th argument.

The generalization does not bring major problems:

$$\mathsf{D}_0 \quad = \quad \mathsf{D}$$

and

$$\mathsf{D}_{i+1}(\lambda x.t) \cdot u \quad = \quad \lambda x.(\mathsf{D}_i t \cdot u)$$

provided $x \notin \mathsf{FV}(u)$.

(+ lots of details. . .)

Terms can be generalized to countable sums of terms,
i.e. formal series $\sum_{k=0}^{\infty} \alpha_k t_k$.

The Taylor formula

Terms can be generalized to countable sums of terms,
i.e. formal series $\sum_{k=0}^{\infty} \alpha_k t_k$.

When we substitute x by u in t, we substitute it a fixed number $n \in \mathbb{N}$ of times (the number of occurrences of x):

Theorem

$$tu \quad \to \quad \sum_{n=0}^{\infty} \frac{1}{n!}(\mathrm{D}^n t \cdot u^n)0$$

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Free algebras

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By definition, we have a bijection (TODO: make this more clear....)

$$A \to B \quad \cong \quad !A \multimap B$$

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By definition, we have a bijection (TODO: make this more clear....)

$$A \to B \quad \cong \quad !A \multimap B$$

So, the differential of a function

$$f \quad : \quad !A \multimap B$$

should be of type

$$\mathrm{d}f \quad : \quad !A \multimap A \multimap B \quad \cong \quad !A \otimes A \multimap B$$

Differential Semantics

The power of analogy

diff. geom.	comp. sci.
manifold	program (cfg)
vector field	choice (in branchings)
1-form	semantics
closed 1-form	local confluence

Graphs as manifolds

Definition

A **manifold** M is a graph with V as vertices, E as edges, $s, t : E \to V$ as source and target maps, possibly with some higher-dimensional cells (such as a precubical set, a polygraph, etc.) and morphisms are graph morphisms (or maybe categorical morphisms ?).

Graphs as manifolds

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Starting from this we get the following.

- The **tangent bundle** is $s : E \to V$.
- The **tangent space** at $x \in V$ is the set

$$T_x M \quad = \quad \{e \in E \mid s(e) = x\}$$

- A **vector field** consists of a choice of edge originating at every vertex.

The state space

Definition

The **state space** R is a (higher) category.

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Example

- We can take R the category of possible memories and memory operations as morphisms. For instance, given a set V of values and a number k of memory cells, R is the simply connected groupoid on V^k (?).
- We can take R the category whose objects are elements of $\mathbb{R}$ and the only morphism $f : x \to y$ is $y - x$.
- Can we think of "non-trivial" examples?

Differential forms

We suppose fixed a set S of **states** (typically the possible states for the memory of the computer). This will replace $\mathbb{R}$ as "negation object".

Definition

The **dual** of a set X is

$$X^* \quad \cong \quad X \to S$$

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- A **function** $f : M^*$ is