Delooping cyclic groups with lens spaces in homotopy type theory

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Abstract

In the setting of homotopy type theory, each type can be interpreted as a space. Moreover, given an element of a type, i.e. a point in the corresponding space, one can define another type which encodes the space of loops based at this point. In particular, when the type we started with is a groupoid, this loop space is always a group. Conversely, to every group we can associate a type (more precisely, a pointed connected groupoid) whose loop space is this group: this operation is called *delooping*. The generic procedures for constructing such deloopings of groups (based on torsors, or on descriptions of Eilenberg-MacLane spaces as higher inductive types) are unfortunately equipped with elimination principles which do not directly allow eliminating to arbitrary types, and are thus difficult to work with in practice. Here, we construct deloopings of the cyclic groups $\mathbb{Z}_m$ which are *cellular*, and thus do not suffer from this shortcoming. In order to do so, we provide type-theoretic implementations of *lens spaces*, which constitute an important family of spaces in algebraic topology. In some sense, this work generalizes the construction of *real projective spaces* by Buchholtz and Rijke in their LICS'17 paper, which handles the case $m = 2$, although the general setting requires more involved tools. Finally, we use this construction to also provide cellular descriptions of dihedral groups, and explain how we can hope to use those to compute the cohomology and higher actions of such groups.

1 Introduction

Homotopy type theory (HoTT), which is based on Martin-Löf type theory [15], was introduced around 2010 [24]. It stems from the idea that types in logic should be interpreted not only as sets, as traditionally done in the semantics of logic, but rather as *homotopy types*, by which we mean spaces considered up to deformation. Namely, the identities between two elements of a type can be thought of as paths between points corresponding to the elements [1], and the fact that two identities between the same points are not necessarily the same [9] means that types can bear a non-trivial geometry. This point of view, which is validated by Voevodsky's simplicial model of univalent type theory [10], allows one to prove properties about spaces by reasoning within type theory. This is satisfactory from a practical point of view, because we have the machine to check our proofs in full details, but also from a theoretical point of view because constructions performed in type theory are homotopy invariant by construction. This last point is actually both a blessing and a curse: it often means that the traditional proofs have to be deeply reworked in order to have a chance to be mechanized.

In this article, we provide a construction of *lens spaces*, which constitute an important family of spaces. Moreover, our construction is performed by iterating pushout constructions. They can thus be thought of as a type-theoretic counterpart of their description as cell complexes, and therefore come equipped with a recursion principle which allows eliminating to arbitrary types.

1.1 Delooping groups. The motivation for defining those spaces stems from the desire to perform computations on groups in homotopy type theory. In this setting, every pointed type A has a loop space ΩA, which is a group when A is a groupoid. Conversely, any group G arises in this way: we can always define a pointed space $\mathrm{B}\,G$ whose loop space is G, which is the reason why $\mathrm{B}\,G$ is called a *delooping* of G. Moreover, this construction induces an equivalence between groups and pointed connected spaces. This construction can be seen as a way to encode a group structure into a space, while retaining the main properties; for instance, the cohomology of $\mathrm{B}\,G$ as a space is the traditional cohomology of G as a group.

There are very general constructions for delooping groups. For instance, those are a particular case of Eilenberg-MacLane spaces, which can be explicitly constructed as higher inductive types in HoTT [12]. There is however an important limitation of such constructions, which make them very difficult to use for some computations: they produce *recursive* higher inductive types (or HITs), due to the use of truncations. More explicitly, given a group G, its delooping can be constructed as the HIT generated by

(0) one point $\star$,
(1) one identity $[a] : \star = \star$ for every element a of the group G,
(2) one relation $[a] \cdot [b] = [ab]$ for every pair of elements a and b of the group, as well as a relation $[1] = 1$,
(3) taking the groupoid truncation of the resulting type.

Because of the last step, we can only easily eliminate to groupoids and we cannot perform computations by induction of the dimension of the constructors, which would be possible if we had a description of the delooping as a non-recursive HIT, what we could call here a *cellular* definition. Namely, such a cellular structure would provide an elimination principle for $\mathrm{B}\,G$ allowing us to effectively construct maps $\mathrm{B}\,G \to X$ for general types X, not necessarily groupoids.

1.2 Constructing cellular deloopings. In order to produce a cellular definition of the delooping of a given group G, one can begin from the above description without performing the last truncation step (3). The pointed connected type $\mathrm{B}_1\,G$ thus defined is not in general a groupoid. In order to obtain one, the general idea consists in progressively adding higher identity generators to the type, in order to make it more and more coherent and construct a family of types $\mathrm{B}_n\,G$ without higher-dimensional "holes" in dimensions $1 < k \leq n$. Taking the inductive limit, we will obtain a space $\mathrm{B}_\infty\,G$ which is a groupoid and thus a delooping of G. The reader familiar with techniques in algebraic topology will observe that this process is very similar to the one of computing resolutions of algebraic structures or spaces.

The way we can perform this construction in general is however not clear at first: we need to find enough cells to provably fill in all the holes in each dimension, and we need to be able to perform this in a systematic way. An approach based on rewriting and a generalization of Squier's theorem was proposed in [11], but it is quite limited: as for now, it is only able to perform the very first steps of the above construction, it is quite involved computationally, and it generally produces types with are not minimal in terms of number of generators. Another way to proceed consists in taking inspiration from geometric constructions where some models of deloopings are known. For instance, the infinite-dimensional real projective space $\mathbb{RP}^\infty$ is known to be a delooping of $\mathbb{Z}_2$ and, moreover, we know a cellular decomposition of this space with one cell in every dimension. We thus can hope to have a description of it in homotopy type theory as a HIT (or similar) with one generating cell in every dimension. In fact, this task was successfully performed by Buchholtz and Rijke in [5].

1.3 Defining lens spaces. Here, we further explore the latest route and compute a model for $\mathrm{B}\,\mathbb{Z}_m$ by using a family of spaces well known in traditional algebraic topology, the *lens spaces*, which are due to Tietze [23]. Our main contribution is to define types corresponding to those spaces, and thus derive a cellular model of $\mathrm{B}\,\mathbb{Z}_m$ with infinite-dimensional lens spaces.

Contrarily to what it might first seem, the task is more difficult than simply replacing 2 by m in the paper [5] defining projective spaces. Firstly, the aforementioned article presents very well the situation with respect to $\mathbb{Z}_2$, but it was unclear (at least for us) that the techniques developed there could be reused in other models. We explain that the construction performed there is a particular instance of a very general construction based on tools developed in Rijke's PhD thesis [21], which generalizes one due to Milnor [17, 18]. Basically, the idea in order to construct a cellular model for $\mathrm{B}\,G$ consists in starting with a map $f : X \to \mathrm{B}\,G$ which is surjective up to homotopy, and compute its iterated join products [20] in order to produce another definition of the delooping which is cellular. In fact, it is so general that it can be performed without starting from a particularly concrete model of the delooping $\mathrm{B}\,G$ as target for f. We also take the opportunity of this paper to rework some of the associated proofs and provide ones based on the flattening lemma (as opposed to descent), which eases mechanization. Secondly, the construction of lens spaces does not actually start from the same data as projective spaces, and is not even a generalization of the latter: it is based on a map $S^1 \to \mathrm{B}\,\mathbb{Z}_m$ rather than $1 \to \mathrm{B}\,\mathbb{Z}_m$, which gives more latitude to the construction (there are actually multiple choices of such maps one could take, and the resulting computations are more involved than in the real projective case).

We believe that the resulting definition of $\mathrm{B}\,\mathbb{Z}_m$ will be useful in order to perform computations on cyclic groups, such as computing their cohomology, as we will explain later. It should also allow defining actions of $\mathbb{Z}_m$ on higher types (as opposed to sets). We leave their use for future work, as well as the exploration of the construction of the delooping of other classical groups.

1.4 Plan of the paper. After recalling some basic notions and constructions in homotopy type theory (Section 2), we introduce the notion of delooping of a group (Section 3). We then define and study the join operation on types and morphisms, which allows computing the image of a map (Section 4), and can be reformulated as an operation on fiber sequences (Section 5). These constructions, along with a correspondence between fiber sequences and group actions, allow us to define a type-theoretic counterpart of lens spaces, and thus a model of $\mathrm{B}\,\mathbb{Z}_m$ (Section 6), which is shown to be cellular (Section 7). We then apply this work in order to define deloopings of dihedral groups (Section 8) and conclude (Section 9).

2 Homotopy type theory

Throughout the article, we suppose the reader is already familiar with homotopy type theory and refer to the book [24] for reference. We only fix here some notations for the classical notions used in the article.

2.1 Types. We write $\mathcal{U}$ for the universe, whose elements are types which are small (for simplicity, we do not detail universe levels throughout the article). We write $x : A$ to indicate that x is an element of a type A. The initial and terminal types are respectively denoted by 0 and 1.

Given two types A and B, we write $A \to B$ for the type of functions between them. A function $f : A \to B$ is an *equivalence* when it admits an inverse up to homotopy. We write $A \simeq B$ for the type of equivalences between A and B. Given a type A and a type family $B : A \to \mathcal{U}$, we write $(x : A) \to B(x)$ or $\Pi(x : A).B(x)$ or $\Pi A.B$ for the induced *dependent product* type, and $\Sigma(x : A).B(x)$ or $\Sigma A.B$ for the induced *dependent sum* type. The two canonical projections are noted $\pi : \Sigma A.B \to A$ and $\pi' : (x : \Sigma A.B) \to B(\pi(x))$. Given morphisms $f : A \to A'$ and $g : (x : A) \to B(x) \to B'(f(x))$, we write $\Sigma f.g : \Sigma A.B \to \Sigma A'.B'$ for the canonically induced morphism.

2.2 Paths. We write $x := y$ to indicate that x is *defined* to be y. Our type theory also features a propositional notion of equality and, given two elements x and y of a type A, we write $x = y$ for their identity type. In particular, an element of $x = x$ is called a *loop* on x. Any element x induces a canonical loop $\mathrm{refl}_x : x = x$. Given paths $p : x = y$ and $q : y = z$, we write $p \cdot q : x = z$ for their concatenation and $p^{-1} : y = x$ for the inverse of p.

Any path $p : A = B$ between types A and B, induces a function $p^{\rightarrow} : A \to B$, called *transport* along p, which is an equivalence whose inverse is denoted $p^{\leftarrow} : B \to A$. The *univalence* axiom states that the map $(A = B) \to (A \simeq B)$ thus induced is an equivalence.

By *congruence*, any function $f : A \to B$ and path $p : x = y$ induce an equality $f^{=}(p) : f(x) = f(y)$. In particular, given a type family $B : A \to \mathcal{U}$ and a path $p : x = y$ in A, we have an induced equality $B(x) = B(y)$ and thus an equivalence $B(x) \simeq B(y)$ whose underlying function is noted $B^{\rightarrow}(p) : B(x) \to B(y)$ (and $B^{\leftarrow}(p)$ for its inverse) and corresponds to *substituting* x by y in B.

Given a type family $B : A \to \mathcal{U}$, a path $p : x = y$, and elements $x' : B(x)$ and $y' : B(y)$, we write $x' =^B_p y'$ (or even $x' =_p y'$ when B is clear from the context) for the type of *paths over p* from x' to y'; by definition, this type is the identity type $B^{\rightarrow}(p)(x') = y'$.

Given functions $f, g : A \to B$, a path $p : f = g$ between those, and $x : A$, we write $p(x) : f(x) = g(x)$ for the induced path obtained by applying p to x pointwise.

2.3 Homotopy levels. A type A is *contractible* when the type $\Sigma(x : A).\Pi(y : A).(x = y)$ is inhabited. A contractible type is also called a (-2)*-type*, and we define by induction an $(n+1)$-type to be a type A such that $x = y$ is an n-type for every elements $x, y : A$. In particular, (-1)-, 0- and 1-types are respectively called *propositions*, *sets* and *groupoids*. Given a type A, we write $\|A\|_n$ for the *n-truncation* of A, which is the universal way of turning a type into an n-type. A type A is *connected* when $\|A\|_0$ is contractible ($\|A\|_0$ being the set of connected components of A).

2.4 Spheres. We write S^n for the type corresponding to the n-sphere (those can for instance be defined inductively by $\mathrm{S}^{-1} = 0$ and $\mathrm{S}^{n+1} = \Sigma\,\mathrm{S}^n$, where Σ is the suspension operation). For S^1, we write $\star$ for the canonical element and $\mathrm{loop} : \star = \star$ for the canonical non-trivial loop.

2.5 Homotopy limits and colimits. All (co)limits considered in the paper are homotopic. Given two maps f, g with the same codomain, we have a pullback square

$$\begin{array}{ccc} \Sigma(x, y : A \times B).(f(x) = g(y)) & \xrightarrow{\pi} & A \\ {\scriptstyle \pi\circ\pi'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

As a particular case, when A and B are the terminal type and $f : 1 \to C$ and $g : 1 \to C$ are maps respectively pointing at elements x and y, their pullback is the identity type $x = y$.

3 Deloopings of groups

3.1 Deloopings. A *group* G is a set equipped with multiplication $m : G \to G \to G$ and unit $e : 1 \to G$ satisfying the usual axioms, and a morphism of groups is a function between the underlying sets preserving the operations.

A *pointed type* A consists of a type A together with an element $\star : A$, and a pointed map $f : A \to_\star B$ consists of a function $f : A \to B$ together with a equality $\star_f : f(\star_A) = \star_B$. The *loop space* of a pointed type A is $\Omega A :\equiv (\star = \star)$, the type of loops on the distinguished element. This operation is functorial: any pointed map $f : A \to B$ canonically induces a map $\Omega f : \Omega A \to \Omega B$ [24, Definition 8.4.2] defined as $\Omega f(p) = \star_f^{-1} \cdot f^{=}(p) \cdot \star_f$, and this construction is compatible with composition and identities. When A is a groupoid (in the sense of a 1-truncated type), ΩA is thus a group with multiplication given by concatenation of paths and unit by constant paths.

A *delooping* of a group G is a pointed connected type $\mathrm{B}\,G$ equipped with an identification $\mathrm{d}_G : \Omega\,\mathrm{B}\,G = G$. Such a type is necessarily a groupoid since G is a set. The following lemma states that we can also deloop morphisms, see [2, Section 4.10] and [25, Corollary 12], as well as the appendix, for a proof:

Lemma 1. *Given two pointed connected groupoids A and B, and a group morphism $f : \Omega A \to \Omega B$, there is a unique pointed morphism $g : A \to_\star B$ such that $\Omega g = f$.*

Given a morphism of groups $f : G \to H$, where G and H admit deloopings, the *delooping* of f is the map

$$\mathrm{B}\,f : \mathrm{B}\,G \to \mathrm{B}\,H$$

associated, by Lemma 1, to the morphism obtained as the composite

$$\Omega\,\mathrm{B}\,G \xrightarrow{\mathrm{d}_G^{\rightarrow}} G \xrightarrow{f} H \xrightarrow{\mathrm{d}_H^{\leftarrow}} \Omega\,\mathrm{B}\,H.$$

This operation is easily shown to be functorial (Lemma 33), in that it preserves composition and identities.

The delooping of a given group G is unique in the sense that any two deloopings of a given group are necessarily equal, thus justifying the notation $\mathrm{B}\,G$ (see Lemma 34 for a more precise statement and proof). It can also be shown that any group G always admits a delooping:

Lemma 2. *Every group G admits a delooping $\mathrm{B}\,G$.*

There are (at least) two generic ways of constructing $\mathrm{B}\,G$ for an arbitrary G. We only briefly mention those here because we will only need to know that the above lemma is true, but will not depend on a particular such construction. The first one consists in defining $\mathrm{B}\,G$ as a truncated HIT, as described in the introduction: this is obtained as the $K(G, 1)$ construction, defined by Finster and Licata in [12]. The second one is the *torsor* construction: it consists in defining $\mathrm{B}\,G$ as the connected component in G-sets of the principal G-set (a G-set is a set equipped with an action of G). More details can be found in [2] or Appendix B. Useful smaller variants of this construction can also be considered [6]. Finally, we should mention that interesting models for deloopings can often be constructed for particular groups. For instance, one of the main early results of homotopy type theory is that the circle S^1 is a delooping of $\mathbb{Z}$ [24, Section 8.1]. Other (new) examples are given in the present paper.

The fundamental theorem about the delooping construction is the following:

Theorem 3. *The functions Ω and B induce an equivalence between the category of groups and the category of pointed connected groupoids.*

The above proposition states that a group is the same as a pointed connected groupoid: the first notion can be thought of as an *external* one (in the sense that we impose axioms on the structures), whereas the second is an *internal* one (in the sense that the structure is generated by the properties of the considered types). It also provides an elimination principle for deloopings. Namely, functions $\mathrm{B}\,G \to A$ correspond to group morphisms $G \to \Omega A$ when A is a pointed connected groupoid. We could easily drop the restriction to connected types, by considering connected components. However, we do not have a direct way of eliminating to types which are not groupoids for now: we will address this point with our construction.

3.2 Actions of groups. Given a group G and a type A, an *action* of G on A is a map $f : \mathrm{B}\,G \to \mathcal{U}$ such that $f(\star) = A$. Namely, writing $A :\equiv f(\star)$, each path $a : \star = \star$ (which can be seen as an element of G), induces by substitution an equivalence $A \to A$, in a way which is compatible with composition and identities. In fact, it can be shown that actions on sets in the above sense correspond to actions of groups in the traditional sense (Lemma 38).

Given an action $f : \mathrm{B}\,G \to \mathcal{U}$, with $A :\equiv f(\star)$, the *homotopy quotient* of A by G (with respect to the action) is the type

$$A /\!\!/ G :\equiv \Sigma(x : \mathrm{B}\,G).f(x)$$

There are canonical *quotient* and *projection* morphisms, respectively defined as

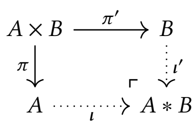
$$\begin{array}{rlcrl} \kappa : A & \to A/\!\!/G & \qquad & \pi : A/\!\!/G & \to A \\ x & \mapsto (\star, x) & & (x,y) & \mapsto x \end{array}$$

where the quotient κ witnesses the inclusion of A as the fiber above $\star$ in the homotopy quotient.

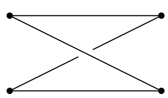
4 The Milnor construction

The *Milnor construction* is a general method, due to Milnor [17, 18], to construct the universal bundle for a fixed topological group G. Its adaptation to the setting of homotopy type theory was done by Rijke in his PhD thesis [20, 21].

4.1 The join construction. Given two types A and B, their *join* $A * B$, see [24, Section 6.8], is the pushout of the product projections:



As any pushout, the join can be constructed as a quotient of the coproduct $A \sqcup B$, and this description provides a way to implement the join of two types as a higher inductive type: $A*B$ is the universal type which contains every element of A, every element of B, and has a path between any element of A and any element of B. For instance, the space corresponding to $S^0 * S^0$ looks like



and is equal to S^1. Namely, for a type A, we have that $S^0 * A$ is the suspension ΣA so that $S^0 * S^0 = \Sigma S^0 = S^1$ and more generally $S^m * S^n = S^{m+n+1}$. The join operation is associative, commutative, and admits the empty type 0 as neutral element [3, Section 1.8].

4.2 Propositional truncation. One of the interests of the join construction is that it results in a type which is more connected than its operands. Namely, recall that a type A is *n-connected* when its n-truncation $\|A\|_n$ is contractible. It can be shown that when A is m-connected and B is n-connected their join is $(m+n+2)$-connected [4, Proposition 3] (in the above example, S^1 is 0-connected, as the join of two (-1)-connected types). If we iterate this construction by computing the join product A^{*n} of n copies of A for n increasing, we obtain more and more connected spaces: if A is k-connected then A^{*n} is $(n(k+2)-2)$-connected. The map ι of the pushout provides us with a canonical inclusion of A^{*n} into $A^{*(n+1)}$ and we write $A^{*\infty}$ for the colimit of the diagram

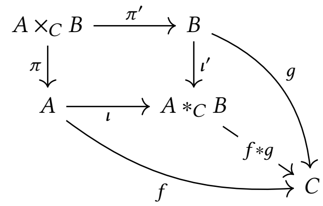
$$0 \longrightarrow A \longrightarrow A * A \longrightarrow A * A * A \longrightarrow \cdots$$

We expect this limit to be ∞-connected, i.e. contractible, excepting in one case: if we begin with A the empty type, the limit will still be the empty type. The proper way to phrase this is as follows [21, Theorem 4.2.7]:

Theorem 4. *The type $A^{*\infty}$ is the propositional truncation of A.*

In particular, with $A := S^0$, we deduce that the infinite-dimensional sphere $S^\infty := (S^0)^{*\infty}$ is contractible.

4.3 Join of maps. The join construction can be generalized to morphisms as follows. Given maps $f : A \to C$ and $g : B \to C$ with the same target, their *join* $f * g$ is the universal map from the pushout of the dependent projections π and π':

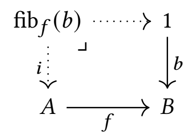


The source is abusively noted $A *_C B$ and we have $A *_1 B = A * B$, so that this generalizes the previous construction which consisted of taking the join of terminal maps.

4.4 Fibers of maps. Given a function $f : A \to B$ and a point $b : B$, the *fiber* of f at b is

$$\mathrm{fib}_f(b) = \Sigma(x : A).(f(x) = b)$$

This type can also be seen as the pullback of the constant arrow at b along f:



where the map i is called here the canonical inclusion. In particular, given a pointed type B and a function $f : A \to B$, we define the *kernel* of f as $\ker f = \mathrm{fib}_f(\star_B)$, see [24, Section 8.4]. When A is pointed and f is a pointed map, $\ker f$ is canonically pointed (by $\star_A$ together with the proof of $f(\star_A) = \star_B$ given by the fact that f is pointed).

It can be shown that the join operation commutes with taking fibers, in the following sense [21, Theorem 2.3.15]:

Theorem 5. *Given $f : A \to X$ and $g : B \to X$ and $x : X$, we have*

$$\mathrm{fib}_{f*g}(x) = \mathrm{fib}_f(x) * \mathrm{fib}_g(x).$$

We provide a proof of a generalization of this result in Section 7.

4.5 Image of maps. If we consider the iterated join f^{*n} of n instances of a map $f : A \to B$, its fibers get more and more connected as n increases, since $\mathrm{fib}_{f^{*n}}(b) = \mathrm{fib}_f(b)^{*n}$. If we take the colimit $f^{*\infty}$, we obtain a map whose fibers are propositions: this is the canonical inclusion of the image of f into the codomain of f. More precisely, writing $\mathrm{im}^n(f)$ for the source of f^{*n}, we have a canonical map $\mathrm{im}^n(f) \to \mathrm{im}^{n+1}(f)$ given by ι, and we write $\mathrm{im}^\infty(f)$ for the colimit of the diagram

$$\mathrm{im}^0(f) \longrightarrow \mathrm{im}^1(f) \longrightarrow \mathrm{im}^2(f) \longrightarrow \cdots$$

The maps $f^{*n} : \mathrm{im}^n \to B$ form a cocone on this diagram and, by universal property, we obtain a map $i : \mathrm{im}^\infty(f) \to B$. It can then be shown, see [21, Theorem 4.2.13] and [20, Theorem 3.3], that $\mathrm{im}^\infty(f)$ coincides with the usual definition of the image of a morphism $\mathrm{im}(f) = \Sigma(y : B).\|\mathrm{fib}_f(y)\|_{-1}$, see [24, Definition 7.6.3]:

Theorem 6. *We have $\mathrm{im}^\infty(f) = \mathrm{im}(f)$ and i is the canonical projection of the image on the first component.*

In the case we start from a map $f : A \to B$ which is surjective, in the sense that we have $\| \mathrm{fib}_f(y) \|_{-1}$ for every $y : B$ [24, Definition 4.6.1], we have $\mathrm{im}(f) = B$, and the colimit of $\mathrm{im}^n(f)$ is B. Each $\mathrm{im}^n(f)$ is obtained from the previous one by a pushout construction and we thus recover analogous in homotopy type theory of well-known cell complexes approximating spaces (see Example 8 below). In practice, the following result is often useful to show that a map is surjective:

Lemma 7. *Given types A and B, with A pointed and B connected, any map $f : A \to B$ is surjective.*

Proof. Write a for the distinguished element of A. Given $b : B$, we know that there merely exists a path $p : f\,a = b$, which we can use to show the proposition $\| \mathrm{fib}_f(b) \|_{-1}$. Namely, the type $\| \mathrm{fib}_f(f(a)) \|_{-1}$ is inhabited (by a) and we conclude by transport along p. □

Example 8 (from [5]). Consider the canonical map $f : 1 \to \mathrm{B}\,\mathbb{Z}_2$. Since $\mathrm{B}\,\mathbb{Z}_2$ is connected, this map is surjective by Lemma 7. Given $n \in \mathbb{N}$, we define the n-th *real projective space* as $\mathbb{R}\mathrm{P}^n = \mathrm{im}^{n+1}(f)$ and the associated *tautological line bundle* $\ell^n : \mathbb{R}\mathrm{P}^n \to \mathrm{B}\,\mathbb{Z}_2$ as $\ell^n = f^{*(n+1)}$ (the shift of indices was chosen in order to match usual conventions). Explicitly, this means that we take $\mathbb{R}\mathrm{P}^{-1} = \mathrm{im}^0(f) = 0$ and define $\mathbb{R}\mathrm{P}^{n+1}$ as the pushout

$$\begin{array}{ccccc}
\mathbb{R}\mathrm{P}^n \times_{B\mathbb{Z}_2} 1 & \longrightarrow & 1 & & \\
\downarrow & & \downarrow & \searrow^{f} & \\
\mathbb{R}\mathrm{P}^n & \longrightarrow & \mathbb{R}\mathrm{P}^{n+1} & & \\
& \searrow_{\ell^n} & & \searrow^{\ell^{n+1} = \ell^n * f} & \\
& & & & B\mathbb{Z}_2
\end{array}$$

where, by definition of the kernel we have $\mathbb{R}\mathrm{P}^n \times_{B\mathbb{Z}_2} 1 = \ker \ell^n$. Since $\ell^{n+1} = \ell^n * f$, by Theorem 5, we have $\ker \ell^{n+1} = \ker \ell^n * \ker f$, and we thus have a pullback

$$\begin{array}{ccc}
\ker \ell^n * \ker f & \longrightarrow & 1 \\
\downarrow & & \downarrow{\scriptstyle f} \\
\mathbb{R}\mathrm{P}^{n+1} & \xrightarrow[\ell^{n+1}]{} & B\mathbb{Z}_2
\end{array}$$

Moreover, $\ker f$ is the pullback of f along itself, which is $\Omega\,\mathrm{B}\,\mathbb{Z}_2$ (see Section 2.5), and thus $\mathbb{Z}_2$ by definition of $\mathrm{B}\,\mathbb{Z}_2$. We have thus shown

$$\ker \ell^{n+1} = \ker \ell^n * \mathbb{Z}_2$$

Since $\ker \ell^{-1} = 0$, and $\mathbb{Z}_2 = \mathrm{S}^0$, it follows by induction that

$$\mathbb{R}\mathrm{P}^n \times_{B\mathbb{Z}_2} 1 = \mathbb{Z}_2^{*(n+1)} = \mathrm{S}^n$$

To sum up, we have a pushout

$$\begin{array}{ccc}
\mathrm{S}^n & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
\mathbb{R}\mathrm{P}^n & \longrightarrow & \mathbb{R}\mathrm{P}^{n+1}
\end{array}$$

If we write $\mathbb{R}\mathrm{P}^\infty = \mathrm{colim}_n\,\mathbb{R}\mathrm{P}^n$ and $\ell^\infty = \mathrm{colim}_n\,\ell^n$ for the colimiting constructions (as described above), we have $\mathbb{R}\mathrm{P}^\infty = \mathrm{im}\,f = \mathrm{B}\,\mathbb{Z}_2$ by Theorem 6.

5 Fiber sequences

5.1 Definition. A *fiber sequence*

$$F \overset{i}{\hookrightarrow} E \overset{f}{\twoheadrightarrow} B \tag{1}$$

consists of three types F (the *fiber*), E (the *total space*) and B (the *base space*), with B pointed, and two maps (i and f) such that F is the kernel of f, with i as canonical inclusion, i.e. such that i is the pullback of the pointing map $b : 1 \to B$ along f.

5.2 Homotopy from fiber sequences. Any fiber sequence induces a long exact sequence of homotopy groups [24, Section 8.4], which in pratice allow the computation of some homotopy groups when a fiber sequence is known. In this article, we will not need to use these constructions in full generality; in fact, we will only need to use the following lemma.

Lemma 9. *Given a fiber sequence of the form (1) with F contractible and B connected, f is an equivalence.*

Proof. Showing that f is an equivalence is equivalent to show that $\mathrm{fib}_f(b)$ is contractible for any $b : B$ [24, Section 4.4]. From the fiber sequence, we have $\mathrm{fib}_f(\star_B) = \ker f = F$ and thus $\mathrm{fib}_f(\star_B)$ is contractible. Since B is connected, the type $\|\star_B = x\|_{-1}$ is inhabited, and, since being contractible is a proposition, we can suppose that we have a path $p : \star_B = x$. By transport, we deduce that $\mathrm{fib}_f(x)$ is contractible. □

We also have the following variant of the above lemma, with a similar proof:

Lemma 10. *Given a fiber sequence (1) with F n-connected, for some $n \in \mathbb{N}$, and B connected, f is n-connected.*

We recall that when $f : E \to B$ is n-connected (i.e. its fibers are n-connected), as in the conclusion of the above lemma, it induces an isomorphism of homotopy groups $\pi_k(E) \simeq \pi_k(B)$ for every natural number $k \leq n$ [24, Corollary 8.4.8].

5.3 Delooping short exact sequences. Given $f : G \to H$ a morphism of groups, we define its *kernel* and *image* respectively as

$$\ker f = \Sigma(x : G).(f(x) = 0)$$
$$\mathrm{im}\,f = \Sigma(y : H).\|\Sigma(x : G).(f(x) = y)\|_{-1}$$

Those are canonically equipped with a group structure induced by the one of G. We write $i : \ker f \to G$ and $j : \mathrm{im}\,f \to H$ for the first projections.

Lemma 11. *With the above notations, we have $\ker \mathrm{B}\,f = \mathrm{B} \ker f$ with $\mathrm{B}\,i$ as canonical inclusion into $\mathrm{B}\,G$.*

Proof. We have $\ker \mathrm{B}\,f = \Sigma(x : \mathrm{B}\,G).(\mathrm{B}\,f(x) = \star_{\mathrm{B}\,H})$, which is pointed by $(\star_{\mathrm{B}\,G}, p)$ with $p : \mathrm{B}\,f(\star_{\mathrm{B}\,G}) = \star_{\mathrm{B}\,H}$ given by the fact that $\mathrm{B}\,f$ is pointed. Thus $\Omega \ker \mathrm{B}\,f$ is the space of loops on $(\star_{\mathrm{B}\,G}, p)$ in this type, i.e. paths $q : \star_{\mathrm{B}\,G} = \star_{\mathrm{B}\,G}$ equipped with a proof of $p =_q^{\mathrm{B}\,f(-) = \star_{\mathrm{B}\,H}} p$. This last type is equivalent to $((\mathrm{B}\,f)^=(q))^{-1} \cdot p = p$ (by [24, Theorem 2.11.3]), and thus to $(\mathrm{B}\,f)^=(q) = \mathrm{refl}$ (since paths are invertible). Therefore,

$$\begin{aligned}\Omega \ker \mathrm{B}\,f &= \Sigma(q : \Omega\,\mathrm{B}\,G).((\mathrm{B}\,f)^=(q) = \mathrm{refl}) \\ &= \Sigma(x : G).(f(x) = 0) \qquad = \ker f\end{aligned}$$

from which we deduce $\ker \mathrm{B} f = \mathrm{B} \ker f$ by uniqueness of deloopings. The fact that $\mathrm{B}\, i$ is the canonical inclusion is routine check. □

A diagram of groups

$$1 \longrightarrow F \xrightarrow{f} G \xrightarrow{g} H \longrightarrow 1$$

is a *short exact sequence* when f is injective, $\ker g = \operatorname{im} f$, and g is surjective. By delooping, such data induces a fiber sequence, which can be though of as stating that the delooping functor is exact:

Lemma 12. *Given a short exact sequence as above, the diagram*

$$\mathrm{B} F \xhookrightarrow{\mathrm{B} f} \mathrm{B} G \xrightarrow{\mathrm{B} g} \mathrm{B} H$$

is a fiber sequence.

Proof. We have

$$\begin{aligned} \ker(\mathrm{B}\, g) &= \mathrm{B}(\ker g) && \text{by Lemma 11} \\ &= \mathrm{B}(\operatorname{im} f) && \text{by exactness in the middle} \\ &= \mathrm{B}\, F && \text{because } f \text{ is injective.} \end{aligned}$$
□

5.4 Join of fiber sequences. The join operation also operates on fiber sequences as follows, which allows reformulating the Milnor construction in a concise way. Any two fiber sequences $F \hookrightarrow A \overset{f}{\twoheadrightarrow} B$ and $F' \hookrightarrow A' \overset{f'}{\twoheadrightarrow} B'$ induce, by the join operation and Theorem 5, a new fiber sequence $F * F' \hookrightarrow A *_B A' \twoheadrightarrow B$. In particular, if we start with a fiber sequence

$$F \hookrightarrow A \xrightarrow{f} B$$

and iterate this operation, we obtain a family of fiber sequences

$$F^{*n} \hookrightarrow \operatorname{im}^n(f) \xrightarrow{f^{*n}} B$$

which converges to the fiber sequence

$$\|F\|_{-1} \hookrightarrow \operatorname{im}(f) \xrightarrow{f^{*\infty}} B$$

Proposition 13. *When F has a point, $f^{*\infty}$ is an equivalence.*

Proof. Since F has a point, we have $\|F\|_{-1} = 1$ and $f^{*\infty}$ is thus an equivalence by Lemma 9. □

Example 14. Consider again the map $f : 1 \to \mathrm{B}\,\mathbb{Z}_2$ of Example 8. We have seen that $\ker f = \mathbb{Z}_2$. By the above reasoning, we thus have exact sequences

$$\mathbb{Z}_2^{*(n+1)} \hookrightarrow \operatorname{im}^{n+1}(f) \xrightarrow{f^{*(n+1)}} \mathrm{B}\,\mathbb{Z}_2$$

which can be rewritten as

$$\mathrm{S}^n \hookrightarrow \mathbb{R}\mathrm{P}^n \xrightarrow{\ell^n} \mathrm{B}\,\mathbb{Z}_2$$

Since S^n is $(n-1)$-connected, by (1) we deduce that ℓ^n is $(n-1)$-connected. Moreover, by Proposition 13, we obtain an equality $\mathbb{R}\mathrm{P}^\infty = \mathrm{B}\,\mathbb{Z}_2$ at the limit.

5.5 Fibrations and type families. There is a well-known correspondence, due to Grothendieck, between fibrations and pseudo-presheaves (pseudo-functors from a small category to **Cat**), which admits many variants and extensions (such as the correspondence between discrete fibrations and presheaves) [22]. This type of correspondence also has an impersonation in the context of type theory as follows.

Given a type B, a *type over B* is a type A together with a map $f : A \to B$. A *type family* indexed by B is a function $B \to \mathcal{U}$. We can define functions between the corresponding types, in both directions:

$$\varphi : \Sigma(A : \mathcal{U}).(A \to B) \rightleftarrows (B \to \mathcal{U}) : \psi$$

respectively defined by $\varphi(A, f) :\equiv \mathrm{fib}_f$, i.e. the function φ associates to any type over B its fiber, and $\psi(F) :\equiv (\Sigma B.F, \pi)$, i.e. the function ψ associates to any type family the first projection from the associated total space. Moreover, these two functions can be checked to form an equivalence [24, Section 4.8]: given (A, f) a type over B, we have

$$\psi(\varphi(A, f)) = (\Sigma B.\, \mathrm{fib}_f, \pi) = (A, f)$$

and given a type family F index by B, we have

$$\varphi(\psi(F)) = \mathrm{fib}_\pi = F$$

with $\pi : \Sigma B.F \to B$ the projection.

Theorem 15. *For any type B, the types $\Sigma(A : \mathcal{U}).(A \to B)$ and $B \to \mathcal{U}$ are equivalent (and thus equal by univalence).*

Suppose given a group G and fix $B :\equiv \mathrm{B}\, G$. The above theorem then states an equivalence between morphisms $f : A \to \mathrm{B}\, G$ and actions of G. In particular, a map $f : E \to \mathrm{B}\, G$ is the same thing as an action of G on $\ker f$, and its homotopy quotient $(\ker f) /\!\!/ G$ is E. This can also be reformulated as follows in terms of fiber sequence:

Proposition 16 (action-fibration duality). *The data of an action of G on a type X whose homotopy quotient is Y is equivalent to the data of a fiber sequence*

$$X \hookrightarrow Y \twoheadrightarrow \mathrm{B}\, G$$

More precisely, given an action $f : \mathrm{B}\, G \to \mathcal{U}$ (on $f(\star)$) we have a fiber sequence

$$f(\star) \xhookrightarrow{\kappa} f(\star) /\!\!/ G \xrightarrow{\pi} \mathrm{B}\, G$$

where κ is the quotient map and π is the canonical projection (see Section 3.2). Conversely, given a fiber sequence

$$X \hookrightarrow Y \xrightarrow{g} \mathrm{B}\, G$$

we have the action $\mathrm{fib}_g : \mathrm{B}\, G \to \mathcal{U}$ of G on X (and Y is the homotopy quotient). The two constructions are mutually inverse of each other.

Proof. Suppose given two types F and B. We write FS_B (resp. FS_B^F) for the type of fiber sequences with B as base space (resp. also with F as fiber). By using the universal property of pullbacks, the type FS_B can be shown to be equivalent to the type $\Sigma(E : \mathcal{U}).(E \to B)$ of types over B, and thus to the type of families $B \to \mathcal{U}$ by Theorem 15, see also [21, Proposition 2.3.9]. This proof can then be refined to show that the type FS_B^F is equivalent to the type $\Sigma(f : E \to B).(f(\star) = F)$. The desired result then follows by taking $F :\equiv X$ and $B :\equiv \mathrm{B}\, G$. □

The join of fiber sequences introduced in Section 5.4, thus induces, when the base is $\mathrm{B}\,G$, an operation which allows combining two actions $f, g : \mathrm{B}\,G \to \mathcal{U}$ as an action which operates on $f(\star) * g(\star)$ by making the two actions operate coordinatewise.

This Grothendieck correspondence between the two points of view is often useful (e.g. the descent property on fibrations corresponds to the flattening lemma for type families), and some operations are more easily performed on one side or the other. In particular, in our context, instead of using Theorem 5, it will turn out to be more convenient to consider a dual version, that we prove in Theorem 24.

5.6 Recovering covering spaces. A classical theorem in algebraic topology states that given an action of a group G on a topological space X, which is free and properly discontinuous, the quotient map $X \to X/G$ is covering with G as fiber [8, Proposition 1.40]. In homotopy type theory, this translates as the existence of the following fiber sequence, which is a variant of Proposition 16 (the freeness and proper discontinuity conditions do not make sense and are not required, intuitively because we are working up to homotopy).

Proposition 17. *Given a group G and an action $f : \mathrm{B}\,G \to \mathcal{U}$ on a pointed type $X := f(\star)$, the fiber of the quotient map $q : X \to X /\!\!/ G$ is G, so we have a fiber sequence*

$$G \xhookrightarrow{i} X \xrightarrow{q} X /\!\!/ G$$

where i is the map defined for $x : G$ as

$$i(x) := f^{\to}(\mathrm{d}_G^{\leftarrow}(x))(\star_X)$$

which sends an element $x : G$, which can be seen as a path $p : \star = \star$ of $\mathrm{B}\,G$, to the element of X obtained by transporting the distinguished element $\star$ of X along this path.

Proof. The homotopy quotient $X /\!\!/ G$ is, by definition, $\Sigma(\mathrm{B}\,G).f$ and the quotient map sends $x : X$ to $q(x) := (\star_{\mathrm{B}\,G}, x)$. We have

$$\begin{aligned}
\mathrm{fib}_q(\star_{\mathrm{B}\,G}, \star_X) &= \Sigma(x : X).((\star_{\mathrm{B}\,G}, \star_X) = q(x)) \\
&= \Sigma(x : X).((\star_{\mathrm{B}\,G}, \star_X) = (\star_{\mathrm{B}\,G}, x)) \\
&= \Sigma(x : X).\Sigma(p : \star_{\mathrm{B}\,G} = \star_{\mathrm{B}\,G}).(\star_X =^f_p x) \\
&= \Sigma(x : X).\Sigma(a : G).(i(a) = x) \\
&= \Sigma(a : G).\Sigma(x : X).(i(a) = x) \\
&= \Sigma(a : G).1 \qquad\qquad = G
\end{aligned}$$

and the projection map is the projection map from $\Sigma X.\,\mathrm{fib}_i$ to X, which is i by Theorem 15. □

6 Construction of lens spaces

6.1 Traditional definition of lens spaces. In this section, we recall the definition of lens spaces (in the setting of classical algebraic topology), which is originally due to Tietze [23]. Modern and accessible definitions of those can also be found in [8, Example 1.43] and [7, Chapter V]. We suppose fixed a natural number m: although the notations do not mention it, the constructions below depend on this natural number.

Given a natural number n, we can view the sphere S^{2n-1} as the subset of $\mathbb{C}^n$ consisting of points whose euclidean norm is 1:

$$\mathrm{S}^{2n-1} = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; |z_1|^2 + \ldots + |z_n|^2 = 1\right\}$$

Given $l_1, \ldots, l_n \in \mathbb{N}$ prime with m, we have a free action

$$\rho_{l_1,\ldots,l_n} : \mathbb{Z}_m \to \mathrm{Aut}(\mathrm{S}^{2n-1})$$

of $\mathbb{Z}_m$ on S^{2n-1} which rotates each z_i by an angle of $2\pi l_i/m$: the image of the generator $1 \in \mathbb{Z}_m$ is

$$\rho_{l_1,\ldots,l_n}(1)(z_1, \ldots, z_n) = (\mathrm{e}^{\frac{2i\pi l_1}{m}} z_1, \ldots, \mathrm{e}^{\frac{2i\pi l_n}{m}} z_n)$$

We then define the *lens space* $L(l_1, \ldots, l_n)$ as the (strict) quotient $\mathrm{S}^{2n-1}/\rho_{l_1,\ldots,l_n}$. The action $\rho_{l_1,\ldots,l_n}$ being free, the quotient map $\mathrm{S}^{2n-1} \to L(l_1, \ldots, l_n)$ is thus covering, with $\mathbb{Z}_m$ as fiber [8, Proposition 1.40], i.e. we have a fiber sequence

$$\mathbb{Z}_m \hookrightarrow L(l_1, \ldots, l_n) \twoheadrightarrow \mathrm{S}^{2n-1}$$

As such, it induces a long exact sequence in homotopy groups [8, Theorem 4.41]:

$$\cdots \to \pi_{k+1}(\mathrm{S}^{2n-1}) \to \pi_k(\mathbb{Z}_m) \to \pi_k(L) \to \pi_k(\mathrm{S}^{2n-1}) \to \cdots$$

where L is a shorthand for $L(l_1, \ldots, l_m)$. We have $\pi_0(\mathbb{Z}_m) = \mathbb{Z}_m$, $\pi_k(\mathbb{Z}_m) = 0$ for $k > 0$, and $\pi_k(\mathrm{S}^{2n-1}) = 0$ for $k < 2n-1$, from which we deduce, using the above exact sequence, that $\pi_1(L) = \mathbb{Z}_m$ and $\pi_k(L) = 0$ for $0 \le k < 2n-1$ with $k \neq 1$.

Moreover, given l_{n+1} prime with m, there is a canonical morphism from $L(l_1, \ldots, l_n)$ to $L(l_1, \ldots, l_n, l_{n+1})$, which adds 0 as last coordinate. This allow us to define L^∞, given an infinite sequence $l_1, l_2, \ldots$ of integers prime with m, as the colimit of the diagram

$$L^0 \longrightarrow L(l_1) \longrightarrow L(l_1, l_2) \longrightarrow \cdots$$

Since taking homotopy groups commutes with sequential colimits [16, Chapter 9, Section 4], we have

$$\pi_k(L^\infty) = \begin{cases} \mathbb{Z}_m & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The space L^∞ is therefore a $K(\mathbb{Z}_m, 1)$, or equivalently a $\mathrm{B}\,\mathbb{Z}_m$. As such, it does not depend on the choice of the parameters l_i, by unicity of the delooping, hence the notation. Lens spaces admit a CW-structure with one cell in every dimension $k \le 2n-1$ [8, Example 2.43], thus L^∞ inherits a CW structure with one cell in every dimension. The associated chain complex is

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\mu_m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mu_m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

where the map μ_m denotes the multiplication by m. Hence, the homology groups $H_n(\mathbb{Z}_m)$ are: $\mathbb{Z}$ for $n = 0$, 0 for n non-zero even, and $\mathbb{Z}_m$ for n odd.

6.2 Definition in homotopy type theory. We now turn to the definition of lens spaces in type theory. Given an integer l, consider the group morphism

$$\varphi^l : \mathbb{Z} \to \mathbb{Z}_m$$

sending 1 to l. When l is prime to m, we have a short exact sequence

$$\mathbb{Z} \xhookrightarrow{\mu_m} \mathbb{Z} \xrightarrow{\varphi^l} \mathbb{Z}_m \tag{2}$$

By delooping, and Lemma 12, we thus have a fiber sequence

$$\mathrm{S}^1 \xhookrightarrow{\mathrm{B}\,\mu_m} \mathrm{S}^1 \xrightarrow{\mathrm{B}\,\varphi^l} \mathrm{B}\,\mathbb{Z}_m \tag{3}$$

(recall that $\mathrm{B}\,\mathbb{Z} = \mathrm{S}^1$). By Proposition 16, we can view (3) as an action of $\mathbb{Z}_m$ on S^1 (which turns the circle by a rotation of $2\pi/m$)

and whose homotopy quotient is S^1. Therefore S^1 is the first lens space.

By iterated join, see Section 5.4, we have, for integers $l_1, \ldots, l_n$, and induced fiber sequence

$$(\mathrm{S}^1)^{*n} \xhookrightarrow{\mathrm{B}\,\mu_m^{*n}} (\mathrm{S}^1)^{*_{\mathrm{B}\,\mathbb{Z}_m} n} \xtwoheadrightarrow{\mathrm{B}\,\varphi^{l_1} * \ldots * \mathrm{B}\,\varphi^{l_n}} \mathrm{B}\,\mathbb{Z}_m$$

Again, this can be viewed as an action of $\mathbb{Z}_m$ on the type S^{2n-1} (which is equal to $(\mathrm{S}^1)^{*n}$), corresponding to the action $\rho_{l_1,\ldots,l_n}$ of Section 6.1. By Proposition 16, its homotopy quotient is the space $(\mathrm{S}^1)^{*_{\mathrm{B}\,\mathbb{Z}_m} n}$. This justifies the following definition of lens spaces.

Definition 18. Given integers $l_1, \ldots, l_n$, relatively prime to m, the associated *lens space* is the type

$$L(l_1, \ldots, l_n) := (\mathrm{S}^1)^{*_{\mathrm{B}\,\mathbb{Z}_m} n}.$$

Suppose given an infinite sequence of integers l_i, with $i \in \mathbb{N}$, all relatively prime to m. For every n there is a canonical inclusion

$$L(l_1, \ldots, l_n) \to L(l_1, \ldots, l_n, l_{n+1})$$

induced by the join and we write L^∞ for the associated inductive limit. By Proposition 13, we have:

Theorem 19. *The type L^∞ is a delooping of $\mathbb{Z}_m$.*

By uniqueness of deloopings (Lemma 34), this type does not depend on the choice of the sequence l_i. In particular, taking $l_i = 1$ for every i is a reasonable canonical choice.

6.3 The case of non-relatively prime parameters. One can wonder what happens when we consider integer parameters l_i which are not supposed to be prime with m. For l non-prime with m, the sequence (2) is not exact, and therefore the sequence (3) is not exact in general. In fact, the proper generalization is the following. Given a natural number k and a space A, we write kA for the space $A \sqcup \ldots \sqcup A$ with k copies of A.

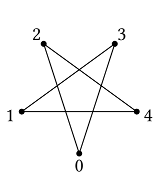
Proposition 20. *Given an arbitrary integer l, with the notations of (3), we have* $\mathrm{fib}_{\mathrm{B}\,\varphi^l} = k\,\mathrm{S}^1$ *with* $k = \mathrm{pgcd}(l, m)$, *i.e. we have a fiber sequence*

$$k\,\mathrm{S}^1 \xhookrightarrow{k\,\mathrm{B}\,\mu_{m/k}} \mathrm{S}^1 \xtwoheadrightarrow{\mathrm{B}\,\varphi^l} \mathrm{B}\,\mathbb{Z}_m$$

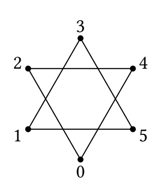
Example 21. Consider situations where $m = 5$ and $m = 6$ respectively, and $l = 2$ in both cases (thus $k = 1$ and $k = 2$, respectively). By (the proof in appendix of) Proposition 20, the kernel of $\mathrm{B}\,\varphi^l : \mathrm{S}^1 \to \mathrm{B}\,\mathbb{Z}_m$ is the coequalizer of

$$\mathbb{Z}_m \underset{\lambda i.(i+l)}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbb{Z}_m$$

For the two values of m, it can be pictured as



$m = 5$



$m = 6$

and is respectively isomorphic to S^1 and $2\,\mathrm{S}^1$.

Contrarily to the situation where l is prime with m, the fiber sequence of Proposition 20 is not necessarily the delooping of an exact sequence of groups. Namely, with $k := \mathrm{pgcd}(l, m)$, the type $k\,\mathrm{S}^1$ has k connected components and is thus not the delooping of a group when $k > 1$. However, the computations still go on. Indeed, by iterated join, any sequence of integers $(l_1, \ldots, l_n)$ induces a fiber sequence

$$k_1\,\mathrm{S}^1 * \ldots * k_n\,\mathrm{S}^1 \xhookrightarrow{k_1\,\mathrm{B}\,\mu_{m/k_1} * \ldots * k_n\,\mathrm{B}\,\mu_{m/k_n}} (\mathrm{S}^1)^{*_{\mathrm{B}\,\mathbb{Z}_m} n} \xtwoheadrightarrow{\mathrm{B}\,\varphi^{l_1} * \ldots * \mathrm{B}\,\varphi^{l_n}} \mathrm{B}\,\mathbb{Z}_m$$

which converges toward the sequence $1 \hookrightarrow \mathrm{B}\,\mathbb{Z}_m \twoheadrightarrow \mathrm{B}\,\mathbb{Z}_m$ (the second map is an identity) when n goes to the infinity, so that the spaces $(\mathrm{S}^1)^{*_{\mathrm{B}\,\mathbb{Z}_m} n}$ still play a role analogous to the one of lens spaces. However, an explicit description of those is difficult to achieve contrarily to the case of lens spaces, see Section 7.

6.4 A non-minimal resolution. One can also wonder what would have happened if we had naively generalized the construction for $\mathrm{B}\,\mathbb{Z}_2$ as the infinite real projective space performed in [5] and recalled in Example 8. We explain here that it also gives rise to a model for $\mathrm{B}\,\mathbb{Z}_m$, although much larger than the one of lens spaces.

The space $\mathrm{B}\,\mathbb{Z}_m$ is pointed and we write $f : 1 \to \mathrm{B}\,\mathbb{Z}_m$ for the pointed map. The kernel $\ker f$ is the pullback of f along itself, and thus $\Omega\,\mathrm{B}\,\mathbb{Z}_m$ (see Section 2.5), hence $\mathbb{Z}_m$. We thus have a fiber sequence

$$\mathbb{Z}_m \hookrightarrow 1 \xtwoheadrightarrow{f} \mathrm{B}\,\mathbb{Z}_m$$

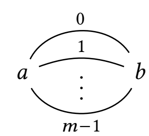
By iterated join on the map f, one obtains a family of maps

$$f^{*n} : X_n \to \mathrm{B}\,\mathbb{Z}_m$$

where the type X_n, which is the source of this map, i.e. the n-th iterated join of 1 relative to f, converges to $\mathrm{B}\,\mathbb{Z}_m$ if we take the inductive limit when n goes to infinity, see Section 5.4.

While this does indeed produce a cellular delooping of $\mathbb{Z}_m$, the resulting model is much larger than the one of lens spaces. Already, for $n = 2$, the space X_2 we obtain is the pushout on the left, which can be pictured as on the right:

$$\begin{array}{ccc} \mathbb{Z}_m & \longrightarrow & 1 \\ \downarrow & & \vdots \\ 1 & \cdots\cdots\to & X_2 \end{array}$$



We can see that it is much "larger" (in terms of number of cells) than the sphere S^1, which is the first lens space. Similarly, higher spaces X_n are much larger than the corresponding lens spaces; in some sense, we are computing the free resolution of $\mathbb{Z}_m$, which is not at all minimal in terms of number of cells, see [14, Section IV.5]. In contrast, lens spaces, which have one cell in every dimension, provide a minimal cellular resolution of $\mathbb{Z}_m$: the homology of $\mathbb{Z}_m$ (see Section 6.1) shows that we cannot build a delooping of $\mathbb{Z}_m$ with less cells. Indeed this homology is non-trivial in degree n for $n = 0$ or n odd, hence it implies that a given cellular model of $\mathrm{B}\,\mathbb{Z}_m$ (whose cellular homology is the same as the homology of $\mathbb{Z}_m$) should have *at least* one cell in each dimension n for $n = 0$ or n odd. Moreover, the fact that these groups have torsion implies that the cellular model should also have *at least* one cell in each dimension n for n even, otherwise the homology groups would be free since no relation in odd degree would be added. Hence a cellular model of $\mathrm{B}\,\mathbb{Z}_m$ have *at least* one cell in every dimension.

7 Cellularity

We have already mentioned that the general constructions for deloopings of groups (by HITs or torsors) do not easily allow eliminating to arbitrary types. We show here that our alternative model of $\mathrm{B}\,\mathbb{Z}_m$, provided by infinite lens spaces constructed above, admits an inductive description. This constitutes a type-theoretic counterpart of the CW-complex structure of topological lens spaces, and induce an induction principle which allows eliminating into general types.

We begin by recalling the following classical flattening lemma for pushouts, see [21, Lemma 2.2.5] for a detailed proof.

Lemma 22 (Flattening for pushouts). *Consider a pushout square*

$$\begin{array}{ccc} X & \xrightarrow{g} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow[i]{} & A \sqcup_X B \end{array}$$

with $p : i \circ f = j \circ g$ witnessing for its commutativity, together with a type family $P : A \sqcup_X B \to \mathcal{U}$. Then the following square of total spaces is also a pushout

$$\begin{array}{ccc} \Sigma X.(P \circ i \circ f) & \xrightarrow{\Sigma g.e} & \Sigma B.(P \circ j) \\ {\scriptstyle \Sigma f.(\lambda\_.\,\mathrm{id})}\downarrow & & \downarrow{\scriptstyle \Sigma j.(\lambda\_.\,\mathrm{id})} \\ \Sigma A.(P \circ i) & \xrightarrow[\Sigma i.(\lambda\_.\,\mathrm{id})]{} & \Sigma(A \sqcup_X B).P \end{array}$$

where $e : (x : X) \to P(j(g(x))) \to P(i(f(x)))$ is the canonical morphism induced by p, i.e. $e(x) :\equiv P^{\to}(p(x))$.

Given a pointed type A, the space $\Sigma(x : A).(\star = x)$ is always contractible. The following result is a dependent generalization of this fact, and is used later on:

Lemma 23. *Given a type X, a family $P : X \to \mathcal{U}$, elements $a, b : X$, a path $p : a = b$, and an element $x : P(a)$, the type $\Sigma(y : P(b)).(x =^P_p y)$ is contractible.*

Proof. By path induction, it is enough to show the contractibility of

$$\Sigma(y : P(a)).(x =^P_{\mathrm{refl}} y)$$

i.e. of the type

$$\Sigma(y : P(a)).(x = y)$$

which is known to be true [24, Lemma 3.11.8]. □

7.1 Sigma types preserve joins. We can now show that sigma types preserve joins relative to morphisms. Under the equivalence of Section 5.5 between fibrations and type families, this is a counterpart of the fact that fibers preserve joins, already encountered in Theorem 5.

Theorem 24. *Suppose given maps $f : A \to X$ and $g : B \to X$, a type family $P : X \to \mathcal{U}$. Then we have*

$$\Sigma(A *_X B).\,(P \circ (f * g)) = (\Sigma A.\,(P \circ f)) *_{\Sigma X.P} (\Sigma B.\,(P \circ g))\,.$$

where $A *_X B$ is the source of the join of f and g, and the type on the right is the source of the join of the maps

$$\Sigma f.(\lambda\_.\,\mathrm{id}) : \Sigma A.(P \circ f) \to \Sigma X.P$$
$$\Sigma g.(\lambda\_.\,\mathrm{id}) : \Sigma B.(P \circ g) \to \Sigma X.P$$

Proof. By definition of the join of f and g, we have a diagram

$$\begin{array}{ccccc} A \times_X B & \xrightarrow{\pi_2} & B & & \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \iota'} & \searrow{\scriptstyle g} & \\ A & \xrightarrow[\iota]{} & A *_X B & & \\ & \searrow{\scriptstyle f} & & \xrightarrow{f * g} & X \end{array}$$

where the outer square is a pullback and the inner square is a pushout. We denote by

$$p_1 : \iota \circ (f * g) = f \quad p_2 : \iota' \circ (f * g) = g \quad p : f \circ \pi_1 = g \circ \pi_2$$

the equalities witnessing for commutativity of various subdiagrams. By the flattening lemma for pushouts (Lemma 22) applied to the type family

$$P \circ (f * g) : A *_X B \to \mathcal{U}$$

we have a pushout

$$\begin{array}{ccc} \Sigma(A \times_X B).P \circ (f * g) \circ \iota \circ \pi_1 & \xrightarrow{\Sigma \pi_2.e} & \Sigma B.P \circ (f * g) \circ \iota' \\ {\scriptstyle \Sigma \pi_1.(\lambda\_.\,\mathrm{id})}\downarrow & & \downarrow{\scriptstyle \Sigma \iota'.(\lambda\_.\,\mathrm{id})} \\ \Sigma A.P \circ (f * g) \circ \iota & \xrightarrow[\Sigma \iota.(\lambda\_.\,\mathrm{id})]{} & \Sigma(A *_X B).P \circ (f * g) \end{array}$$

where e is the canonical function induced by transport along the canonical composite equality

$$(f * g) \circ \iota \circ \pi_1 = f \circ \pi_1 = g \circ \pi_2 = (f * g) \circ \iota' \circ \pi_2$$

induced by the above ones. By using the identities p_1, p_2 and p, this simplifies as

$$\begin{array}{ccc} \Sigma(A \times_X B).P \circ f \circ \pi_1 & \xrightarrow{\Sigma \pi_2.(\lambda x.P^{\to}(p(x)))} & \Sigma B.P \circ g \\ {\scriptstyle \Sigma \pi_1.(\lambda\_.\,\mathrm{id})}\downarrow & & \downarrow{\scriptstyle \Sigma \iota'.P^{\leftarrow}(p_2)} \\ \Sigma A.P \circ f & \xrightarrow[\Sigma \iota.P^{\leftarrow}(p_1)]{} & \Sigma(A *_X B).P \circ (f * g) \end{array} \tag{4}$$

In order to conclude, it is enough to show that the upper-left span coincides with the upper-left span of the following pullback diagram:

$$\begin{array}{ccc} (\Sigma A.(P \circ f)) \times_{\Sigma X.P} (\Sigma B.(P \circ g)) & \xrightarrow{\pi_2} & \Sigma B.(P \circ g) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \Sigma g.(\lambda\_.\,\mathrm{id})} \\ \Sigma A.(P \circ f) & \xrightarrow[\Sigma f.(\lambda\_.\,\mathrm{id})]{} & \Sigma X.P \end{array} \tag{5}$$

Namely, the pushout of the upper-left span of (5) is, by definition of the join,

$$(\Sigma A.(P \circ f)) *_{\Sigma X.P} (\Sigma B.(P \circ g))$$

and it will coincide with the pushout of the upper-left span of (4), which is

$$\Sigma(A *_X B).P \circ (f * g).$$

We show that the upper-left objects coincide (the fact that the map coincide can be shown by transporting along the resulting equality, which is left to the reader). We namely have

$$\begin{aligned}
&(\Sigma A.(P \circ f)) \times_{\Sigma X.P} (\Sigma B.(P \circ g)) \\
&\quad = \Sigma((a,x) : \Sigma A.(P \circ f)).\Sigma((b,y) : \Sigma B.(P \circ g)). \\
&\qquad (\Sigma f.(\lambda_.\,\mathrm{id}))(a,x) = (\Sigma g.(\lambda_.\,\mathrm{id}))(b,y) \\
&\quad = \Sigma(a : A).\Sigma(x : P(f(a))).\Sigma(b : B).\Sigma(y : P(g(b))). \\
&\qquad \Sigma(p : f(a) = g(b)).(x =^P_p y) \\
&\quad = \Sigma(a : A).\Sigma(b : B).\Sigma(p : f(a) = g(b)).\Sigma(x : P(f(a))). \\
&\qquad \Sigma(y : P(g(b))).(x =^P_p y) \\
&\quad = \Sigma(a : A).\Sigma(b : B).\Sigma(p : f(a) = g(b)).\Sigma(x : P(f(a))).1 \\
&\quad = \Sigma(a : A).\Sigma(b : B).(f(a) = g(b)) \times P(f(a)) \\
&\quad = \Sigma(A \times_X B).(P \circ f \circ \pi_1)
\end{aligned}$$

Most of the steps of this reasoning can easily be justified by associativity and commutativity properties of Σ-types. The only step which is not of this nature, in the middle, uses the fact that the type

$$\Sigma(y : P(g(b))).(x =^P_p y)$$

is contractible for $a : A$, $b : B$, $p : f(a) = g(b)$ and $x : P(f(a))$, which follows from Lemma 23. ◻

With the notations of the above theorem, we have the following interesting particular case. When $\Sigma X.P$ is contractible (i.e. when P is the universal cover of X), we have

$$\Sigma(A *_X B).\,(P \circ (f * g)) = (\Sigma A.\,(P \circ f)) * (\Sigma B.\,(P \circ g))$$

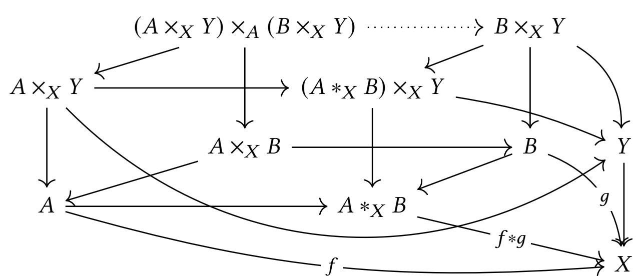
This is essentially the result which is used in [5, Theorem III.4].

Under the fibration-family correspondence of Section 5.5, the above theorem can be reformulated as follows:

Theorem 25. *Suppose given maps $f : A \to X$ and $g : B \to X$ and a map $h : Y \to X$, we have*

$$(A *_X B) \times_X Y = (A \times_X Y) *_Y (B \times_X Y)$$

where the pullbacks are taken over the expected maps. This can be summarized by the diagram



where the bottom diagram is the pushout of pullback defining the join, the vertical squares are pullbacks, the dotted arrow is obtained by universal property of pullback, and the theorem states that the top square is a pushout.

The Theorem 25, which will not be needed in the following, can also directly be proved by descent. As a particular case, when Y is a point (or, more generally, contractible), the theorem simplifies as

$$\mathrm{fib}_{f*g}(x) = \mathrm{fib}_f(x) * \mathrm{fib}_g(x)$$

where x is the image by h of the point of Y, and we recover Theorem 5.

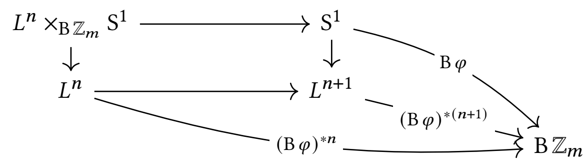
7.2 A characterization of relations. We have seen at the end of Section 6.2 that the construction of lens spaces does not depend on the choice of parameters l_i, so that we can always take $l_i = 1$. In order to simplify notations, we thus write

$$L^n \quad := \quad L(1,\ldots,1)$$

(with n occurrences of 1 on the right) and will only work with such lens spaces in the following (but all constructions would generalize in the expected way to varying parameters l_i). We also write $\varphi : \mathbb{Z} \to \mathbb{Z}_m$ instead of φ^1, so that $L^0 = 0$ and L^{n+1} is obtained as the join

$$L^{n+1} \quad := \quad L^n *_{\mathrm{B}\,\mathbb{Z}_m} \mathrm{S}^1$$

i.e. as a pushout of the following pullback:



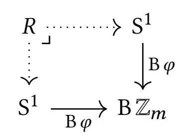
(6)

Our goal in this section is to provide a concrete description of the upper left space, which will play a central role in our recursion principle for $\mathrm{B}\,\mathbb{Z}_m$.

We write

$$R \quad := \quad \Sigma(x : \mathrm{S}^1).\Sigma(y : \mathrm{S}^1).(\mathrm{B}\,\varphi(x) = \mathrm{B}\,\varphi(y))$$

for the pullback $\mathrm{S}^1 \times_{\mathrm{B}\,\mathbb{Z}_m} \mathrm{S}^1$, i.e.



This type comes equipped with a canonical map $\psi : R \to \mathrm{B}\,\mathbb{Z}_m$ (the diagonal of the square). In order to characterize R, we first show that it is a delooping (of its loop space).

Lemma 26. *The type R is a pointed connected groupoid.*

Proof. The distinguished elements of S^1 and $\mathrm{B}\,\mathbb{Z}_m$ induce a distinguished element

$$(\star_{\mathrm{S}^1}, \star_{\mathrm{S}^1}, \mathrm{refl}\ \star_{\mathrm{B}\,\mathbb{Z}_m})$$

in R, making it pointed. Also the type R is a groupoid as a pullback of groupoids [24, Theorem 7.1.8]. We are left with showing that it is connected. Suppose given a point (x, y, r) of R: we want to show that there merely exists a path from the distinguished point of R to it. Since this is a proposition and S^1 is connected, we can suppose given paths $\star = x$ and $\star = y$ in S^1, and, by path induction, those paths can be supposed to be refl . We are thus left with proving $(\star, \star, \mathrm{refl}\ ) = (\star, \star, r)$ for an arbitrary path $r : \Omega\,\mathrm{B}\,\mathbb{Z}_m$. By the characterization of paths in Σ-types [24, Theorem 2.7.2] and transport along identity types [24, Theorem 2.11.3], this amounts to find paths $p, q : \Omega\,\mathrm{S}^1$ together with a proof that

$$r = (\mathrm{B}\,\varphi)^=(p)^{-1} \cdot \mathrm{refl}\ \cdot (\mathrm{B}\,\varphi)^=(q)$$

Since $\Omega\,\mathrm{S}^1 = \mathbb{Z}$ and $\Omega\,\mathrm{B}\,\mathbb{Z}_m = \mathbb{Z}_m$, this amounts to show that the canonical morphism $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}_m$ sending (i, j) to $j - i$ is surjective, which is the case: it admits the map $i \mapsto (0, i)$ as a section. ◻

Remark 27. The property of being connected is not preserved in general under pullbacks. For instance, the pullback of the map $1 \to \mathrm{S}^1$ along itself is $\Omega\,\mathrm{S}^1$, i.e. $\mathbb{Z}$, which is not connected. The proof of the connectedness of R thus had to be specific.

By Theorem 3, we thus have $R = \mathrm{B}\,\Omega\,R$, i.e. R is determined by its loop space, which is a group that can be computed in the following way.

Lemma 28. *The type $\Omega\,R$ is obtained as the following pullback:*

$$\begin{array}{ccc} \Omega\,R & \dashrightarrow & \mathbb{Z} \\ \vdots & \lrcorner & \downarrow{\scriptstyle q} \\ \mathbb{Z} & \xrightarrow[q]{} & \mathbb{Z}_m \end{array}$$

where q is the canonical quotient map. Hence R is a delooping of the following subgroup of $\mathbb{Z}^2$

$$\{(j,k) \in \mathbb{Z} \times \mathbb{Z} \mid k - j = 0 \bmod m\}\,.$$

Proof. Given a pointed type A, the loop space functor can be described as a covariant hom functor (we have $\Omega\,A = \mathrm{S}^1 \to_\star A$, see [24, Lemma 6.2.9]), so it preserves limits. The loop space of R is thus

$$\begin{aligned} \Omega\,R &= \Omega(\mathrm{S}^1 \times_{\mathrm{B}\,\mathbb{Z}_m} \mathrm{S}^1) && \text{by definition of } R \\ &= \Omega\,\mathrm{S}^1 \times_{\Omega\,\mathrm{B}\,\mathbb{Z}_m} \Omega\,\mathrm{S}^1 && \text{by commutation of } \Omega \text{ with pullbacks} \\ &= \mathbb{Z} \times_{\mathbb{Z}_m} \mathbb{Z} \end{aligned}$$

From which we conclude. □

An even more explicit description of this group is:

Lemma 29. $\Omega\,R = \mathbb{Z}^2$.

Proof. $\Omega\,R$ is a commutative subgroup of $\mathbb{Z}^2$, i.e. a sub-$\mathbb{Z}$-module of $\mathbb{Z}^2$. Since $\mathbb{Z}$ is principal and $\mathbb{Z}^2$ is free of rank 2 on $\mathbb{Z}$, $\Omega\,R$ is a free $\mathbb{Z}$-module of rank at most 2 ; but it is also of rank at least 2 because we have a free familly of $\Omega\,R$ with 2 elements (consider for instance $(m,0)$ and $(0,m)$). Thus $\Omega\,R$ is a free $\mathbb{Z}$-module of rank 2, i.e. $\Omega\,R = \mathbb{Z}^2$. □

Corollary 30. *The type R is isomorphic to the torus.*

Proof. The torus is also a delooping of $\mathbb{Z}^2$, so we conclude by unicity of the delooping of a given group. □

Note that once we fix a basis of $\Omega\,R$ (for instance $\{(1,1),(0,m)\}$), we have an explicit isomorphism between the torus and $\Omega\,R$ given by the delooping of the associated "base change" morphism $\mathbb{Z}^2 \to \Omega\,R$.

We write R^{*n} for the iterated product of n instances of the diagonal map ψ:

$$R^{*n} \quad := \quad R *_{\mathrm{B}\,\mathbb{Z}_m} R *_{\mathrm{B}\,\mathbb{Z}_m} \cdots *_{\mathrm{B}\,\mathbb{Z}_m} R$$

(note that this is a dependent join, we should write $R^{*_{\mathrm{B}\,\mathbb{Z}_m} n}$ but this is ugly). As a consequence of Theorem 24, we have the following theorem.

Theorem 31. *We have, for every $n \in \mathbb{N}$,*

$$L^n \times_{\mathrm{B}\,\mathbb{Z}_m} \mathrm{S}^1 = R^{*n}$$

i.e. the upper-left square of (6) is the pushout

$$\begin{array}{ccc} R^{*n} & \longrightarrow & \mathrm{S}^1 \\ \downarrow & \ulcorner & \downarrow \\ L^n & \longrightarrow & L^{n+1} \end{array}$$

Proof. For the base case, we have $L^0 = 0$ and the result is immediate. For the inductive case, we have

$$\begin{aligned} &L^{n+1} \times_{\mathrm{B}\,\mathbb{Z}_m} \mathrm{S}^1 \\ &\quad = \Sigma(x : L^{n+1}).\Sigma(y : \mathrm{S}^1).(\mathrm{B}\,\varphi^{*(n+1)}(x) = \mathrm{B}\,\varphi(y)) \\ &\quad = \Sigma(x : (L^n *_{\mathrm{B}\,\mathbb{Z}_m} \mathrm{S}^1)).\Sigma(y : \mathrm{S}^1).(\mathrm{B}\,\varphi^{*(n+1)}(x) = \mathrm{B}\,\varphi(y)) \\ &\quad = \Big(\Sigma(x : L^n).\Sigma(y : \mathrm{S}^1).(\mathrm{B}\,\varphi^{*n}(x) = \mathrm{B}\,\varphi(y))\Big) *_{\mathrm{B}\,\mathbb{Z}_m} \\ &\qquad \Big(\Sigma(x : \mathrm{S}^1).\Sigma(y : \mathrm{S}^1).(\mathrm{B}\,\varphi(x) = \mathrm{B}\,\varphi(y))\Big) \\ &\quad = (L^n \times_{\mathrm{B}\,\mathbb{Z}_m} \mathrm{S}^1) *_{\mathrm{B}\,\mathbb{Z}_m} R \\ &\quad = R^{*n} *_{\mathrm{B}\,\mathbb{Z}_m} R \\ &\quad = R^{*(n+1)} \end{aligned}$$

Above, in order to go from the second to the third line, we apply Theorem 24 with $A = L^n$, $B = \mathrm{S}^1$, $C = \mathrm{B}\,\mathbb{Z}_m$, $f = (\mathrm{B}\,\varphi)^{*n}$, $g = \mathrm{B}\,\varphi$ and $P : \mathrm{B}\,\mathbb{Z}_m \to \mathcal{U}$ defined by $P(x) := \Sigma(y : \mathrm{S}^1).(x = \mathrm{B}\,\varphi(y))$ which allows to conclude. □

7.3 A recursion principle. Fix an arbitrary type X. As a direct consequence of the above theorem, maps $L^{n+1} \to X$ correspond to triples consisting of a map $L^n \to X$, a map $\mathrm{S}^1 \to X$ and an equality witnessing the commutation of the square

$$\begin{array}{ccc} R^{*n} & \longrightarrow & \mathrm{S}^1 \\ \downarrow & \ulcorner & \downarrow \\ L^n & \longrightarrow & X \end{array}$$

When X is an n-type, we have that maps $\mathrm{B}\,\mathbb{Z}_m \to X$ correspond to maps $L^n \to X$, which can be defined as above, by induction on n. For general types X, we have $\mathrm{B}\,\mathbb{Z}_m = \mathrm{colim}_n\, L^n$ and thus, since hom functors commutes with colimits, we have the following result, which allows computing maps out of the delooping of $\mathbb{Z}_m$.

Theorem 32. *Given a type X, the maps $\mathrm{B}\,\mathbb{Z}_m \to X$ are the limits of maps $L^n \to X$.*

7.4 Applications. Let us mention some possible applications of this result. Given a group G, its n-th integral *cohomology group* is the space $H_n(G) := \|\mathrm{B}\,G \to K(\mathbb{Z},n)\|_0$ of homotopy classes of maps from $\mathrm{B}\,G$ to the space $K(\mathbb{Z},n)$, which is an n-type [12]. Theorem 32 should thus allow to perform explicit computations of cohomology classes of $\mathbb{Z}_m$ of degree n, even when $n > 1$.

As another application, the above theorem allows defining maps from $\mathrm{B}\,\mathbb{Z}_m$ to n-types or types (whereas the HIT definition of [12] only allows eliminating to sets, the case $n = 0$), thus enabling us to define actions of $\mathbb{Z}_m$ on higher homotopy types. Indeed, recall that maps $f : \mathrm{B}\,G \to \mathcal{U}$ with $f(\star) = X$ encodes actions of G on the type X, thus being able to eliminate to universes of (n-)types provide a way to define actions on (n-)types.

8 Delooping dihedral groups

In this section, we explain how to construct the delooping of a group P which admits a group G as (strict) quotient for which we already know a delooping. This situation is very general: in particular, it includes the case where P is the semidirect product of G with another group H. As an illustration, we explain how to construct deloopings for dihedral groups.

8.1 Delooping short exact sequences. Recall from Lemma 12 that any short exact sequence of groups

$$1 \longrightarrow H \longrightarrow P \xrightarrow{f} G \longrightarrow 1 \qquad (7)$$

induces, by delooping, a fiber sequence

$$\mathrm{B}\,H \hookrightarrow \mathrm{B}\,P \xrightarrow{\mathrm{B}\,f} \mathrm{B}\,G$$

(note that this includes in particular the case where P is the semidirect product of G and H, i.e. when the short exact sequence is split). As explained in Section 5.5, in such a situation, we have an action of G on $\mathrm{B}\,H$ given by $\mathrm{fib}_f : \mathrm{B}\,G \to \mathcal{U}$, whose homotopy quotient $\Sigma(x : \mathrm{B}\,G).\,\mathrm{fib}_f(x)$ is $\mathrm{B}\,P$. This thus gives us a way to construct a delooping of P from a delooping of G, as illustrated below.

8.2 Dihedral groups. Given a natural number m, we write D_m for m-th *dihedral group*, which is the group of symmetries of a regular polygon with m sides. This group is generated by a rotation r and a symmetry s, and admits the presentation

$$D_m \quad := \quad \left\langle r, s \mid r^m = 1, s^2 = 1, rs = sr^{m-1} \right\rangle$$

From this presentation, it can be seen that we have a short exact sequence

$$\mathbb{Z}_m \xrightarrow{\kappa} D_m \xrightarrow{\pi} \mathbb{Z}_2 \qquad (8)$$

where the maps are characterized by $\kappa(1) = r$, $\pi(r) = 0$ and $\pi(s) = 1$, see for instance [13, Chapter XII].

By the reasoning of Section 8.1, we have the following delooping for D_m:

$$\mathrm{B}\,D_m \quad := \quad \Sigma(x : \mathrm{B}\,\mathbb{Z}_2).\,\mathrm{fib}_\pi(x)$$

By curryfication, we thus have a recursion principle for $\mathrm{B}\,D_m$. Namely, given a type X, we have

$$\mathrm{B}\,D_m \to X \qquad = \qquad (x : \mathrm{B}\,\mathbb{Z}_2) \to \mathrm{fib}_\pi(x) \to X$$

Here, the arrow on the right can be computed by eliminating from $\mathrm{B}\,\mathbb{Z}_2$ as explained in Theorem 32.

In order to be fully satisfied, we still need a usable description of $\mathrm{fib}_\pi(x)$. This can be achieved as follows. The short exact sequence (8) is split, with the section $\sigma : \mathbb{Z}_2 \to D_n$ such that $\sigma(1) = s$ (the existence of such a splitting corresponds to the fact that D_m is a semidirect product of $\mathbb{Z}_m$ and $\mathbb{Z}_2$). This splitting induces an action $\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_m)$ which is the group morphism determined by $\varphi(1)(k) = -k$. Moreover, writing $\mathcal{U}_{\mathrm{B}\,\mathbb{Z}_m}$ for $\mathcal{U}$ pointed at $\mathrm{B}\,\mathbb{Z}_m$, we have $\mathrm{Aut}(\mathbb{Z}_m) = \Omega\,\mathcal{U}_{\mathrm{B}\,\mathbb{Z}_m}$, thus φ can be seen as an action of G on $\mathrm{B}\,\mathbb{Z}_m$. Its delooping can be shown to coincide with fib_π, so that the action given by the fiber sequence induced by (8) is the delooping of the action induced by the splitting. The point here is that, up to the correspondence between actions on $\mathbb{Z}_m$ and actions on $\mathrm{B}\,\mathbb{Z}_m$, $\mathrm{B}\,\varphi$ is an action on $\mathbb{Z}_m$ (a set), so it can be explicitely defined by using the Finster-Licata construction of $\mathrm{B}\,\mathbb{Z}_2$ [12]. Indeed, Set is a groupoid, so we can define $\mathrm{B}\,\varphi : \mathrm{B}\,\mathbb{Z}_2 \to \mathrm{Set}$ by giving its value on generators and relations. Hence, we have obtained a description of fib_π which is relevant from a computational point of view, allowing us to make use of the recursion principle for $\mathrm{B}\,D_m$. This will be detailed in another paper, along with a general study of delooping of semidirect products.

We should finally mention that the above construction can directly be generalized in order to construct deloopings for all (split) *metacyclic groups*: those are groups P for which there is a short exact sequence (7) with G and H cyclic, and include dihedral, quasidihedral and dicyclic groups as particular cases.

9 Conclusion and future work

We believe that the general recursion principles we gave for $\mathrm{B}\,\mathbb{Z}_m$ in Section 7 and for $\mathrm{B}\,D_m$ in Section 8 will be useful to perform computations with cyclic and dihedral groups involving non-trivial higher geometry: for instance, computing cohomology classes or defining higher actions of these groups, as explained in Section 7.4. This would be a great step forward in synthetic group theory since homotopy type theory, as opposed to plain type theory, is mainly about reasoning with objects which are non-trivial from a homotopical perspective. This is left for future work.

We also plan to use other geometric models in order to obtain cellular models of delooping for other classical groups, while defining in the process some important spaces of algebraic topology which are not formalized yet. In particular, we are working on the construction in homotopy type theory of the hypercubical manifold (a space studied by Poincaré during his quest to define homology spheres [19]), and higher variants of this space using the join construction and the action-as-fibration paradigm.

As mentioned at the end of Section 8, we will further investigate the delooping of semidirect products of groups. This should provide us with a modular way of constructing cellular models of deloopings from already known ones.

Finally, we plan to have an Agda formalization of our work (the implementation of lens spaces in homotopy type theory and the recursion principle of $\mathrm{B}\,\mathbb{Z}_m$ associated). This would first require developing basic constructions such as the join of maps, which are unfortunately not present at the moment in the standard cubical library.

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A Additional properties on deloopings

We begin by detailing the proof of Lemma 1.

Proof of Lemma 1. We write $\Omega^{-1} f$ for the type of deloopings of f, i.e. the type of pointed maps $g : A \to_\star B$ whose looping is f, i.e.

$$\Sigma(g : A \to B).\Sigma(\star_g : g(\star_A) = \star_B).\,\Omega\, g = f$$

This type can be shown to be equivalent to the type

$$\Sigma(g : A \to B).\Pi(a : A).C(a, g(a)) \tag{9}$$

where, for $a : A$ and $b : B$, the type $C(a, b)$ is defined as

$$\Sigma(\pi : (a = \star_A) \to (b = \star_B)).D(a, \pi)$$

with $D(a, \pi)$ defined as

$$\Pi(p : a = \star_A).\Pi(q : \star_A = \star_A).\pi(p \cdot q) = \pi(p) \cdot f(q) \tag{10}$$

see [25, Lemma 9]. This can be explained as follows. Any delooping $g : A \to_\star B$ of f induces a function $\pi : (a = \star_A) \to (b = \star_B)$, defined by $\pi(p) = g^=(p) \cdot \star_g$, which satisfies, for $p : a = \star_A$ and $q : \star_A = \star_A$,

$$\begin{aligned}\pi(p \cdot q) &= g^=(p \cdot q) \cdot \star_g\\ &= g^=(p) \cdot g^=(q) \cdot \star_g\\ &= g^=(p) \cdot \star_g \cdot \Omega\, g(q)\\ &= \pi(p) \cdot \Omega\, g(q)\end{aligned}$$

and thus the type $C(a, g(a))$ is inhabited for any $a : A$. The above equivalence shows that this characterizes the deloopings of f: those are precisely the functions g such that, for any $a : A$, there exists a function π satisfying the above property.

From the equivalence with (9), by type-theoretic choice, $\Omega^{-1} f$ is also equivalent to

$$\Pi(a : A).\Sigma(b : B).C(a, b) \tag{11}$$

Because B is a groupoid, the last equality in the definition (10) of $D(a, \pi)$ is a proposition and thus also $D(a, \pi)$ itself. We show below that we have, for any $b : B$,

$$C(\star_A, b) \quad = \quad (b = \star_A) \tag{12}$$

From there we deduce that $\Sigma(b : B).C(\star_A, b)$ is equivalent to $\Sigma(b : B).(b = \star_A)$ and is thus contractible. By path induction, we thus have

$$\Pi(a : A).(a = \star_A) \to \mathrm{isContr}(\Sigma(b : B).C(a, b))$$

and thus

$$\Pi(a : A).\|a = \star_A\|_{-1} \to \mathrm{isContr}(\Sigma(b : B).C(a, b))$$

because being contractible is a proposition. Since A is connected, we deduce that $\Sigma(b : B).C(a, b)$ is contractible for every $a : A$. The type (11) is thus contractible and thus also $\Omega^{-1} f$, which is what we wanted to show.

We are left with showing (12). It can be shown that for any suitably typed function π, the type $D(\star_A, \pi)$ is equivalent to

$$\Pi(q : \star_A = \star_A).\pi(q) = \pi(\mathrm{refl}) \cdot f(q)$$

Namely the former implies the later as a particular case and, conversely, supposing the second one, we have for $p : \star_A = \star_A$ and $q : \star_A = \star_A$,

$$\pi(p \cdot q) = \pi(\mathrm{refl}) \cdot f(p \cdot q) = \pi(\mathrm{refl}) \cdot f(p) \cdot f(q) = \pi(p) \cdot f(q)$$

because f preserves composition. From there follows easily (12), i.e. that

$$\Sigma(\pi : (\star_A = \star_A) \to (b = \star_B)).\pi(q) = \pi(\mathrm{refl}) \cdot f(q)$$

is equivalent to $b = \star_B$, since, in the above type, the second component expresses that the function π in the first component is uniquely determined by $\pi(\mathrm{refl})$, which is an element of $b = \star_B$. A similar approach is developed in [2, Section 4.10]. This result also follows from [25, Corollary 12]. □

Lemma 33. *The delooping of morphisms is functorial: given two morphisms $f : \Omega A \to \Omega B$ and $g : \Omega B \to \Omega C$, we have $\mathrm{B}(g \circ f) = \mathrm{B}\, g \circ \mathrm{B}\, f$ and $\mathrm{B}\, \mathrm{id}_{\Omega A} = \mathrm{id}_A$.*

Proof. By functoriality of Ω, we have $\Omega(\mathrm{B}\, g \circ \mathrm{B}\, f) = \Omega\,\mathrm{B}\, g \circ \Omega\,\mathrm{B}\, f$ and thus $\mathrm{B}(g \circ f) = \mathrm{B}\, g \circ \mathrm{B}\, f$ by Lemma 1. Preservation of identities can be proved similarly. □

The following lemma states that deloopings of groups are unique, in a strong sense. We write PCGpd for the type of pointed connected groupoids.

Lemma 34. *Given a group G, type $\Sigma(A : \mathrm{PCGpd}).(\Omega A = G)$ is contractible. In particular, for any two deloopings $\mathrm{B}\, G$ and $\mathrm{B}'\, G$, we have $\mathrm{B}\, G = \mathrm{B}'\, G$.*

Proof. Suppose given two deloopings of G, i.e. two types $A, A' : \mathrm{PCGpd}$ equipped with identities $p : \Omega A = G$ and $q : \Omega A' = G$. We thus have a morphism $f : \Omega A \to \Omega A'$, defined as $f :\equiv q^{\leftarrow} \circ p^{\rightarrow}$, which induces, by Lemma 1, a morphism $\mathrm{B}\, f : A \to A'$ such that $\Omega\,\mathrm{B}\, f = f$. By Lemma 1, $\mathrm{B}\, f$ is an isomorphism and thus induces an identity $A = A'$ by univalence. We finally show that $\Sigma(A : \mathrm{PCGpd}).(\Omega A = G)$ is a proposition. Given two elements (A, p) and (B, q) of this type, by [24, Theorem 2.7.2 and Theorem 2.11.3], an equality between them consists of an equality $r : A = B$ such that $\Omega\, r^{-1} \cdot p = q$. This amounts to show $p^{\rightarrow} = \Omega\, r^{\rightarrow} \circ q^{\rightarrow}$, i.e. $p^{\rightarrow} = \Omega\,\mathrm{B}\, f \circ q^{\rightarrow}$, which follows easily from the equality $\Omega\,\mathrm{B}\, f = f$. □

B Delooping with torsors

We recall here a classical torsor construction in order to define the delooping of a group G. A *G-type* is a type A together with a (left) action of G on A, i.e. a morphism $G \to \mathrm{Aut}(A)$ to the group of automorphisms of A. We write $g \cdot a$ for the action of an element of $g : G$ on $a : A$. A *G-set* is the particular case of an action of G on a set A. Any group G acts on itself by multiplication, and thus canonically induces a G-set P_G called the *principal G-set*. A morphism of G-sets is a morphism $f : A \to B$ between the underlying sets which preserves the action: $f(g \cdot a) = g \cdot f(a)$ for every $g : G$ and $a : A$. We write Set_G for the type of G-sets, $A \to_G B$ for the type of morphisms of G-sets and $A \simeq_G B$ for the type of isomorphisms of G-sets.

The main property of connected components is the following one.

Lemma 35. *For any type A with a distinguished element $\star$, the type $\mathrm{Comp}(A, \star)$ is connected and the first projection $\iota : \mathrm{Comp}(A, \star) \to A$ is an embedding.*

Proof. Easy, left to the reader. □

We can then show any group admits a delooping (Lemma 2) as follows:

Theorem 36. *Every group G admits a delooping $\mathrm{B}\,G$, which can be defined as* $\mathrm{B}\,G = \mathrm{Comp}(\mathrm{Set}_G, P_G)$.

Proof. By Lemma 35, it is enough to show that $(P_G = P_G) = G$. By univalence, see [24, Section 2.14], the type $P_G = P_G$ is equivalent to $P_G \simeq_G P_G$, and we are left with showing that it is in turn equivalent to G. Any element $g : G$ induces an isomorphism $[g] : P_G \to_G P_G$ defined by $[g](h) = hg$. Conversely, to any isomorphism $f : P_G \to_G P_G$, we associate the element $f(1) : G$. The two operations are mutually inverse group morphisms: given $g : G$, we have $[g](1) = 1g = g$; given an isomorphism $f : P_G \to_G P_G$, we have $f(g) = f(g1) = gf(1) = [f(1)](g)$. Thus inducing the required equivalence. □

The fundamental theorem of deloopings (Theorem 3) can then be shown as follow for this model:

Theorem 37. *The functions Ω and B induce an equivalence between the type of groups and the type of pointed connected groupoids.*

Proof. We have seen that, for any type A, ΩA has a canonical group structured induced by path composition and constant paths. The type of sets is a groupoid [24, Theorem 7.1.11], and given a group G and a set A, the type of morphisms $G \to \mathrm{Aut}(A)$ is a set and thus a groupoid; therefore the type of G-sets is a groupoid [24, Theorem 7.1.8], and thus also $\mathrm{B}\,G$ by Lemma 35. Given a group G, we have $\Omega\,\mathrm{B}\,G = G$ by definition of deloopings. Given a pointed connected groupoid A, A and $\mathrm{B}\,\Omega\,A$ are both deloopings of ΩA (with the canonical identities $\Omega\,A = \Omega A$ and $\Omega\,\mathrm{B}\,\Omega A = \Omega A$) and are thus equal by Lemma 1. □

Finally, we show that external and internal group actions on sets coincide:

Lemma 38. *We have* $\mathrm{Set}_G = (\mathrm{B}\,G \to \mathrm{Set})$.

Proof. Fix a set A. The type Set is a groupoid [24, Theorem 7.1.11] and therefore, by Lemma 35, $\mathrm{Comp}(\mathrm{Set}, A)$ is a pointed connected groupoid. By Theorem 3, actions on A, i.e. morphisms of groups $\Omega\,\mathrm{B}\,G \to \Omega\,\mathrm{Comp}(\mathrm{Set}, A)$ correspond to morphisms of pointed connected groupoids $\mathrm{B}\,G \to_\star \mathrm{Comp}(\mathrm{Set}, A)$. The result follows by summing the correspondence over $A : \mathrm{Set}$. □

Following what we have explained in Appendix B, the above correspondence can be interpreted as the fact that *external* actions (elements of Set_G) correspond to *internal* ones (morphisms in $\mathrm{B}\,G \to \mathrm{Set}$).

C Proof of Proposition 20

The proof below requires the following classical result, called the *flattening lemma* [24, Section 6.12], which we first recall (a variant of this lemma for pushouts was given in Lemma 22):

Lemma 39 (Flattening for coequalizers). *Suppose given a coequalizer*

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \overset{h}{\dashrightarrow} C$$

with $p : h \circ f = h \circ g$, and a type family $P : C \to \mathcal{U}$. Then the diagram

$$\Sigma A.(P \circ h \circ f) \underset{\Sigma g.e}{\overset{\Sigma f.(\lambda\_.\,\mathrm{id})}{\rightrightarrows}} \Sigma B.(P \circ h) \overset{\Sigma h.(\lambda\_.\,\mathrm{id})}{\dashrightarrow} \Sigma C.P$$

is a coequalizer, where the map

$$e : (a : A) \to P(h(f(a))) \to P(h(g(a)))$$

is induced by transport along p by

$$e\,a\,x := P^{\to}(p(a))(x).$$

Note that there is a slight asymmetry: we could have formulated a similar statement with $\Sigma A.(P \circ h \circ g)$ as left object.

We can now prove Proposition 20 as follows.

Proof of Proposition 20. Let us compute the fiber of the map $\mathrm{B}\,\varphi^l$, which is

$$\mathrm{B}\,\varphi^l := \Sigma(x : \mathrm{S}^1).(\star = \mathrm{B}\,\varphi^l(x))$$

We write $P : \mathrm{S}^1 \to \mathcal{U}$ for the map defined by $P(x) := (\star = \mathrm{B}\,\varphi^l(x))$ whose total space is the above fiber. The sphere S^1 can be obtained as the coequalizer

$$1 \underset{g}{\overset{f}{\rightrightarrows}} 1 \overset{h}{\dashrightarrow} \mathrm{S}^1$$

where $h(\star) = \star$ is the base point of the circle and the equality $p : h \circ f = h \circ g$ is such that $p(\star) : \star = \star$ is the canonical non-trivial loop of the circle, which is denoted q below. The flattening lemma for coequalizers (Lemma 39) ensures that that total space of P is the following coequalizer

$$\Sigma 1.(P \circ h \circ f) \underset{\Sigma g.e}{\overset{\Sigma f.(\lambda\_.\,\mathrm{id})}{\rightrightarrows}} \Sigma 1.(P \circ h) \overset{\Sigma h.(\lambda\_.\,\mathrm{id})}{\dashrightarrow} \Sigma\,\mathrm{S}^1.P$$

where the map e is induced by transport along p, i.e.

$$e\,x\,r := P^{\to}(p(x))(r).$$

By using the fact that for every $A : 1 \to \mathcal{U}$, we have $\Sigma(x : 1).A(x) = A(\star)$, the above coequalizer can be rewritten as

$$P(\star) \underset{e(\star)}{\overset{\mathrm{id}}{\rightrightarrows}} P(\star) \dashrightarrow \Sigma\,\mathrm{S}^1.P$$

with, for $r : \star = \mathrm{B}\,\varphi^l(\star)$ in $\mathrm{B}\,\mathbb{Z}_m$,

$$\begin{aligned} e(\star)(r) &= P^{\to}(p(\star))(r) && \text{by definition of } e \\ &= P^{\to}(q)(r) && \text{by definition of } p \\ &= r \cdot (\mathrm{B}\,\varphi^l)^{=}(q) && \text{by path transport [24, Theorem 2.11.4]} \end{aligned}$$

Moreover, we have

$$\begin{aligned} P(\star) &= (\star = \mathrm{B}\,\varphi^l(\star)) && \text{by definition of } P \\ &= (\star = \star) && \text{because } \mathrm{B}\,\varphi^l \text{ is pointed} \\ &= \Omega\,\mathrm{B}\,\mathbb{Z}_m && \text{by definition of } \Omega \\ &= \mathbb{Z}_m && \text{because } \mathrm{B}\,\mathbb{Z}_m \text{ is a delooping of } \mathbb{Z}_m \end{aligned}$$

and the above equalizer can be rewritten as

$$\mathbb{Z}_m \underset{\lambda i.(i+l)}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbb{Z}_m \dashrightarrow \Sigma\,\mathrm{S}^1.P$$

Finally, it can be decomposed as the coproduct of k copies of the coequalizer

$$\mathbb{Z}_{m/k} \underset{\lambda i.(i+1)}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbb{Z}_{m/k} \dashrightarrow \mathrm{S}^1$$

with $k = \mathrm{pgcd}(l, m)$. Namely, we have an isomorphism

$$\begin{aligned} f : \mathbb{Z}_m &\to k\mathbb{Z}_{m/k} \\ i &\mapsto (i \bmod k, i/k) \end{aligned}$$

(an element of $k\mathbb{Z}_{m/k}$ can be seen as a pair (i, j) with $0 \leq i < k$ and $j \in \mathbb{Z}_{m/k}$), whose inverse sends (i, j) to $kj + i$, which makes the diagram

$$\begin{array}{ccc} \mathbb{Z}_m & \underset{\lambda i.(i+l)}{\overset{\mathrm{id}}{\rightrightarrows}} & \mathbb{Z}_m \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ k\mathbb{Z}_{m/k} & \underset{k(\lambda i.i+1)}{\overset{k\,\mathrm{id}}{\rightrightarrows}} & k\mathbb{Z}_{m/k} \end{array}$$

commute. □