Notes on Koszul duality (for quadratic algebras)

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These are notes about Loday and Vallette's point of view about Koszul duality for algebras [LV12]. Most of what is in here comes from there (excepting errors and some detailed examples). Two introductory articles about Koszul duality where also used [Frö99, Krä].

1 Homology of algebras

We suppose fixed a field $\Bbbk$ over which vector spaces will be considered. We do not recall basic definitions about vector spaces. The free vector space over a (finite) set X is denoted $\Bbbk X$. It is quite useful to keep in mind the following isomorphisms:

$$\Bbbk(X \uplus Y) \cong \Bbbk X \oplus \Bbbk Y \qquad\qquad \Bbbk(X \times Y) \cong \Bbbk X \otimes \Bbbk Y$$

1.1 Algebras

Definition 1. An **algebra** (A, μ) consists of a vector space A together with a *multiplication*

$$\mu \quad : \quad A \otimes A \quad \to \quad A$$

which is associative

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes A} & A \otimes A \\ {\scriptstyle A \otimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array}$$

(all algebras considered here are associative). We often write ab instead of $\mu(a \otimes b)$. A **unital algebra** (A, μ, η) consists of an algebra together with a unit

$$\eta \quad : \quad \Bbbk \quad \to \quad A$$

satisfying

$$\begin{array}{ccccc} \Bbbk \otimes A & \xrightarrow{\eta \otimes A} & A \otimes A & \xleftarrow{A \otimes \eta} & A \otimes \Bbbk \\ & {\scriptstyle \cong}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \cong} & \\ & & A & & \end{array}$$

An **augmented algebra** (A, ε) is an algebra equipped with a morphism of algebras $\varepsilon : A \to \Bbbk$ (in particular, for unital algebras $\varepsilon \circ \eta = \mathrm{id}_{\Bbbk}$).

Remark 2. More abstractly an algebra can be defined as a monoid in the monoidal category **Vect**.

Remark 3. There is an isomorphism between non-unital algebras and augmented unital algebras. Namely, given (A, ε) augmented, we have the isomorphism of vector spaces

$$A \quad \cong \quad \Bbbk 1 \oplus \ker \varepsilon \quad = \quad \Bbbk 1 \oplus \overline{A}$$

So, to an algebra A, we associate the augmented unital algebra $\Bbbk 1 \oplus A$ (with obvious multiplication) and A augmented unital we associated $\overline{A}$.

Definition 4. The **tensor algebra** TV over a vector space V is

$$TV = \Bbbk 1 \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

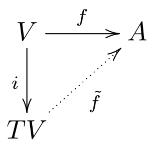
with tensor as multiplication and the inclusion $\eta : \Bbbk \to TV$ as unit. The **reduced tensor algebra** $\overline{T}V$ is

$$TV = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

(i.e. $TV = \Bbbk 1 \oplus \overline{T}V$, i.e. $\overline{TV} = \overline{T}V$).

Remark 5. The tensor algebra is canonically augmented by $\varepsilon(1) = 1$ and $\varepsilon(u) = 0$ for $u \in \overline{T}V \subseteq TV$.

Proposition 6. *The tensor algebra TV is the free unital algebra over V, i.e. any linear map $f : V \to A$ where A is an unital algebra extends uniquely as an algebra map $\tilde{f} : TV \to A$:*

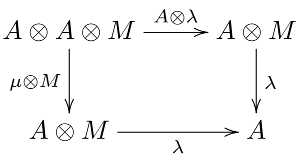


Similarly, the reduced tensor algebra $\overline{T}V$ is the free (non-unital) algebra over V.

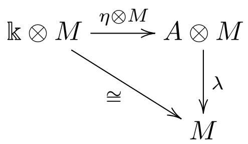
Definition 7. A **(left) module** (M, λ) over an algebra A is a vector space M equipped with an *action*

$$\lambda : A \otimes M \to M$$

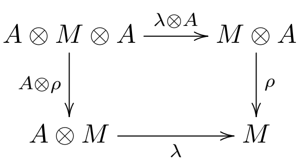
satisfying



This definition extends to unital algebras by also requiring



The notion of a right module (M, ρ) with $\rho : M \otimes A \to M$ is defined similarly. A bimodule (M, λ, ρ) is both a left and a right module such that



Remark 8. Given an augmented $\Bbbk$-algebra A, $\Bbbk$ is canonically a left (or similarly for right/bi) A-module by

$$\lambda(a,k) \quad = \quad \varepsilon(a)k$$

Proposition 9. *The free left (resp. right, resp. bi) A-module over V is $A \otimes V$ (resp. $V \otimes A$, resp. $A \otimes V \otimes A$).*

Remark 10. Given an algebra $A = (A, \mu)$, we define $A^{\mathrm{op}} = (A, \mu^{\mathrm{op}})$ where $\mu^{\mathrm{op}} = \mu \circ \tau$ with $\tau : A \otimes A \to A \otimes A$ the canonical switching. The category of right A-modules is isomorphic to the category of left A^{op}-modules, A-bimodules can be seen as left $A^{\mathrm{op}} \otimes A$-modules, etc. In this sense, the distinction between left, right and bi-modules is not fundamental (but of course we won't get the same results if we compute homology in left, right or bimodules...). In the following, we sometimes use the notation

$$A^e \quad = \quad A \otimes A^{\mathrm{op}}$$

1.2 Homology of free modules and algebras

Definition 11. A **chain complex**

$$\ldots \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

consists of a (left/right/bi) modules $(C_i)_{i \geq 0}$ and module maps $d_i : C_i \to C_{i-1}$ such that

$$d_i \circ d_{i+1} \quad = \quad 0$$

Notice that this implies $\operatorname{im} d_{i+1} \subseteq \ker d_i$, so that we can define the n-th **homology group** by

$$H_n(C_\bullet) \quad = \quad \ker d_i / \operatorname{im} d_{i+1}$$

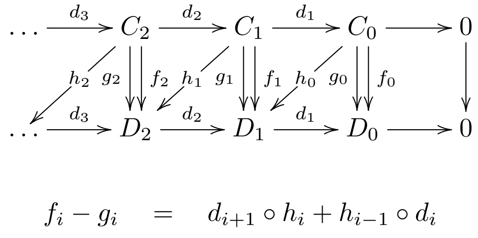
The complex is **exact** when we have $\operatorname{im} d_{i+1} = \ker d_i$. A **morphism** between to such chain complexes $f : C_\bullet \to D_\bullet$ consists of maps $f_i : C_i \to D_i$ such that

$$f_i \circ d_i \quad = \quad d_{i+1} \circ f_{i+1}$$

i.e. the following diagram commutes

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{d_3} & C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow \\
\ldots & \xrightarrow{d_3} & D_2 & \xrightarrow{d_2} & D_1 & \xrightarrow{d_1} & D_0 & \longrightarrow & 0
\end{array}$$

An **homotopy** $h : f \Rightarrow g : C_\bullet \to D_\bullet$ between two such morphisms consists of linear (not module) maps $h_i : C_i \to D_{i+1}$



such that

$$f_i - g_i \quad = \quad d_{i+1} \circ h_i + h_{i-1} \circ d_i$$

The operation H extends to a functor, i.e. a morphism $f : P \to Q$ of chain complexes induces morphisms

$$H_n(f) \quad : \quad H_n(P) \quad \to \quad H_n(Q)$$

Proposition 12. *Two homotopic maps induce the same morphism in homology.*

Definition 13. An **augmented chain complex** $C_\bullet$ of a chain complex together with a linear map

$$\varepsilon \quad : \quad C_0 \to M$$

to a module M, such that the sequence

$$C_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is exact, i.e. $\operatorname{im} \varepsilon = M$, and

$$\varepsilon \circ d_1 \quad = \quad 0$$

It is a **resolution** of M when the complex C is acyclic, i.e. the complex

$$\ldots \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

has vanishing homology groups (i.e. the sequence is exact).

Remark 14. The right way to see a an augmented chain complex is as a morphism of chain complexes

$$\begin{array}{ccccccccc}
\ldots & \xrightarrow{d_3} & C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varepsilon} & & \downarrow \\
\ldots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

and the condition for it to be a resolution is equivalent to requiring that the above morphism is a **quasi-isomorphism**, i.e. induces an isomorphism in homology.

Definition 15. A module P is **projective** if for any morphism $f : P \to M$ and surjective morphism $p : N \twoheadrightarrow M$ there exists a unique morphism $g : P \to N$ such that

$$\begin{array}{ccc}
 & & N \\
 & {\scriptstyle g} \nearrow & \downarrow{\scriptstyle p} \\
P & \xrightarrow[f]{} & M
\end{array}$$

Lemma 16. *Every free module is projective.*

Proof. Since p is surjective, it admits a section $s : M \to N$. Since P is free, the function which to a generator x of P associates $s \circ f(x)$ extends to a morphism of modules $g : P \to N$. □

Remark 17. Conversely, every projective module is a direct summand of a free module.

Proposition 18. *Every finitely generated module over a local ring is free.*

Proof. Use Nakayama's lemma. See also [Krä]. □

Proposition 19. *Between any two projective resolutions of M there is a morphism, whose component on M is id_M, and between any two such morphisms there is a homotopy.*

Proof. Consider two projective resolutions P and P' of M. We build a morphism between them, by induction, as follows:

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{d_3} & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow \mathrm{id} & & \downarrow \\
\dots & \xrightarrow{d'_3} & P'_2 & \xrightarrow{d'_2} & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\varepsilon'} & M & \longrightarrow & 0
\end{array}$$

Since ε is surjective and P_0 is projective, we have f_0. Suppose the morphisms constructed up to f_i. We have that $d'_i \circ f_i \circ d_{i+1} = f_{i-1} \circ d_i \circ d_{i+1} = 0$, and therefore $\mathrm{im}(f_i \circ d_{i+1}) \subseteq \ker d'_i = \mathrm{im}\, d'_{i+1}$. Therefore f_{i+1} exists since P_{i+1} is projective and d'_{i+1} is surjective on its image. The construction of the homotopy is similar. □

Suppose given a $\Bbbk$-module A and consider a resolution of a right A-module M by projective right A-modules P_i

$$\dots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

Tensoring it by a left A-module N yields a chain complex which is not exact anymore

$$\dots \xrightarrow{d_{n+1} \otimes_A N} P_n \otimes_A N \xrightarrow{d_n \otimes_A N} \dots \xrightarrow{d_2 \otimes_A N} P_1 \otimes_A N \xrightarrow{d_1 \otimes_A N} P_0 \longrightarrow 0$$

The **torsion functor**

$$\mathrm{Tor}^A_n(M, N) \quad = \quad H_n(P_\bullet \otimes_A N)$$

is the n-th homology group of this chain complex (the above tensor notation is explained in Definition 42). Since we have chosen projective resolutions, the homology does not depend on the choice of the resolution by the above propositions. These constructions can be developed similarly for resolutions by left, and bi A-modules.

In the case where we have a group (or a monoid) G, we consider the free ring $A = \Bbbk G$, and define

$$H_n(G) \quad = \quad \mathrm{Tor}^A_n(\Bbbk, \Bbbk)$$

where $\Bbbk$ is seen as a trivial left A-module, i.e. $g \cdot k = k$ for $g \in G$ and $k \in \Bbbk$. This construction works more generally for augmented algebras with the "trivial" A-module structure on $\Bbbk$ being given by $a \cdot k = \varepsilon(a)k$.

Example 20. Consider the monoid $M = \langle x, y \mid xx = yy, xy = yx \rangle$. We write $A = \Bbbk M$ with $\Bbbk = \mathbb{Z}$. We can start building a resolution of the trivial left A-module $\Bbbk$ by free left A-modules as follows

$$\ldots \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} \Bbbk \longrightarrow 0$$

We begin by $C_0 = \Bbbk M \otimes \Bbbk \cong \Bbbk M$ and $\varepsilon : \Bbbk M \to \Bbbk$ such that $\varepsilon(m) = 1$ for $m \in M$. Now, we have $\ker d_1$ generated by elements of the form $m-1$ for $m \in M$. So we could take $C_1 = \Bbbk M \otimes \Bbbk M$ with $d_1(1 \otimes m) = m - 1$. However, we can do better since $\ker d_1$ is also generated by $\{x-1, y-1\}$ as a left $\Bbbk M$-module: we have

$$ma - 1 = m(a-1) + (m-1)$$

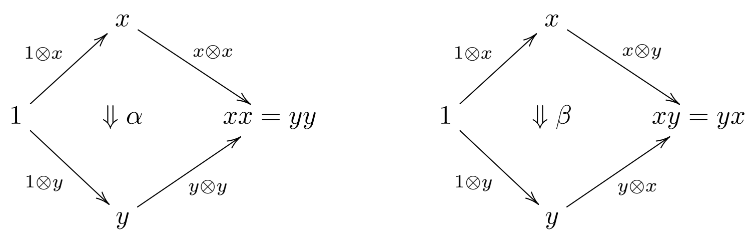
and we can conclude by induction. So, we can take $C_1 = \Bbbk M \otimes \Bbbk\{x, y\}$ with $d_1(1 \otimes x) = x - 1$ and $d_1(1 \otimes y) = y - 1$. Geometrically, this means that we have a path from 1 to every word $m \in M$ in the graph with M as vertices and $m \otimes x : m \to mx$ as edges

$$1 \xrightarrow{1 \otimes x} x \xrightarrow{x \otimes y} xy = yx$$

i.e. it is connected. In order to go on, we should take in account the two rules $\alpha : xx = yy$ and $\beta : xy = yx$. Namely, considering β, we have

$$d_1(1 \otimes x + x \otimes y) \quad = \quad x - 1 + xy - x \quad = \quad yx - 1 \quad = \quad d_1(1 \otimes y + y \otimes x)$$

Geometrically, this corresponds to the fact that we have to add 2-cells in order to make the "graph" simply connected



We can thus take $C_2 = \Bbbk M \otimes \Bbbk\{\alpha, \beta\}$ with

$$\begin{aligned} d_2(\alpha) &= 1 \otimes x + x \otimes x - 1 \otimes y - y \otimes y \\ d_2(\beta) &= 1 \otimes x + x \otimes y - 1 \otimes y - y \otimes x \end{aligned}$$

(in order to formally prove that the chain complex constructed so far is exact, one can for example construct a contracting homotopy). Notice that from what we have constructed, we have $H_0(M) = \Bbbk$ (as always). Moreover, on the trivialized complex $H_1(M)$ is computed from

$$\Bbbk\{\alpha, \beta\} \xrightarrow{A \otimes_{\Bbbk} d_2} \Bbbk\{x, y\} \xrightarrow{A \otimes_{\Bbbk} d_1} \Bbbk$$

with

$$(A \otimes d_1)(x) = 1 \qquad (A \otimes d_1)(y) = 1 \qquad (A \otimes d_2)(\alpha) = 2x - 2y \qquad (A \otimes d_2)(\beta) = 0$$

So, in characteristic 0, we have $\ker(A \otimes d_1) = \Bbbk\{x - y\}$ and $\operatorname{im}(A \otimes d_2) = \Bbbk\{x - y\}$, and therefore

$$H_1(M) = 0$$

When the monoid is presented by a convergent rewriting system, we can take C_3 to be the free $\Bbbk M$-module on critical pairs, C_4 the free $\Bbbk M$-module on critical triples, and so on, in order to make the "graph" contractile. This is nicely explained in [LP91]. These ideas are the basis of the partial resolution constructed by Squier [Squ87] and Anick's resolution [Ani86] for algebras, rediscovered for monoids by Kobayashi [Kob90]...

In the above example, we have been particularly smart and have built a quite reasonably small resolution. However, if we only want to theoretically build a resolution we can use a much more canonical way, at the expense of constructing a much bigger complex. In the above situation notice that $\Bbbk M$ is not only a ring, but actually an algebra $A = \Bbbk M$ which is augmented as described above.

Definition 21. Suppose given an augmented algebra A. The **bar resolution** of the trivial A-module $\Bbbk$ by free left A-modules is

$$\ldots \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} \Bbbk \longrightarrow 0$$

with

$$C_n = A \otimes A^{\otimes n}$$

An element of $A \otimes A^{\otimes n}$ is often denoted

$$a[a_1|a_2|\ldots|a_n]$$

and the differential is defined by

$$d_n([a_1|a_2|\ldots|a_n]) = a_1[a_2|\ldots|a_n] + \sum_{1 \leq i < n} (-1)^i [a_1|\ldots|a_i a_{i+1}|\ldots|a_n] + (-1)^n [a_1|a_2|\ldots|a_{n-1}]$$

The **normalized bar resolution** is defined similarly, without taking the unit in account:

$$C_n = A \otimes \overline{A}^{\otimes n}$$

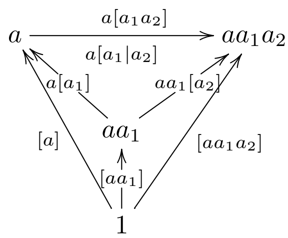
There are of course right and bimodule versions of this: in the bimodule case, we have

$$C_n = A \otimes A^{\otimes n} \otimes A$$

the augmentation is $\mu : A \otimes A \to A$ and

$$d_n([a_1|a_2|\ldots|a_n]) = a_1[a_2|\ldots|a_n] + \sum_{1 \leq i < n} (-1)^i [a_1|\ldots|a_i a_{i+1}|\ldots|a_n] + (-1)^n [a_1|a_2|\ldots|a_{n-1}]a_n$$

Remark 22. Geometrically, this corresponds to constructing a complex with the elements of A as vertices and making it contractible:



which explains graphically why $d(a[a_1]) = aa_1 - a$ and

$$d(a[a_1|a_2]) \quad = \quad aa_1[a_2] - a[a_1a_2] + a[a_1]$$

We refer to [MLM95, Lod98] for a presentation of the homology of algebras.

Definition 23. The **Hochschild homology** of an algebra with coefficients in a A-bimodule M is

$$H_n(A, M) \quad = \quad \mathrm{Tor}_n^{A^e}(A, M)$$

where A is considered to be a bimodule over itself using multiplication. More explicitly, it is the homology of the Hochschild complex

$$\ldots \xrightarrow{d_3} M \otimes A^{\otimes 2} \xrightarrow{d_2} M \otimes A \xrightarrow{d_1} M$$

with

$$d_n(m[a_1|\ldots|a_n]) \quad = \quad ma_1[a_2|\ldots|a_n] + \sum_{1\leq i<n} (-1)^i[a_1|\ldots|a_ia_{i+1}|\ldots|a_n] + (-1)^n a_n m[a_1|\ldots|a_{n-1}]$$

Definition 24. Given an augmented chain complex $\varepsilon : (A \otimes C_\bullet) \to M$ of free left A-modules

$$\ldots \xrightarrow{d_3} A \otimes C_2 \xrightarrow{d_2} A \otimes C_1 \xrightarrow{d_1} A \otimes C_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

a **contracting homotopy** consists of $\mathbb{k}$-linear (generally not A-linear!) morphisms $\eta : M \to A \otimes C_0$ and $s_i : A \otimes C_i \to A \otimes C_{i+1}$

$$\ldots \underset{s_2}{\overset{d_3}{\rightleftarrows}} A \otimes C_2 \underset{s_1}{\overset{d_2}{\rightleftarrows}} A \otimes C_1 \underset{s_0}{\overset{d_1}{\rightleftarrows}} A \otimes C_0 \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} M \longrightarrow 0$$

such that

$$\varepsilon \circ \eta = \mathrm{id}_M \qquad d_1 \circ s_0 + \eta \circ \varepsilon = \mathrm{id}_{A\otimes C_0} \qquad d_{i+1} \circ s_i + s_{i-1} \circ d_i = \mathrm{id}_{A\otimes C_i}$$

Proposition 25. *An augmented chain complex equipped with a contacting homotopy as above is a resolution of M.*

Example 26. We can define a contracting homotopy on the left bar resolution by

$$\eta(k) = k[] \qquad\qquad s_n(a[a_1|\ldots|a_n]) = [a|a_1|\ldots|a_n]$$

(this works more generally for augmented unital algebras).

We are often interested in building smallest possible resolutions.

Definition 27. A resolution P of M is **minimal** if $H_n(P) = 0$ for $n > 1$ (and $H_0(P) = M$).

Remark 28. From [Krä], a free finitely generated resolution is minimal when the matrices encoding the differentials

$$P_{n+1} \cong A^{b_{n+1}} \xrightarrow{d_{n+1}} A^{b_n} \cong P_n$$

have coefficients in $\overline{A}$. Namely, when we $- \otimes_A \mathbb{k}$ to compute the homology, with the usual augmentation $\varepsilon : A \to \mathbb{k}$ (sending everything excepting 1 to 0), the maps become 0.

1.3 Anick's resolution for algebras

We describe in this section the resolution introduced by Anick in [Ani86]. It can be seen as a generalization of the resolution for monoids described by Kobayashi [Kob90]. Consider an (augmented unital) algebra $A = A(X, R)$ where R is a reduced Gröbner basis (see Definition 97). We are going to build a resolution

$$\ldots \underset{i_{n+1}}{\overset{d_{n+1}}{\rightleftarrows}} C_n \otimes_{\Bbbk} A \underset{i_n}{\overset{d_n}{\rightleftarrows}} \ldots \underset{i_2}{\overset{d_2}{\rightleftarrows}} C_1 \otimes_{\Bbbk} A \underset{i_1}{\overset{d_1}{\rightleftarrows}} C_0 \otimes_{\Bbbk} A \underset{i_0}{\overset{d_0}{\rightleftarrows}} A \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} \Bbbk \longrightarrow 0$$

of the trivial right A-module $\Bbbk$ by free right A-modules. Above, the d_n and ε are A-linear whereas the i_n and η are $\Bbbk$-linear. These should moreover satisfy

$$\varepsilon d_0 = 0 \qquad\qquad d_{n+1} d_n = 0$$

and

$$\varepsilon\eta = \mathrm{id}_{\Bbbk} \qquad\qquad d_1 i_1 + \eta\varepsilon = \mathrm{id}_A \qquad\qquad d_{n+1} i_{n+1} + i_n d_n = \mathrm{id}_{C_n \otimes_{\Bbbk} A}$$

It is actually simpler to construct

$$i_n \quad : \quad \ker d_{n-1} \quad \to \quad C_n \otimes_{\Bbbk} A$$

and replace last axiom by

$$d_n i_n \quad = \quad \mathrm{id}_{\ker d_{n-1}}$$

(any extension to the domain $C_{n-1} \otimes_{\Bbbk} A$ will suit in the previous sense, typically by 0 in the following).

Definition 29. An **n-chain** is a sequence of $n+1$ words defined inductively by

- there is a unique (-1)-chain
- a 0-chain is any letter
- a $(n+1)$-chain $x|u_1|\ldots|u_{n+1}$ is a sequence of words such that
  - $x|u_1|\ldots|u_n$ is an n-chain,
  - u_{n+1} is a non-empty word in normal form,
  - the word $u_n u_{n+1}$ is reducible, in a unique way, at a rightmost position.

Lemma 30. *The forgetful function from n-chain to words, sending $x|u_1|\ldots|u_n$ to $xu_1 \ldots u_n$ is injective. We can therefore omit the "$|$" if we feel like to.*

We define the C_n as the free $\Bbbk$-modules generated by the n-chains. The complicated part is to define the maps, which by are going to do by induction on both homological degree and monomials: the monomials in C_n are totally ordered since we have a Gröbner basis and we extend this order on $C_n \otimes A$ by $u \otimes v < u' \otimes v'$ if $u\hat{v} < u'\hat{v}'$ in C_n. In addition to the required axioms, it will be checked inductively that d_n is weakly decreasing ($d_n(u) \leq u$) and i_n does not change the order (so that $i_n d_n$ is weakly decreasing).

We define $d_0 : C_0 \otimes A \to A$ on $x \otimes 1 \in C_0 \otimes A$ by

$$d_0(x \otimes 1) \quad = \quad x - \eta\varepsilon(x)$$

and $i_0 : A \to C_0 \otimes A$ on $u \in A$ by

$$i_0(u) \quad = \quad x \otimes v$$

if $xv = \hat{u}$ is the normal form of the word u. Those morphisms are easily verified to satisfy the required axioms.

We now suppose defined $d_n : C_n \otimes A \to C_{n-1} \otimes A$ and $i_n : C_{n-1} \otimes A$ (up to n) and we construct d_{n+1} and i_{n+1}. The morphism $d_{n+1} : C_{n+1} \otimes A \to C_n \otimes A$ is defined by

$$d_{n+1}(u \otimes 1) \quad = \quad u' \otimes u_{n+1} - i_n d_n(u' \otimes u_{n+1})$$

with $u = x|u_1|\ldots|u_n|u_{n+1} = u'|u_{n+1} \in C_{n+1}$. We have that $d_n d_{n+1} = 0$:

$$d_n d_{n+1}(u \otimes 1) \quad = \quad d_n(u' \otimes u_{n+1}) - d_n i_n d_n(u' \otimes u_{n+1})$$

and $d_n i_n d_n(u' \otimes u_{n+1}) = d_n(u' \otimes u_{n+1})$ because $d_n(u' \otimes u_{n+1}) \in \ker_{d_{n-1}}$ since $d_{n-1}d_n = 0$ and $d_n i_n = \mathrm{id}_{\ker d_{n-1}}$. Moreover, we have

$$d_n(u' \otimes u_{n+1}) \quad = \quad x|u_1|\ldots|u_{n-1} \otimes u_n u_{n+1} + lt$$

where lt designates strictly lower terms so $i_n d_n(u' \otimes u_{n+1}) < u' \otimes u_{n+1}$ because i_n does not change the order, d_n is decreasing, and $u_n u_{n+1}$ is reducible by definition of chains, so that

$$i_n(x|u_1|\ldots|u_{n-1} \otimes u_n u_{n+1}) \quad < \quad u' \otimes u_{n+1}$$

We define $i_{n+1} : \ker d_n \to C_n \otimes A$ as follows. Suppose given a term

$$t \quad = \quad \lambda u \otimes v + lt$$

in C_n with $u = x|u_1|\ldots|u_n$ (we have identified the summand of maximal degree). We know that

$$\begin{aligned} d_n(t) &= \lambda d_n(u \otimes v) + d_n(lt) \\ &= \lambda x|u_1|\ldots|u_{n-1} \otimes u_n v + d_n(lt) \\ &= 0 \end{aligned}$$

This implies that $u_n v$ is reducible (otherwise $\lambda x|u_1|\ldots|u_{n-1} \otimes u_n$ could not be canceled in $d_n(lt) \leq lt$). Therefore, we can write $u_n v = v' w v''$ with w a left member of a rule, and we choose the leftmost decomposition (with v' as small as possible). We finally define

$$i_{n+1}(t) \quad = \quad \lambda u|v'w \otimes v'' + i_{n+1}(t - \lambda d_n(u|v'w \otimes v''))$$

where $i_{n+1}(t - \lambda d_n(u|v'w \otimes v''))$ is defined inductively. Namely,

$$d_n(u|v'w \otimes v'') \quad = \quad u \otimes u_{n+1} - i_{n-1}d_{n-1}(u \otimes u_{n+1})$$

which shows that the leading term of t gets canceled. It can be checked that $d_{n+1}i_{n+1} = \mathrm{id}_{\ker d_n}$: we the above notations, we have by induction

$$\begin{aligned} d_{n+1}i_{n+1}(t) &= \lambda d_{n+1}(u|v'w \otimes v'') + d_n i_n(lt) \\ &= \lambda d_{n+1}(u \otimes v'wv'') + lt \\ &= \lambda u \otimes u_{n+1} + lt \\ &= t \end{aligned}$$

Example 31. Consider

$$A \quad = \quad \langle x, y \mid xx - yy, xy - yx \rangle$$

with $x > y$ and the usual augmentation for graded algebras. We have

$$C_{-1} = \{*\} \qquad C_0 = \{x, y\} \qquad C_1 = \{xx, xy\} \qquad \dots \qquad C_n = \{x^n x, x^n y\}$$

with

$$\begin{aligned} d(x \otimes 1) &= x & d(y \otimes 1) &= y \\ d(xx \otimes 1) &= x \otimes x - y \otimes y & d(xy \otimes 1) &= x \otimes y - y \otimes x \\ &\vdots & &\vdots \\ d(x^n x \otimes 1) &= x^n \otimes x - x^{n-1} y \otimes y & d(x^n y \otimes 1) &= x^n \otimes y - y^{n-1} y \otimes x \end{aligned}$$

2 Bar, cobar, and twisting morphisms

2.1 Algebras and coalgebras

The right bar resolution can be seen as

$$BA \otimes_\pi A$$

where BA denotes the collection of all the $A^{\otimes n}$, i.e. TA as a vector space. First, it can be noticed that BA has a structure of free dg coalgebra. The differential on this complex is the sum of two distinct contributions

- $d([a_1|\dots|a_n]) = [a_1|\dots|a_i a_{i+1}|\dots|a_n]$ comes from the multiplication extended as a coderivation on the coalgebra
- $d([a_1|\dots|a_n]) = [a_1|\dots|a_{n-1}]a_n$ derives from a morphism $\pi : BA \to a$ called "twisting"

Then we are going to study when we can find a "small" coalgebra C, which can play the role of BA in our constructions. This will typically be the case for Koszul algebras.

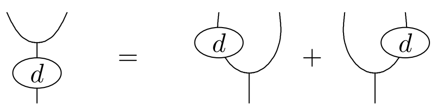
Definition 32. A **derivation** $d : A \to M$ from an algebra A to a bimodule M over A is a linear map such that

$$d \circ \mu \quad = \quad \rho \circ (d \otimes \mathrm{id}_A) + \lambda \circ (\mathrm{id}_A \otimes d)$$

i.e.

$$d(ab) \quad = \quad d(a)\, b + a\, d(b)$$

Graphically,



Proposition 33. *A derivation on TV is completely determined by its restriction to V:*

$$\mathrm{Hom}(V, M) \quad \cong \quad \mathrm{Der}(TV, M)$$

and given $f : V \to M$,

$$d_f(v_1 \dots v_n) \quad = \quad \sum_{1 \leq i \leq n} v_1 \dots f(v_i) \dots v_n$$

Remark 34. Given a smooth manifold M, we write $\mathcal{C}^\infty(M)$ for the ring of smooth functions $M \to \mathbb{R}$. The tangent space $T_x M$ at a point x can be defined as the vector space of derivations of $\mathcal{C}^\infty(M)$ at x, i.e. its elements are $d : \mathcal{C}^\infty(M) \to \mathbb{R}$ such that

$$d(fg) \quad = \quad d(f)g(x) + f(x)d(g)$$

Example 35. Let us give an example in **Set**, which is not exactly in the context we describe, but is enlightening. Consider the monoid $M = \{a, b\}^*$ and suppose that we want to count the number of as before a b in words u. We consider the monoid $\mathbb{N}_\perp = \mathbb{N} \uplus \{\perp\}$ whose addition extends the one on natural numbers by $\perp + n = n = n + \perp$. Since M is free, we can define a (left) action of M on $\mathbb{N}_\perp$ by its effect on generators. We define

$$a \cdot \perp = \perp \qquad a \cdot n = (n+1) \qquad b \cdot \perp = \perp \qquad b \cdot n = n$$

This can be extended as a biaction, whose action on the right is trivial. Consider the map $f : \{a, b\} \to \mathbb{N}_\perp$ defined by

$$a \mapsto \perp \qquad\qquad b \mapsto 0$$

This extends as a derivation d_f on words by

$$d_f(x_1 \dots x_n) \quad = \quad \sum_{1 \leq i \leq n} x_1 \dots f(x_i) \dots x_n$$

A term $x_1 \dots f(x_i) \dots x_n$ in this sum is $\perp$ if $x_i = b$ and is equal to the number of as in $x_1 \dots x_{i-1}$ otherwise. The derivation thus computes the number of as before a b in a word. It can be for instance used to show that the rewriting system

$$\langle a, b \mid ab \to bba \rangle$$

terminates.

Definition 36. A **coalgebra** (C, Δ) is a comonoid in **Vect**: $\Delta : C \to C \otimes C$ such that

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta \otimes C} \\ C \otimes C & \xrightarrow[C \otimes \Delta]{} & C \otimes C \otimes C \end{array}$$

Variants are also defined: counital ($\varepsilon : C \to \Bbbk$ linear), coaugmented ($\eta : \Bbbk \to C$ coalgebra morphism). In the counital coaugmented case we have

$$C \quad = \quad \overline{C} \oplus \Bbbk 1$$

with $\overline{C} = \operatorname{im} \eta$. The *reduced coproduct* $\overline{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}$ is given by

$$\overline{\Delta}(x) \quad = \quad \Delta(x) - 1 \otimes x - x \otimes 1$$

We write $\overline{\Delta}^n : \overline{C} \to \overline{C}^{\otimes n}$ for the iterated reduced coproduct. An augmented coalgebra C is **conilpotent** when for every $x \in C$, there exists $n \in \mathbb{N}$ such that

$$\overline{\Delta}^n(x) \quad = \quad 0$$

Proposition 37. *The* ***cofree conilpotent coalgebra*** T^cV *on a vector space* V *is*

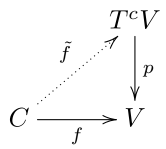
$$T^cV \quad = \quad \Bbbk 1 \oplus V \oplus \ldots \oplus V^{\otimes n} \oplus \ldots$$

equipped with the deconcatenation tensor product $\Delta : T^cV \to T^cV \otimes T^cV$ *defined by*

$$\Delta(v_1 \ldots v_n) \quad = \quad \sum_{0 \leq i \leq n} v_1 \ldots v_i \otimes v_{i+1} \ldots v_n$$

(in particular $\Delta(1) = 1 \otimes 1$*). By this we mean that any linear map* $f : C \to V$ *from a conilpotent coalgebra* C *to* V *extends uniquely as a morphism of coalgebras*

TODO: in [LV12] they impose that morphisms such as f satisfy $f(1) = 0$, why?...



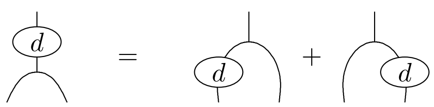
In the non-unital context, we have $T^cV = \bigoplus_{n>0} V^{\otimes n}$ *equipped with the reduced deconcatenation coproduct*

$$\overline{\Delta}(v_1 \ldots v_n) \quad = \quad \sum_{0 < i < n} v_1 \ldots v_i \otimes v_{i+1} \ldots v_n$$

Definition 38. A **coderivation** is a linear map $d : C \to C$ such that $d(1) = 0$ and

$$\Delta \circ d \quad = \quad (d \otimes \mathrm{id}_C) \circ \Delta + (\mathrm{id}_C \otimes d) \circ \Delta$$

Graphically,



Proposition 39. *A coderivation on* T^cV *is determined by its corestriction to* V*. To* $f : T^cV \to V$ *corresponds the coderivation whose corestriction to any* $V^{\otimes n}$ *is*

$$d_f(x)^{(n)} \quad = \quad \sum_{1 \leq i \leq n} \sum_{(x)} x_{(1)} \otimes \ldots \otimes f(x_{(i)}) \otimes \ldots \otimes x_{(n)}$$

with $\overline{\Delta}^n(x) = \sum_{(x)} x_{(1)} \otimes \ldots \otimes x_{(n)}$.

Example 40. Consider an algebra (A, μ). The multiplication μ can be extended as a coderivation d_μ on T^cA. If we write $[a_1 | \ldots | a_n]$ for an element of $A^{\otimes n}$ we have

$$d_\mu([a_1 | \ldots | a_n]) \quad = \quad \sum_{i=1}^{n-1} [a_1 | \ldots | a_i a_{i+1} | \ldots | a_n]$$

which is almost the bar complex (see Definition 21), excepting for the signs and the fact that we do not really remember about the grading, see Definition 52 for details in the graded case.

2.2 Differential graded vector spaces

Definition 41. A **graded vector space** is a family $(V_n)_{n\in\mathbb{Z}}$ of vector spaces whose direct sum is denoted $V_\bullet$, elements $v \in V_n$ are of degree $|v| = n$. A *morphism* $f : V \to W$ of degree $|f| = r$ is a family of maps $f_n : V_n \to W_{n+r}$.

Definition 42. The **tensor product** $V \otimes W$ of graded vector spaces V and W is

$$(V \otimes W)_n \quad = \quad \bigoplus_{i+j=n} V_i \otimes W_j$$

Notice that given a graded vector space V, the tensor TV thus admits two gradings: given $v \in V^{\otimes n}$, its

- **degree** is $|v| = |v_1| + \ldots + |v_n|$
- **weight** is n

We write $\Bbbk s$ for the graded vector space generated by s with $|s| = 1$. The **suspension** sV of V is

$$sV \quad = \quad \Bbbk s \otimes V$$

so that we have $(sV)_i \cong V_{i-1}$. The **desuspension** $s^{-1}V$ is defined similarly with s^{-1} in degree -1.

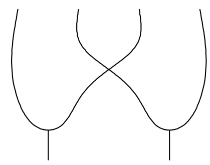
The category of vector spaces is equipped with the symmetric structure induced by the natural family of maps $\tau : V \otimes W \to W \otimes V$ defined by

$$\tau(v \otimes w) \quad = \quad (-1)^{|v||w|} w \otimes v$$

This has a number of consequences. For instance, given an algebra A, $A \otimes A$ is canonically equipped with a structure of algebra with

$$(\mu \otimes \mu) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \quad : \quad (A \otimes A) \otimes (A \otimes A) \to A \otimes A$$

as multiplication:



This means that the sign rule is

$$(x \otimes y)(x' \otimes y') \quad = \quad (-1)^{|y||x|}(xx' \otimes yy')$$

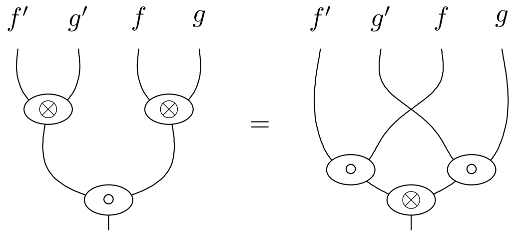
For similar reasons, we follow the **Koszul sign rule**, defining the tensor product $f \otimes g : V \otimes V' \to W \otimes W'$ of two maps $f : V \to V$ and $g : W \to W'$ of graded vector spaces by

$$(f \otimes g)(v \otimes w) \quad = \quad (-1)^{|g||x|} f(v) \otimes g(w)$$

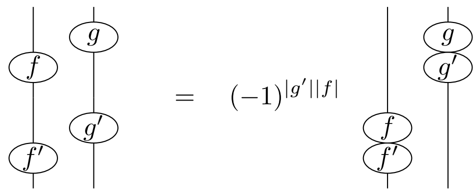
and given maps $f' : V' \to V''$ and $g' : W' \to W''$, we have

$$(f' \otimes g') \circ (f \otimes g) \quad = \quad (-1)^{|g'||f|}(f' \circ f) \otimes (g' \circ g) \tag{1}$$

which is better understood if we look at the graphical version of the exchange law



Another way to represent the graded exchange law (1) is by using "leveled" string diagrams



Definition 43. A **differential graded vector space** (or *dg vector space* or **chain complex**) (V, d) consists of a graded vector space together with a map $d : V_\bullet \to V_{\bullet-1}$ of degree -1 such that

$$d^2 \quad = \quad 0$$

A *morphism* of degree r between two chain complexes is a map $f : V_\bullet \to W_{\bullet+r}$ such that

$$d_W \circ f \quad = \quad (-1)^r f \circ d_V$$

The tensor product $V \otimes W$ of two chain complexes is equipped with the differential

$$d_{V \otimes W} \quad = \quad d_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes d_W$$

i.e. for $v \otimes w \in V_p \otimes W_q$

$$d_{V\otimes W}(v \otimes w) \quad = \quad d_V(v) \otimes w + (-1)^{|p|} v \otimes d_W(w)$$

The suspension is defined as before, which implies

$$d_{sV} \quad = \quad -d_V$$

Definition 44. The **derivative** of a map $f : V_\bullet \to W_{\bullet+r}$ of degree r is

$$\partial(f) \quad = \quad [d, f] \quad = \quad d_W \circ f - (-1)^r f \circ d_v$$

Remark 45. f is a morphism of chain complexes if and only if $\partial(f) = 0$. Notice that $\partial^2 = 0$, i.e. we have a differential on hom-dg vector spaces.

Remark 46. We will mostly consider dg vector spaces in which $V_n = 0$ for $n < 0$. However, keeping the gradation in $\mathbb{Z}$ ensure that morphisms of negative degree are properly defined, etc.

Remark 47. Given two graded vector spaces V and W, $\mathrm{Hom}(V, W)$ is graded (by the degree of morphisms) and differential graded by the derivative (Definition 44) when V and W are dg.

Definition 48. A **homotopy** between two maps $f, g : V \to W$ of degree 0 is $h : V \to W$ of degree $+1$ such that

$$f - g \quad = \quad \partial(h) \quad = \quad d_W \circ h - h \circ d_V$$

Definition 49. Given a chain complex (V, d), the homology groups are

$$H_n(V) \quad = \quad \ker(d : V_n \to V_{n+1}) / \operatorname{im}(d : V_{n-1} \to V_n)$$

We write $H_\bullet(V)$ for the associated graded vector space. A **quasi-isomorphism** is a morphism of chain complexes which induces an isomorphism in homology. A chain complex V is **acyclic** when, for $n \neq 0$,

$$H_0(V) = \Bbbk \qquad\qquad H_n(V) = 0$$

which means that $\ker(d : V_n \to V_{n+1}) = \operatorname{im}(d : V_{n-1} \to V_n)$ for $n \geq 1$.

2.3 Differential graded (co)algebras

Definition 50. A **graded algebra** is a graded vector space $(A_n)_{n\geq 0}$ equipped with a product $\mu : A \otimes A \to A$ of degree zero: it is thus a family of maps $\mu_{p,q} : V_p \otimes V_q \to V_{p+q}$. A **differential graded algebra** is a dg vector space A such that the multiplication $\mu : A \otimes A \to A$ is a morphism of chain complexes. More explicitly, it consists of a graded vector space $(A_n)_{n\geq 0}$ which is a graded algebra such that differential is a derivation for the product

$$d \circ \mu \quad = \quad \mu \circ (d \otimes \mathrm{id}_A) + \mu \circ (\mathrm{id}_A \otimes d)$$

i.e.

$$d(ab) \quad = \quad d(a)\, b + (-1)^{|a|} a\, d(b)$$

It is **connected** when $A_0 = \Bbbk 1$.

Definition 51. A **differential graded coalgebra** is defined similarly.

Definition 52. Consider an augmented algebra $A = \Bbbk 1 \oplus \overline{A}$ with multiplication $\mu : A \otimes A \to A$. This algebra can be thought as a dg algebra concentrated in degree 0 with trivial differential, the general case being given below. The **bar construction** associates to it a coalgebra BA. As a differential algebra, we have $BA = T^c(s\overline{A})$. Notice that $s\overline{A} = \Bbbk s \otimes \overline{A}$ is equipped with a "multiplication"

$$s\mu \quad : \quad s\overline{A} \otimes s\overline{A} \to s\overline{A}$$

of degree -1 defined as

$$\Bbbk s \otimes \overline{A} \otimes \Bbbk s \otimes \overline{A} \xrightarrow{\mathrm{id}_{\Bbbk s} \otimes \tau \otimes \mathrm{id}_{\overline{A}}} \Bbbk s \otimes \Bbbk s \otimes \overline{A} \otimes \overline{A} \xrightarrow{\mu_s \otimes \mu} \Bbbk s \otimes \overline{A}$$

where $\mu_s : \mathbb{k}s \otimes \mathbb{k}s \to \mathbb{k}s$ is the map of degree -1 defined by $\mu_s(s \otimes s) = s$. This induces a map

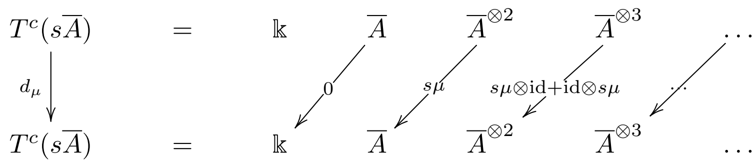
$$f_\mu \quad : \quad T^c(s\overline{A}) \quad \to \quad s\overline{A}$$

by precomposing $s\mu$ by the canonical projection $T^c(s\overline{A}) \twoheadrightarrow s\overline{A}$. Since $T^c(s\overline{A})$ is cofree, by Proposition 39 it extends as a unique coderivation

$$d_\mu \quad : \quad T^c(s\overline{A}) \quad \to \quad T^c(sA)$$

which satisfies $d_\mu^2 = 0$ because μ is associative.

The coderivation can be pictured as follows



Concretely, it can be identified with the (non-unital) Hochschild complex of $\overline{A}$

$$\ldots \longrightarrow \overline{A}^{\otimes 2} \longrightarrow \overline{A} \longrightarrow \mathbb{k}$$

with

$$d_\mu([a_1|\ldots|a_n]) \quad = \quad \sum_{i=1}^{n} (-1)^{i-1}[a_1|\ldots|\mu(a_i, a_{i+1})|\ldots|a_n]$$

For instance, in the case $[a_1|a_2|a_3] = (sa_1 \otimes sa_2 \otimes sa_3)$

$$\begin{aligned} d_\mu(sa_1 \otimes sa_2 \otimes sa_3) &= f_\mu(sa_1) \otimes sa_2 \otimes sa_3 - sa_1 \otimes f_\mu(sa_2) \otimes sa_3 + sa_1 \otimes sa_2 \otimes f_\mu(sa_3) \\ &\quad + f_\mu(sa_1 \otimes sa_2) \otimes sa_3 - sa_1 \otimes f_\mu(sa_2 \otimes sa_3) + f_\mu(sa_1 \otimes sa_2 \otimes sa_3) \\ &= s\mu(sa_1 \otimes sa_2) \otimes sa_3 - sa_1 \otimes s\mu(sa_2 \otimes sa_3) \end{aligned}$$

By the sign rule, we get a minus in the last line because μ is switched with sa_1, etc.

When A is a graded algebra the same construction works and we get

$$d_\mu(sa_1 \otimes \ldots \otimes sa_n) \quad = \quad \sum_{i=1}^{n} (-1)^{i-1+|a_1|+\ldots+|a_{i-1}|} sa_1 \otimes \ldots \otimes s\mu(a_i \otimes a_{i+1}) \otimes \ldots \otimes a_n$$

Lemma 53. *The map $d_\mu : T^c(s\overline{A}) \to T^c(s\overline{A})$ is a differential: $d_\mu^2 = 0$.*

When A is differential graded, the differential $d_A : A \to A$ induces a differential on $A^{\otimes n}$ (and thus on $T^c(sA)$) by

$$\sum_{i=1}^{n} \mathrm{id}_A \otimes \ldots \otimes d_A \otimes \ldots \otimes \mathrm{id}_A \quad : \quad A^{\otimes n} \quad \to \quad A^{\otimes n}$$

that we still write d_A. The chain complex BA then has the total differential

$$d_{BA} \quad = \quad d_\mu + d_A$$

Lemma 54. *The map d_{BA} satisfies $d_{BA}^2 = 0$.*

Proof. Since μ_A is a morphism of dgvs, we have $d_\mu \circ d_A + d_A \circ d_\mu = 0$. □

Proposition 55. *Any quasi-isomorphism $f : A \to A'$ of aug. dga induces a quasi-isomorphism $Bf : BA \to BA$.*

Proof. Spectral sequence. □

Definition 56. The **cobar construction** ΩC on a coaugmented dg coalgebra C is defined similarly. We start from the free dg algebra $T(s^{-1}\overline{C})$ and add to the differential, the differential obtained by extending $\overline{\Delta}$ by derivation in $T(s^{-1}\overline{C})$, seen as a bimodule over itself.

Theorem 57. *We have an adjunction*

$$\Omega \quad : \quad \{\text{con. dg coalg.}\} \quad \dashv \quad \{\text{aug. dg alg.}\} \quad : \quad B$$

Proof. See Theorem 67. □

2.4 Twisting morphisms

So far we have defined the bar complex BA. How do we define the bar resolution? We could take something like $BA \otimes A$. The boundary is almost what we expect, excepting that we miss the last term in the sum:

$$d([a_1|\dots|a_n]a_{n+1}) \quad = \quad \sum_{i=1}^{n}(-1)^{i-1}[a_1|\dots|a_i a_{i+1}|\dots|a_n] + (-1)^{n-1}[a_1|\dots|a_{n-1}]a_n a_{n+1}$$

The proper differential can be achieved by "twisting" the tensor product along a morphism π with suitable properties.

Definition 58. Given a coalgebra (C, Δ, ε) and an algebra (A, μ, η), the **convolution** product on $\mathrm{Hom}(C, A)$ defined by

$$f \star g \quad = \quad \mu \circ (f \otimes g) \circ \Delta$$

is associative, with $\eta \circ \varepsilon$ as unit, i.e. we have a **convolution algebra**

$$(\mathrm{Hom}(C, A), \star, \eta \circ \varepsilon)$$

Definition 59. Suppose given a dg algebra A. A **derivation** on a right A-module M is a linear $d_M : M \to M$ such that

$$d_M \circ \rho \quad = \quad \rho \circ (d_M \otimes \mathrm{id}_A) + \rho \circ (\mathrm{id}_M \otimes d_A)$$

i.e. for $m \in M$ and $a \in A$,

$$d_M(ma) \quad = \quad d_M(m)a + (-1)^{|m|} m d_A(a)$$

A **coderivation** of a left comodule is defined in a similar way.

Proposition 60. *Given a dg algebra A a dg coalgebra C and a chain complex N, we have*

$$\mathrm{Der}(N \otimes A) \quad \cong \quad \mathrm{Hom}(N, N \otimes A)$$

and

$$\mathrm{Coder}(C \otimes N) \quad \cong \quad \mathrm{Hom}(C \otimes N, N)$$

A linear map $\alpha : C \to A$ defines

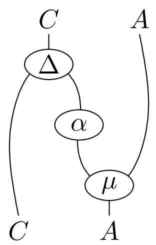
$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\mathrm{id}_C \otimes \alpha} C \otimes A$$

which extends as a derivation on $C \otimes A$ and a morphism

$$C \otimes A \xrightarrow{\alpha \otimes \mathrm{id}_A} A \otimes A \xrightarrow{\mu} A$$

which extends as a coderivation on $C \otimes A$. Both extensions are equal to

$$d_\alpha \quad = \quad C \otimes A \xrightarrow{\Delta \otimes \mathrm{id}_A} C \otimes C \otimes A \xrightarrow{\mathrm{id}_C \otimes \alpha \otimes \mathrm{id}_A} C \otimes A \otimes A \xrightarrow{\mathrm{id}_C \otimes \mu} C \otimes A$$



Example 61. Given an algebra A, consider $C = T^c A$. We have a projection $\pi : T^c A \twoheadrightarrow A$. The associated differential on $C \otimes A$ is given on $A^n \otimes A$ by

$$d_\pi(a_1 \otimes \ldots \otimes a_n)a_{n+1} \quad = \quad (a_1 \otimes \ldots \otimes a_{n-1})a_n a_{n+1}$$

which is almost (up to sign and grading) the missing part from the differential on $BA \otimes A$.

Lemma 62. *Given $\alpha, \beta : C \to A$, we have*

$$d_\alpha \circ d_\beta = d_{\alpha \star \beta} \qquad \textit{and} \qquad d_{\varepsilon \circ \eta} = \mathrm{id}_{c \otimes A}$$

Thus if $\alpha \star \alpha = 0$, we have $d_\alpha^2 = 0$ and we get a chain complex $(C \otimes A, d_\alpha)$.

Remark 63. In the (non-differential) graded setting, the convolution algebra $(\mathrm{Hom}(C, A), \star, \partial)$ is a dg algebra equipped with the derivative ∂ of graded linear maps (Definition 44): given $f : C \to A$, we recall $\partial f = d_A \circ f - (-1)^{|f|} f \circ d_A$.

In the dg setting, things goes on as follows. We would like $(C \otimes A, d_\alpha)$ to be a chain complex with

$$d_\alpha \quad = \quad d_{C \otimes A} + d'_\alpha$$

where $d_{C \otimes A} = d_C \otimes \mathrm{id}_A + \mathrm{id}_C \otimes d_A$ and

$$d'_\alpha \quad = \quad (\mathrm{id}_C \otimes \mu) \circ (\mathrm{id}_C \otimes \alpha \otimes \mathrm{id}_A) \circ (\Delta \otimes \mathrm{id}_A)$$

is the previously defined differential.

Proposition 64. *We have*

$$d_\alpha^2 \quad = \quad d'_{\partial(\alpha) + \alpha \star \alpha}$$

It is a derivation if and only if α satisfies the ***Maurer-Cartan equation***

$$\partial(\alpha) + \alpha \star \alpha \quad = \quad 0$$

Such an $\alpha : C \to A$ of degree -1 is called a ***twisting morphism****.*

Remark 65. In the dg setting, when $\alpha : C \to A$ is twisting we get a dg algebra $(\mathrm{Hom}(C, A), \star, \partial_\alpha)$ with perturbed differential

$$\partial_\alpha(f) \quad = \quad \partial(f) + [\alpha, f]$$

Definition 66. Given a twisting morphism $\alpha : C \to A$, the twisted tensor product is

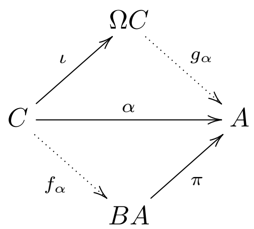
$$(C \otimes_\alpha A, d_\alpha)$$

Theorem 67. *The adjunction of Theorem 57 can be factored through twisting morphisms:*

$$\mathrm{Hom}_{\text{dg alg}}(\Omega C, A) \quad \cong \quad \mathrm{Hom}_{\text{twisting}}(C, A) \quad \cong \quad \mathrm{Hom}_{\text{dg coalg}}(C, BA)$$

Proof. Consider the first bijection. A dg alg morphism $f : \Omega C \to A$ is characterized by its restriction to $\overline{C}$ since $\Omega C \cong T(s^{-1}\overline{C})$ (as an algebra) is free. And since we consider C coaugmented and A augmented, α sends $\Bbbk$ to 0 and $\overline{C}$ to $\overline{A}$. By commutation to differentials, we get the fact that the morphism is actually twisting. Conversely, a twisting morphism $\alpha : C \to A$ extends by derivation as a morphism of algebras $\Omega C \to A$. □

Proposition 68. *By the universal property of the adjunction $\Omega \dashv B$, any twisting morphism $\alpha : C \to A$ factorizes uniquely as*



where g_α is a dg algebra morphism and f_α is a dg coalgebra morphism.

Proof. We show this for the lower triangle (the upper one is similar). Consider a dg coalgebra C and the dg algebra morphism $\mathrm{id}_{\Omega C} : \Omega C \to \Omega C$. Since we have $\mathrm{Hom}_{\text{dg alg}}(\Omega C, \Omega C) \cong \mathrm{Hom}_{\text{twisting}}(C, \Omega C)$, we get the twisting morphism $\pi : C \to \Omega C$ and the natural bijection gives the universal property. □

Concretely, π is the composite

$$BA = T^c(s\overline{A}) \twoheadrightarrow s\overline{A} \cong \overline{A} \rightarrowtail A$$

of degree -1 (and ι is similar).

Definition 69. Given a dg algebra A, its (right) **bar resolution** is

$$BA \otimes_\pi A$$

Proof. This is the bar (or Hochschild) resolution, acyclicity can be checked as usual, for instance using a contracting homotopy. □

3 Koszul duality

3.1 Quadratic algebras

Given an algebra A, we know that we can always compute the bar resolution $BA \otimes_\pi A$. However, BA is quite big and often impractical for computation purposes, so we would like to find a smaller dg coalgebra C such that $C \otimes_\pi A$ is acyclic. Koszul duality provides a way to do this in the quadratic case (there are some extensions to quadratic + linear and + constants). First, to any algebra A we associate a "natural" coalgebra $A^{¡}$, called the Koszul dual coalgebra, which is a good candidate for a coalgebra such that $A^{¡} \otimes A$ is acyclic. When it is the case, the algebra A is called Koszul and various properties to show that an algebra is Koszul are studied.

In the following, we will consider quadratic algebras, i.e. of the form

$$A(V,R) \quad = \quad TV/(R)$$

with $R \subseteq V^{\otimes 2}$. If we do this by hand, we are lead to consider C of the form

$$C \quad = \quad V \oplus R \oplus (R \otimes V \cap V \otimes R) \oplus \dots$$

In good cases (when the algebra is Koszul), this will actually prove to provide a resolution.

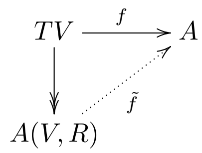
Definition 70. Given $R \subseteq V^{\otimes 2}$ the algebra

$$A(V,R) \quad = \quad TV/(R)$$

is the quotient of TV by the two-sided ideal generated by R. It is universal among algebras such that

$$R \rightarrowtail TV \twoheadrightarrow A \quad = \quad 0$$

i.e. for any such algebra A, an algebra morphism $f : TV \to A$ factors uniquely through the quotient $TV \twoheadrightarrow A(V,R)$:



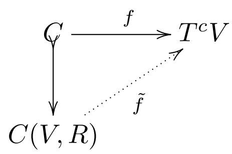
Notice that $A(V,R)$ is graded by the *weight* (length of words) and augmented:

$$A \quad = \quad \bigoplus_{n\geq 0} A^{(n)} \quad = \quad \Bbbk 1 \oplus V \oplus (V^{\otimes 2}/R) \oplus \dots \oplus \left(V^{\otimes n} / \sum_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right) \oplus \dots$$

Definition 71. The same game can be played for coalgebras. The coalgebra $C(V,R)$ is the subcoalgebra of T^cV which is universal among subcoalgebras C of T^cV such that

$$C \rightarrowtail T^cV \twoheadrightarrow V^{\otimes 2}/R \quad = \quad 0$$

For any such coalgebra C a coalgebra morphism $f : C \to T^c V$ factors uniquely as



Explicitly,

$$C = \bigoplus_{n\geq 0} C^{(n)} = \Bbbk 1 \oplus V \oplus R \oplus (R\otimes V \cap V \otimes R) \oplus \ldots \oplus \left(\bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \right) \oplus \ldots$$

3.2 Dual coalgebra and algebra

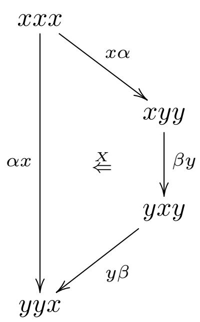
Definition 72. The **Koszul dual coalgebra** $A^{¡}$ of a quadratic algebra $A = A(V, R)$ is the coalgebra

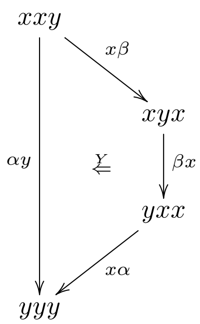
$$A^{¡} = C(sV, s^2 R)$$

Example 73. Consider the algebra $\langle x, y \mid xx - yy, xy - yx\rangle$ with the augmentation $\varepsilon(x) = \varepsilon(y) = 0$. Notice that with $x > y$, we have a Gröbner basis:

$$\alpha : xx \to yy \qquad \beta : xy \to yx$$

Namely,





The dual coalgebra is given by

$$A^{¡(0)} = \Bbbk \qquad A^{¡(1)} = \Bbbk\{x, y\} \qquad A^{¡(2)} = \Bbbk\{xx - yy, xy - yx\}$$

Next, $A^{¡(3)} = V \otimes R \cap R \otimes V$ is given by solutions of

$$\begin{aligned} & a(xxx - xyy) + b(yxx - yyy) + c(xxy - xyx) + d(yxy - yyx) \\ = \;& e(xxx - yyx) + f(xxy - yyy) + g(xyx - yxx) + h(xyy - yxy) \end{aligned}$$

i.e.

$$\begin{aligned} & axxx + cxxy - cxyx - axyy + byxx + dyxy - dyyx - byyy \\ = \;& exxx + fxxy + gxyx + hxyy - gyxx - hyxy - eyyx - fyyy \end{aligned}$$

Therefore

$$\begin{array}{cccc} a = e & c = f & -c = g & -a = h \\ b = -g & d = -h & d = e & b = f \end{array}$$

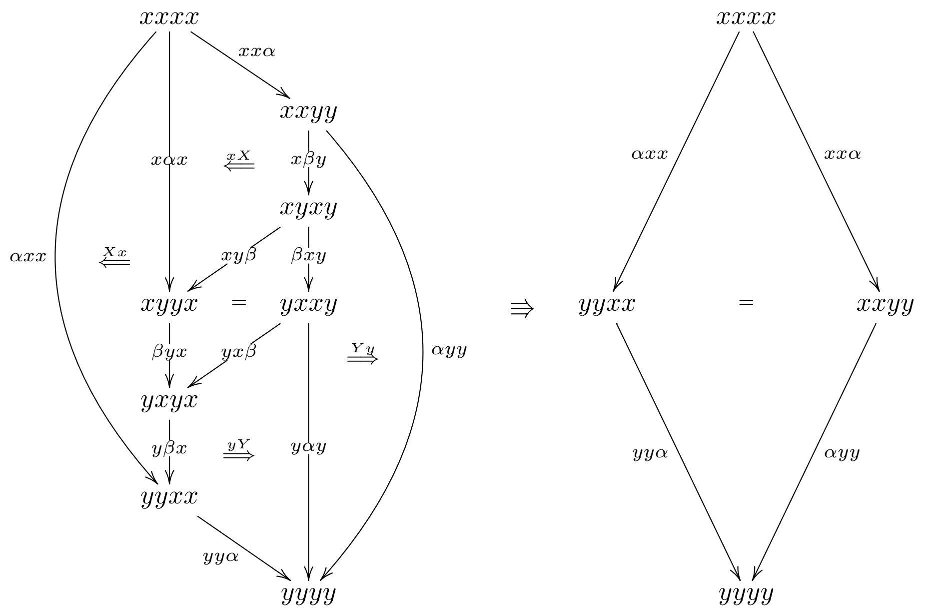
i.e.

$$a = d = e = -h \qquad\qquad b = c = f = -g$$

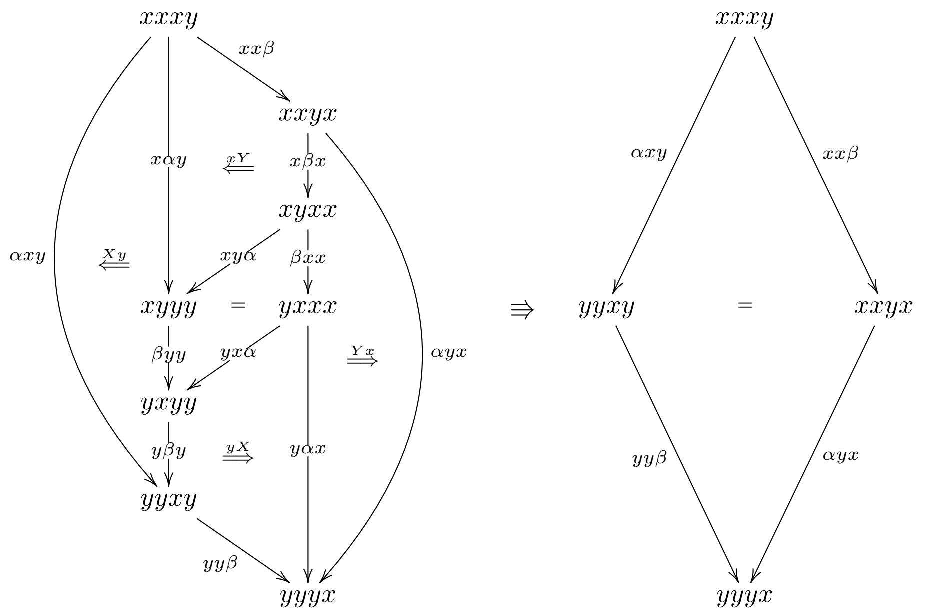
and we have

$$A^{\mathsf{i}(3)} \quad = \quad \Bbbk\{xxx - xyy + yxy - yyx, xxy - xyx + yxx - yyy\} \quad = \quad \Bbbk\{X, Y\}$$

(we write X and Y for the two elements of the basis generated by xxx and xxy respectively). Similarly, critical triples are



and



We have

$$A^{¡(4)} \quad = \quad R \otimes V \otimes V \cap V \otimes R \otimes V \cap V \otimes V \otimes R$$

and its elements are given by solutions of

$$\begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ -e \\ -f \\ -g \\ -h \\ -a \\ -b \\ -c \\ -d \end{pmatrix} = \begin{pmatrix} a' \\ b' \\ e' \\ f' \\ -e' \\ -f' \\ -a' \\ -b' \\ c' \\ d' \\ g' \\ h' \\ -g' \\ -h' \\ -c' \\ -d' \end{pmatrix} = \begin{pmatrix} a'' \\ e'' \\ -e'' \\ -a'' \\ b'' \\ f'' \\ -f'' \\ -b'' \\ c'' \\ g'' \\ -g'' \\ -c'' \\ d'' \\ h'' \\ -h'' \\ -d'' \end{pmatrix} \qquad \begin{matrix} xxxx \\ xxxy \\ xxyx \\ xxyy \\ xyxx \\ xyxy \\ xyyx \\ xyyy \\ yxxx \\ yxxy \\ yxyx \\ yxyy \\ yyxx \\ yyxy \\ yyyx \\ yyyy \end{matrix}$$

and therefore

$$a = a' = -g = g' = a'' = -f' = f'' = -d = -d'' = f = \dots$$
$$b = b' = -h = h' = e'' = -e' = b'' = -c = h'' = e = \dots$$

i.e. restricting to first column

$$a = -d = f = -g \qquad b = -c = e = -h$$

A basis for $A^{¡(4)}$ is thus

$$xxxx - xxyy + xyxy - xyyx - yyxx + yyyy - yxxy + yxyx$$
$$xxxy - xxyx + xyxx - xyyy - yyxy + yyyx - yxxx + yxyy$$

Notice that we could have been a little smarter and used

$$A^{¡(4)} \quad = \quad A^{¡(3)} \otimes V \cap V \otimes A^{¡(3)}$$

but anyway this is starting to get quite boring. We will see a much more direct way of computing $A^{¡}$ in Example 76, via the dual algebra.

Since people are more used to algebras than coalgebras, the dual algebra is more often considered.

Definition 74. The **Koszul dual algebra** $A^!$ of an algebra A is the suspended linear dual of the dual coalgebra $A^{¡}$:

$$(A^!)^{(n)} \quad = \quad s^n (A^{¡*})^{(n)}$$

equipped with "obvious" multiplication coming from the comultiplication of $A^{¡}$.

In order to compute a presentation for it, notice that dualizing the exact sequence

$$0 \to R \rightarrowtail V^{\otimes 2} \twoheadrightarrow V^{\otimes 2}/R \to 0$$

yields the exact sequence

$$0 \leftarrow R^* \twoheadleftarrow (V^*)^{\otimes 2} \leftarrowtail R^\perp \leftarrow 0$$

where $R^\perp$ is the image of $(V^{\otimes 2}/R)^*$ in $(V^*)^{\otimes 2}$ through the iso $(V^{\otimes 2})^* \cong (V^*)^{\otimes 2}$ (there is a canonical map $(V^*)^{\otimes 2} \to (V^{\otimes 2})^*$, which is an isomorphism in finite dimension). The notation $R^\perp$ comes from the fact that $R^\perp$ is the vector subspace of $(V^{\otimes 2})^*$ of functions vanishing on R, i.e. is orthogonal to R.

Proposition 75. *The Koszul dual algebra of $A(V, R)$ is*

$$A^! \quad = \quad A(V^*, R^\perp)$$

Example 76. Going back to Example 73. We have the two rules

$$\alpha = xx - yy \qquad \text{and} \qquad \beta = xy - yx$$

Notice that $V^{\otimes 2}/R$ admits $u = yy$ and $v = yx$ as (PBW) basis. So, we have made explicit the first exact sequence

$$0 \to \Bbbk\{\alpha, \beta\} \rightarrowtail \Bbbk\{x, y\}^{\otimes 2} \twoheadrightarrow \Bbbk\{u, v\} \to 0$$

Writing ϕ and ψ for the two non-trivial morphisms, we have

$$\phi(\alpha) = xx - yy \quad \phi(\beta) = xy - yx \quad \psi(xx) = u \quad \psi(xy) = v \quad \psi(yx) = v \quad \psi(yy) = u$$

Thus, dualizing we get the map $\psi^* : \Bbbk\{u, v\}^* \to (\Bbbk\{x, y\}^{\otimes 2})^*$ defined by

$$\psi^*(u^*) = u^* \circ \psi \qquad \psi^*(v^*) = v^* \circ \psi$$

Therefore,

$$\psi^*(u^*)(xx) = \psi^*(u^*)(yy) = 1 \qquad \psi^*(u^*)(xy) = \psi^*(u^*)(yx) = 0$$

(and similarly for v^*), and we deduce

$$\psi^*(u^*) = (xx)^* + (yy)^* \qquad \psi^*(v^*) = (xy)^* + (yx)^*$$

By postcomposing with the isomorphism $(\Bbbk\{x, y\}^{\otimes 2})^* \cong (\Bbbk\{x, y\}^*)^{\otimes 2}$, we get

$$\psi^*(u^*) = x^* \otimes x^* + y^* \otimes y^* \qquad \psi^*(v^*) = x^* \otimes y^* + y^* \otimes x^*$$

which means

$$A^! \quad = \quad \langle x^*, y^* \mid x^*x^* + y^*y^*, x^*y^* + y^*x^* \rangle$$

A simpler way to perform this computation is to consider the matrix associated to ψ:

$$M_\psi \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad : \quad \Bbbk\{xx, xy, yx, yy\} \quad \to \quad \Bbbk\{u, v\}$$

and the matrix for the dual is obtained by transposition

$$M_{\psi^*} \quad = \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \quad : \quad \Bbbk\{u^* v^*\} \quad \to \quad \Bbbk\{xx^*, xy^*, yx^*, yy^*\}$$

Let us describe $A^!$ more explicitly. We can orient the relations by $x^* < y^*$:

$$y^* y^* \to -x^* x^* \qquad y^* x^* \to -x^* y^*$$

We have a Gröbner basis and thus

$$A^! \quad = \quad \Bbbk\{x^{*n}, x^{*n} y^*\}$$

with multiplication

$$\mu(x^{*m}, x^{*n}) = x^{*m+n} \qquad \mu(x^{*m}, x^{*n} y) = x^{*m+n} y$$
$$\mu(x^{*m} y, x^{*n}) = (-1)^n x^{*m+n} y \qquad \mu(x^{*m} y, x^{*n} y) = (-1)^{n+1} x^{*m+n+2}$$

From which, we immediately deduce that $\dim A^{¡(n)} = \dim A^{!(n)} = 2$, i.e.

$$A^{¡} \quad = \quad \Bbbk\{x^n, x^n y\}$$

with comultiplication

$$\begin{aligned} \Delta(x^n) \quad &= \quad \sum_{i=0}^{n} x^i \otimes x^{n-i} + \sum_{i=0}^{n-2} (-1)^{n-i} x^i y \otimes x^{n-i-2} y \\ &\quad + \sum_{i=1}^{n-1} (-1)^{n-i-1} x^{i-1} y \otimes x^{n-i-1} y + \sum_{i=2}^{n} (-1)^{n-i} x^{i-2} y \otimes x^{n-i} y \\ \Delta(x^n y) \quad &= \quad \sum_{i=0}^{n} (-1)^{n-i} x^i y \otimes x^{n-i} + x^i \otimes x^{n-i} y \end{aligned}$$

For instance

$$\begin{aligned} \Delta(xx) \quad &= \quad 1 \otimes xx + x \otimes x - y \otimes y + xx \otimes 1 \\ \Delta(xy) \quad &= \quad 1 \otimes xy + x \otimes y - y \otimes x + xy \otimes 1 \\ \Delta(xxx) \quad &= \quad 1 \otimes xxx + x \otimes xx + xx \otimes x + xxx \otimes 1 - xy \otimes y + y \otimes xy \\ \Delta(xxy) \quad &= \quad 1 \otimes xxy + x \otimes xy + xx \otimes y + y \otimes xx - xy \otimes x + xxy \otimes 1 \end{aligned}$$

We recover the previous presentation given in Example 73 with the following renaming of generators:

above presentation	previous presentation (Ex 73)	
x	x	$\in V$
y	y	$\in V$
xx	$xx - yy$	$\in R$
xy	$xy - yx$	$\in R$
xxx	$xxx - xyy + yxy - yyx$	$\in VR \cap RV$
xxy	$xxy - xyx + yxx - yyy$	$\in VR \cap RV$
$\vdots$	$\vdots$	

3.3 Some more examples

We give here some more examples.

3.3.1 The free algebra

Consider the free algebra

$$A(V, 0)$$

The dual coalgebra is

$$A^{¡} \quad = \quad C(sV, 0) \quad = \quad \mathbb{k}1 \oplus V$$

The dual algebra is given by $R^{\perp} = V^{\otimes 2}$ and therefore

$$A^{!} \quad = \quad A(V^*, (V^*)^{\otimes 2}) \quad = \quad \mathbb{k} \oplus V$$

Given a basis ε_i of V, multiplication is

$$\left(a + \sum_i b_i \varepsilon_i\right)\left(a' + \sum_i b'_i \varepsilon_i\right) \quad = \quad aa' + \sum_i (b_i a' + a b'_i)\, \varepsilon_i$$

$A^{!}$ is called the **algebra of dual numbers**.

3.3.2 An example from Fröberg

This comes from [Frö99]. Consider

$$A \quad = \quad \mathbb{k}[x, y, z]/(x^2, yz, xz - z^2)$$

i.e.

$$A \quad = \quad \left\langle x, y, z \;\middle|\; x^2, yz, xz - z^2, xy - yx, xz - zx, yz - zy \right\rangle$$

(the generators for R are independent). If we orient them as rules and complete in order to remove inclusion critical pairs we get

$$x^2 \Rightarrow 0 \qquad yz \Rightarrow 0 \qquad xz \Rightarrow z^2 \qquad xy \Rightarrow yx \qquad zx \Rightarrow z^2 \qquad zy \Rightarrow 0$$

A basis for $A^{\otimes 2}/R$ is thus

$$R \quad = \quad \mathbb{k}\{yx, yy, zz\}$$

and the quotient matrix $A \to A^{\otimes 2}/R$ is

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

The dual algebra is thus

$$A^{!} \quad = \quad \langle x^*, y^*, z^* \mid x^*y^* + y^*x^*, y^*y^*, x^*z^* + z^*x^* + z^*z^* \rangle$$

3.3.3 The symmetric algebra

Consider the **symmetric algebra** over $V = \Bbbk\{x_i\}$

$$SV \quad = \quad \langle x_i \mid x_i x_j - x_j x_i \rangle$$

A basis for R is thus $[x_i, x_j]$ with $i < j$. The Koszul dual $A^{¡} = C(sV, s^2R)$:

$$A^{¡} \quad = \quad \Bbbk \oplus V \oplus R \oplus (R \otimes V \cap V \otimes R) \oplus \dots$$

We have that R is spanned in $V^{\otimes 2}$ by

$$\left\{ \sum_{\sigma \in \Sigma_2} \operatorname{sgn}(\sigma) s x_{\sigma(1)} + s x_{\sigma(2)} \;\middle|\; x_1, x_2 \in V \right\}$$

and more generally $A^{¡(n)}$ is spanned in $V^{\otimes n}$ by

$$\left\{ \sum_{\sigma \in \Sigma_n} \operatorname{sgn}(\sigma) s x_{\sigma(1)} + s x_{\sigma(2)} + \dots + s x_{\sigma(n)} \;\middle|\; x_1, \dots, x_n \in V \right\}$$

For instance, consider $A^{¡(3)} = R \otimes V \cap V \otimes R$, an element of the basis is of the form

$$\begin{aligned} & x_i x_j x_k - x_j x_i x_k + x_j x_k x_i - x_k x_j x_i + x_k x_i x_j - x_i x_k x_j \\ = \quad & x_i(x_j x_k - x_k x_j) + x_j(x_k x_i - x_i x_k) + x_k(x_i x_j - x_j x_i) \\ = \quad & (x_j x_k - x_k x_j)x_i + (x_k x_i - x_i x_k)x_j + (x_i x_j - x_j x_i)x_k \end{aligned}$$

This coalgebra is called the **exterior coalgebra** $\Lambda^c(sV)$. Its Koszul dual is the **exterior algebra**

$$SV^! \quad = \quad \Lambda(V^*) \quad = \quad \left\langle x_i^* \;\middle|\; x_i^* x_j^* + x_j^* x_i^* \right\rangle$$

3.3.4 Quantum stuff

The **quantum plane** is a variant over the preceeding example (see e.g. [Man87]):

$$A \quad = \quad \langle x, y \mid xy - qyx \rangle$$

The Koszul dual coalgebra is

$$A^{¡} \quad = \quad \Bbbk \oplus \Bbbk\{x, y\} \oplus \Bbbk\{xy - qyx\}$$

i.e. $(A^{¡})^{(n)} = 0$ for $n \geq 3$ and the Koszul dual algebra is

$$A^! \quad = \quad \langle x^*, y^* \mid x^* x^*, (1-q) y^* x^*, y^* y^* \rangle$$

This could be called a **quantized exterior algebra** [Krä].

We can also consider **quantum matrices**

$$\left\langle a, b, c, d \;\middle|\; ab = qba, ac = qca, ad - da = (q - q^{-1})bc, bc = cb, bd = qdb, cd = qdc \right\rangle$$

3.3.5 Limits of Gröbner basis

In [Frö99] is recalled an example from [ERT94]:

$$A \quad = \quad \mathbb{k}[x,y,z]/\mathbb{k}\{xx+xy, yy+yz, zz+zx\}$$

which is a Koszul commutative quadratic algebra, but admits no quadratic Gröbner basis, for any coordinates and any monomial order (see also Section 3.6).

TODO: the following explanation is messed up because we are in the commutative case... Notice that it is "obviously" Kosuzl if we could allow rewriting systems instead of Gröbner basis since the rewriting system

$$xx \to xy \qquad yy \to yz \qquad zz \to zx$$

has no critical pairs. And indeed, we have the following linear resolution by free left A-modules

$$0 \to A^{\otimes 3} \xrightarrow{\begin{pmatrix} x+y & 0 & 0 \\ 0 & y+z & 0 \\ 0 & 0 & z+x \end{pmatrix}} A^{\otimes 3} \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} A \xrightarrow{\varepsilon} \mathbb{k} \to 0$$

3.3.6 The Sklyanin algebra

In the same vein consider the Sklyanin algebra

$$\left\langle x,y,z \;\middle|\; xyz = x^3+y^3+z^3 \right\rangle$$

It is Koszul [TVdB96] but admits no Gröbner basis [Ber98].

3.3.7 No bounds for linear resolutions

In [Frö99] is recalled an example from [FL91] showing that there is no bound until which it is enough to check that an algebra admits a linear resolution to be Koszul. Consider

$$A \quad = \quad \left\langle a,b,c,d \;\middle|\; ab-ac, bc-cb-\lambda c^2, bd \right\rangle$$

In characteristic 0, if $\lambda^{-1} = l \in \mathbb{N}$ then $\mathrm{Tor}_n^{A^!}(\mathbb{k},\mathbb{k})$ is concentrated in weight n if $n \leq l+2$, but not for $n = l+3$.

3.4 Koszul at the bar

Given an algebra $A = A(V,R)$, consider the bar construction $BA = T^c(s\overline{A})$. It is bigraded: an element $[a_1|\ldots|a_n]$ of $(s\overline{A})^n$ has

- a **homological degree**: n
- a **weight**: $\omega([a_1|\ldots|a_n]) = \omega(a_1) + \ldots + \omega(a_n)$

where the weight of an element of A is the grading coming from the grading of TA since $A = TV/(R)$ and R is homogeneous. Bigraded in this way, the bar chain complex $B_\bullet A$ looks like

$$
\begin{array}{ccccccccl}
\vdots & & \vdots & & \vdots & & \vdots & & (4) \\
V\otimes V\otimes V & \longrightarrow & (V^2/R\otimes V)\oplus(V\otimes V^2/R) & \longrightarrow & V^3/(VR+RV) & \longrightarrow & 0 & & (3) \\
 & & V\otimes V & \longrightarrow & V^2/R & \longrightarrow & 0 & & (2) \\
 & & & & V & \longrightarrow & 0 & & (1) \\
 & & & & & & \Bbbk & & (0) \\
3 & & 2 & & 1 & & 0 & &
\end{array}
$$

Notice that the diagonal is T^cV. In order to make this diagonal into a column, we now define

- a **syzygy degree**: $\omega([a_1|\dots|a_n]) - n$

Since A has trivial differential, the differential of BA is d_μ which is of weight degree 0 and homological degree -1 (of course since it's a differential). Therefore it is of syzygy degree $+1$. So, we have a cochain complex $B^\bullet A$ (this notation with index as exponent of B is for the syzygy graduation) which splits wrt weight. It therefore looks like

$$
\begin{array}{ccccccccl}
\vdots & & \vdots & & \vdots & & \vdots & & (4) \\
0 & \longleftarrow & V^3/(VR+RV) & \longleftarrow & (V^2/R\otimes V)\oplus(V\otimes V^2/R) & \longleftarrow & V\otimes V\otimes V & & (3) \\
 & & 0 & \longleftarrow & V^2/R & \longleftarrow & V\otimes V & & (2) \\
 & & & & 0 & \longleftarrow & V & & (1) \\
 & & & & & & \Bbbk & & (0) \\
3 & & 2 & & 1 & & 0 & &
\end{array}
$$

Columns are syzygy degrees and lines are weight degree. Notice that the Koszul dual coalgebra is a subspace of the first column.

Proposition 77. *Given $A = A(V,R)$ and its dual $A^{\text{!`}} = C(sV, s^2R)$, the natural inclusion $A^{\text{!`}} \rightarrowtail BA$ (in degree 0) induces an isomorphism of graded coalgebras*

$$A^{\text{!`}} \quad \xrightarrow{\sim} \quad H^0(B^\bullet A)$$

i.e.

$$(A^{\text{!`}})^{(n)} \quad \cong \quad H^0(B^\bullet A)^{(n)}$$

3.5 The Koszul resolution

Given an algebra $A(V,R)$, we define a twisting morphism $\kappa : A^{¡} \to A$ from the Koszul dual by

$$A^{¡} = C(sV, s^2R) \twoheadrightarrow sV \xrightarrow{s^{-1}} V \rightarrowtail A(V,R) = A$$

which is of degree -1.

Definition 78. The **Koszul complex** is $A^{¡} \otimes_\kappa A$.

Notice that the summand of weight n of this complex is

$$0 \to A^{¡(n)} \to A^{¡(n-1)} \otimes A^{(1)} \to \ldots \to A^{¡(1)} \otimes A^{(n-1)} \to A^{(n)} \to 0$$

Definition 79. An algebra is **Koszul** when its Koszul complex is acyclic.

Proposition 80. *An algebra is Koszul iff*

$$A^{¡} \quad \cong \quad H^0(B^\bullet A)$$

i.e. $A^{¡} \cong H^0(B^\bullet A)$ *as in Proposition 77, and* $H^n(BA) = 0$ *for* $n > 0$.

This means that the cohomology of BA is concentrated in syzygy degree 0, which can be equivalently rephrased as homological degree n is concentrated in weight n, i.e. the homology of its bar complex is diagonal. Since the homology does not really depends on the complex but on A,

Proposition 81. *An algebra A is Koszul iff it has diagonal homology:*

$$\mathrm{Tor}_n^A(\Bbbk, \Bbbk)^{(m)} \quad = \quad 0 \qquad\qquad \textit{for } n \neq m$$

Proposition 82. *An algebra A is Koszul iff it admits a linear minimal graded resolution of* $\Bbbk$ *by free A-modules. We recall that* linear *means that P_n is concentrated in degree n and* minimal *means that* $H_n(P_\bullet \otimes_A \Bbbk) = 0$ *(and actually a linear resolution is always minimal).*

Remark 83. In [Krä], this is reformulated by saying that A admits a minimal resolution of $\Bbbk$, such that the matrices of the differentials have coefficients in $A^{(1)}$. For instance, the quantum plane

$$\langle x, y \mid xy - qyx \rangle$$

(see Section 3.3.4) is Koszul because we have the following resolution of $\Bbbk$ by left A-modules:

$$0 \to A \xrightarrow[d_2]{\begin{pmatrix} -qy \\ x \end{pmatrix}} A^{\otimes 2} \xrightarrow[d_1]{\begin{pmatrix} x & y \end{pmatrix}} A \xrightarrow{\varepsilon} \Bbbk \to 0$$

Namely, $\ker \varepsilon$ is generated (as a left A-module) by x and y. To compute $\ker d_1$, notice that $x^i y^j$ forms a basis of A and

$$\begin{aligned} d_1\left(\sum_{ij} \lambda_{ij} x^i y^j \otimes \sum_{ij} \rho_{ij} x^i y^j\right) &= \sum_{ij} \lambda_{ij} x^i y^j x + \sum_{ij} \rho_{ij} x^i y^{j+1} \\ &= \sum_{ij} \lambda_{ij} q^{-j} x^{i+1} y^j + \sum_{ij} \rho_{ij} x^i y^{j+1} \\ &= \sum_{i \geq 0} \lambda_{i0} q x^{i+1} + \sum_{i \geq 0} \rho_{0i} y^{i+1} + \sum_{i,j>0} \left(\lambda_{(i-1)j} q^{-j} + \rho_{i(j-1)}\right) x^i y^j \end{aligned}$$

and $\ker d_1$ is generated by elements which satisfy $\lambda_{i0}q = 0$, $\rho_{0i} = 0$ and $\lambda_{(i-1)j}q^{-j} + \rho_{i(j-1)} = 0$, i.e. spanned over $\Bbbk$ by

$$-qx^iy^{j+1} \otimes q^{-j}x^{i+1}y^j \quad = \quad x^iy^j(-qy \otimes x)$$

i.e. generated by $(-qy, x)$ as a left A-module. Notice that we get precisely Koszul and Anick resolutions. In the general case, we do understand the criterion from the Anick resolution when the presentation is quadratic, since in this case the coefficients of the differential are in $A^{(1)}$.

Proposition 84. *An algebra A is Koszul iff the dual algebra $A^!$ is Koszul.*

3.6 Koszulity and rewriting

One way to show that a quadratic algebra is Koszul is to

- order the generators
- consider the associated deglex ordering, this gives an orientation of the generators of R as rewriting rules
- check that critical pairs are confluent

Theorem 85. *If a quadratic algebra admits an ordering on generators, for which the associated rewriting system is confluent then the algebra is Koszul.*

Proof. In this case the Anick resolution is diagonal. □

In the following, we suppose fixed $A(V, R)$, with $A = \Bbbk X$, and a total ordering on the generators. This ordering is extended (by deglex for instance) as a total ordering on X^*. We consider the filtration of A

$$F_uA \quad = \quad \operatorname{im}\left(\left(\bigoplus_{v \leq u} \Bbbk\{v\}\right) \rightarrowtail TV \twoheadrightarrow A\right)$$

(F_uA are elements of A which can be written by words $\leq u$) and define the associated graded algebra

$$\operatorname{gr}_u A \quad = \quad F_uA/F_{u^-}A$$

where u^- is the immediate predecessor of u, whose multiplication

$$\mu \quad : \quad \operatorname{gr}_u A \otimes \operatorname{gr}_v A \quad \to \quad \operatorname{gr}_{uv} A$$

is given by original multiplication in A. If we have a Gröbner basis, $\operatorname{gr}_u A$ is the set of polynomials in normal form whose leading term is u.

Example 86. In $\langle x, y \mid xx - yy, xy - yx\rangle$ with $x > y$, we have

$$\begin{array}{rrrcl} \mu & : & \operatorname{gr}_x \otimes \operatorname{gr}_x & \to & \operatorname{gr}_{xx} \\ & & x \otimes x & \mapsto & 0 \end{array} \qquad \begin{array}{rrrcl} \mu & : & \operatorname{gr}_y \otimes \operatorname{gr}_y & \to & \operatorname{gr}_{yy} \\ & & y \otimes y & \mapsto & yy \end{array}$$

Theorem 87. *A is Koszul iff $\operatorname{gr} A$ is Koszul.*

Proof. Spectral sequence. □

We write

$$R_{\mathrm{lead}} \quad = \quad \ker\left(V^{\otimes 2} \rightarrowtail TV \twoheadrightarrow \operatorname{gr} A\right)$$

(the space generated by leading terms of elements of R) and define

$$\mathring{A} \quad = \quad TV/(R_{\mathrm{lead}})$$

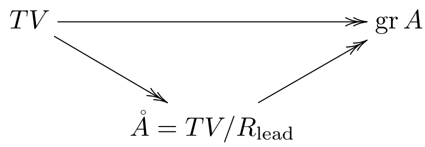
Example 88. For $A = \langle x, y \mid xx - yy, xy - yx \rangle$ with $x > y$,

$$\mathring{A} \quad = \quad \Bbbk\{y^n x, y^n y\}$$

with multiplication

$$\mu(y^n x, y^m x) = \mu(y^n x, y^m y) = 0 \quad \mu(y^n y, y^m y) = y^{n+m+1} y \quad \mu(y^n y, y^m x) = y^{n+m+1} x$$

We have a commutative diagram of epimorphisms of graded modules (but not algebras in general) which respects the grading in X^*:

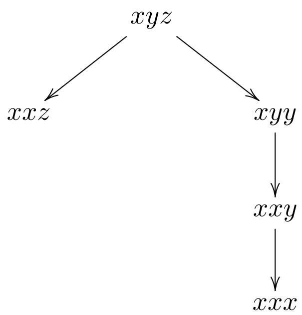


Notice that $\mathring{A} \twoheadrightarrow \operatorname{gr} A$ is bijective in weights 0, 1 and 2.

Example 89. Consider $\langle x, y, z \mid xy \Rightarrow xx, yz \Rightarrow yy \rangle$ with $x < y < z$. Notice that

$$\mathring{A}^{(3)} \quad = \quad \Bbbk\{xxx, xxz, xzx, xzy, xzz, yxx, yxz, yyx, yyy\}$$

We have a critical pair which is not confluent



and therefore xxz is killed by xxx in $\operatorname{gr} A$:

$$(\operatorname{gr} A)^{(3)} \quad = \quad \Bbbk\{xxx, xzx, xzy, xzz, yxx, yxz, yyx, yyy\} \quad \neq \quad \mathring{A}^{(3)}$$

Lemma 90. *Consider $A(V, R)$ with a monomial ordering. If the algebra $\mathring{A} = TV/R_{\mathrm{lead}}$ is Koszul and if the canonical projection $\mathring{A} \twoheadrightarrow \operatorname{gr} A$ is an isomorphism of algebras then A is Koszul.*

Proof. Immediate by Theorem 87. □

Lemma 91. *If generators in X are totally ordered (which is the case we are usually considering), i.e. the decomposition*

$$V = \bigoplus_{x \in X} V_x$$

consists of one-dimensional vector spaces $V_x = \Bbbk\{x\}$, the algebra $\mathring{A}$ is monomial and quadratic, and therefore always Koszul.

Proof. The Koszul complex $\mathring{A}^{¡} \otimes_\kappa \mathring{A}$ is described explicitly below, and can be checked to be acyclic (by constructing a contracting homotopy). □

Proposition 92. *Consider $A(V, R)$. If the canonical projection $\mathring{A} \twoheadrightarrow \operatorname{gr} A$ is injective in weight 3 then it is an isomorphism.*

Proof. Spectral sequence. □

In the case where $V = \Bbbk X$ and $X = \{x_i \mid i \in I\}$ is totally ordered, all this is more easily understood through rewriting. We write:

- *reducible pairs*: $\overline{L}^{(2)} \subseteq I \times I$ for the set of pairs (i, j) such that $x_i x_j$ is a leading term of a relation
- *irreducible pairs*: $L^{(2)} = I^2 \setminus \overline{L}^{(2)}$
- *reducible uples*: $\overline{L}^{(n)} \subseteq I^n$ for the set of $(i_1, \ldots, i_n)$ such that for every k, $(i_k, i_{k+1}) \in \overline{L}^{(2)}$
- *irreducible uples*: $L^{(n)} \subseteq I^n$ for the set of $(i_1, \ldots, i_n)$ such that for every k, $(i_k, i_{k+1}) \in L^{(2)}$

Given $\iota = (i_1, \ldots, i_n) \in I^n$, we write x_ι for $x_{i_1} \ldots x_{i_n}$, and

$$L = \biguplus_n L^{(n)} \qquad \overline{L} = \biguplus_n \overline{L}^{(n)}$$

Lemma 93. *We have*

$$\mathring{A} = \Bbbk\{x_\iota \mid \iota \in L\}$$

and

$$\mathring{A}^{¡} = \Bbbk\left\{x_\iota \mid \iota \in \overline{L}\right\}$$

and $\mathring{A}^! = A(V^*, R^\perp)$ with

$$R^\perp = \Bbbk\left\{x_i^* x_j^* \;\middle|\; (i, j) \in \overline{L}^{(2)}\right\}$$

Definition 94. The image of the basis $\left\{x_\iota \mid \iota \in \overline{L}\right\}$ of $\mathring{A}$ under the surjection $\mathring{A} \twoheadrightarrow \operatorname{gr} A$ spans $\operatorname{gr} A$. When these are linearly independent they are called a **Poincaré-Birkhoff-Witt basis** (or PBW basis).

Proposition 95. *An algebra A equipped with a quadratic* PBW *basis is Koszul.*

Proof. The monomial algebra $\mathring{A}$ is Koszul (as a quadratic monomial algebra), and $\mathring{A} \cong \operatorname{gr} A$. We conclude using Theorem 87. □

Proposition 96. *Given an algebra $A(V, R)$, with an ordering of the basis X of V, if critical pairs are confluent then $\{x_\iota \mid \iota \in L\}$ forms a* PBW *basis of A and A is therefore Koszul.*

Definition 97. A **Gröbner basis** of an ideal I is a set $G \subseteq I$ such that

1. G generates the ideal I: $(G) = I$
2. leading terms generate the same ideal: $(\mathrm{lt}(G)) = (\mathrm{lt}(I))$

Proposition 98. *Given an algebra $A(V, R)$ equipped with an ordered basis, the terms*

$$\{x_\iota \mid \iota \in L\}$$

form a PBW *basis iff the elements*

$$\left\{ x_i x_j - \sum_{\substack{(k,l)\in L^{(2)} \\ (k,l)<(i,j)}} \lambda_{k,l}^{i,j} x_k x_l \;\middle|\; (i,j) \in \overline{L}^{(2)} \right\}$$

spanning R form a Gröbner basis of (R) in TV.

3.7 Hilbert series

Definition 99. Given a weight-graded algebra A such that $A_0 = \Bbbk 1$ and $A^{(n)}$ is finite-dimensional, its **generating series** or **Hilbert-Poincaré series** is

$$f^A(x) \quad = \quad \sum_{n=0}^{\infty} \dim A^{(n)} x^n$$

Theorem 100. *Given a Koszul algebra A,*

$$f^{A^!}(x) f^A(-x) \quad = \quad 1$$

Remark 101. Given a quadratic algebra A, if the series $1/f^A(-x)$ contains negative coefficient then the algebra A is not Koszul.

3.8 Quadratic-linear algebras

Consider $A = A(V, R)$ with $R \subseteq V \oplus V^{\otimes 2}$. We consider

$$q \quad : \quad TV \quad \twoheadrightarrow \quad V^{\otimes 2}$$

the projection. In particular, $A(V, qR)$ is quadratic. We suppose that our algebra satisfies the following two conditions:

1. There are no superfluous generators:
$$R \cap V \quad = \quad \{0\}$$
2. There are no "critical pairs":
$$(R \otimes V + V \otimes R) \cap V^{\otimes 2} \quad \subseteq \quad R \cap V^{\otimes 2}$$

Example 102. The presentation

$$\langle x, y, z \mid x + y \rangle$$

does not satisfy condition 1. The presentation

$$\langle x, y, z \mid xy - x, yz - y \rangle$$

does not satisfy condition 2 since

$$xz - xy \in (R \otimes V + V \otimes R)$$

which comes from the fact that the critical pair

$$\begin{array}{ccc} & xyz & \\ \swarrow & & \searrow \\ xz & & xy \end{array}$$

is not confluent.

By condition 1, we can define

$$\varphi \quad : \quad qR \quad \to \quad V$$

which to a quadratic relation associates its linear part in R. Now, the map

$$(qA)^{\text{¡}} = C(sV, s^2qR) \twoheadrightarrow s^2qR \xrightarrow{s^{-1}\varphi} sV$$

extends as a coderivation

$$d_\varphi \quad : \quad (qA)^{\text{¡}} \quad \to \quad T^c(sV)$$

By condition 2, its image is actually in $(qA)^{\text{¡}} \subseteq T^c(sV)$ and (still by condition 2) it squares to 0. By definition, the **Koszul dual dg coalgebra** of A is

$$A^{\text{¡}} \quad = \quad \left((qA)^{\text{¡}}, d_\varphi\right)$$

It gives rises to a quasi-isomorphism $\Omega A^{\text{¡}} \twoheadrightarrow A$.

Example 103. Consider

$$A \quad = \quad \langle x, y \mid xx - x, yy - y \rangle$$

We have

$$qA \quad = \quad \langle x, y \mid xx, yy \rangle$$

and therefore

$$(qA)^{\text{¡}} \quad = \quad \mathbb{k} \oplus \mathbb{k}\{x, y\} \oplus \mathbb{k}\{xx, yy\}$$

with

$$d(xx) = x \qquad\qquad d(yy) = y$$

3.9 Minimal models

Definition 104. A **model** for a dg algebra A is a surjective map of dg algebras

$$p \quad : \quad M \quad \twoheadrightarrow \quad A$$

which is a quasi-isomorphism, such that M is **quasi-free**, i.e. $M \cong TV$ (M is free as a graded algebra). It is **minimal** when

1. differential is *decomposable*:

$$d \quad : \quad V \quad \to \quad (TV)^{\otimes \geq 2}$$

2. the generating graded module V admits a decomposition into

$$V \quad = \quad \bigoplus_{k \geq 1} V^{(k)}$$

   such that

$$d\left(V^{(k+1)}\right) \quad \subseteq \quad T\left(\bigoplus_{1 \leq i \leq k} V^{(i)}\right)$$

Proposition 105. *Given a connected wdg algebra A and connected wdg coalgebra C, the following are equivalent*

1. *$C \otimes_\alpha A$ is acyclic*
2. *$A \otimes_\alpha C$ is acyclic*
3. *the dg coaglebra morphism $f_\alpha : C \to BA$ is a quasi-isomorphism*
4. *the dg algebra morphism $g_\alpha : \Omega A \to C$ is a quasi-isomorphism*

where f_α and g_α are the liftings of α defined in Proposition 68.

Proof. (1) $\Leftrightarrow$ (3) Consider $f_\alpha \otimes \mathrm{id}_A : C \otimes A \to BA \otimes A$. It is a morphism of chain complexes $C \otimes_\alpha A \to BA \otimes_\pi A$ and the second one is acyclic. Therefore $C \otimes_\alpha A$ is acyclic iff $f_\alpha \otimes \mathrm{id}_A$ is a quasi-iso, which is the case iff f_α is a quasi-iso (this last step requires a spectral sequence). □

Remark 106. In particular, since the bar complex $BA \otimes_\pi A$ is acyclic, we get a resolution (in algebras, i.e. a model)

$$\Omega BA \quad \twoheadrightarrow \quad A$$

called the **cobar-bar resolution**.

Proposition 107. *An algebra A is Koszul iff the projection $\Omega A^{¡} \twoheadrightarrow A$ is a quasi-isomorphism, i.e. provides a minimal resolution of A.*

Proof. It is a minimal resolution: it is quasi-free by construction, differential is decomposable by construction, $H_0(\Omega_\bullet A^{¡}) = A$ by (dual of) Proposition 77 and $\Omega A^{¡} \twoheadrightarrow A$ is a quasi-isomorphism. □

Corollary 108. *A is Koszul iff*

$$A^{\text{¡}} \quad \cong \quad H^\bullet(BA)$$

Remark 109. In the case of operads, this resolution provides the right notion of *operad up to homotopy* or ∞*-operad*. For instance,

$$A_\infty \quad = \quad \Omega Ass^{\text{¡}}$$

Example 110. Consider the algebra $A = \langle x, y \mid xx - yy, xy - yx\rangle$. In Example 76, we have seen that

$$A^{\text{¡}} \quad = \quad \Bbbk\{x^n, x^n y\}$$

equipped with suitable comultiplication..................

TODO: $\Omega A^{\text{¡}}$

3.10 A_∞-algebras

Definition 111. An A_∞**-algebra** A is a graded vector space A equipped with a codifferential

$$m \quad : \quad T^c(s\overline{A}) \quad \to \quad T^c(s\overline{A})$$

(i.e. a coderivation of degree $|m| = -1$ with $m \circ m = 0$).

Since $T^c(s\overline{A})$ is cofree, the codifferential is determined by its corestriction to degree 1, and we have the following equivalent definition:

Definition 112. An A_∞**-algebra** A is a graded vector space A equipped with

$$m_n \quad : \quad A^{\otimes n} \quad \to \quad A$$

for $n \geq 1$, of degree

$$|m_n| \quad = \quad n - 2$$

such that

$$\sum_{p+q+r=n} (-1)^{p+qr} m_{(p+1+r)} \circ (\mathrm{id}^{\otimes p} \otimes m_q \otimes \mathrm{id}^{\otimes r}) \quad = \quad 0 \qquad (2)$$

It is interesting to have a look at the relation (2) for low values of n:

1. $n = 1$: m_1 is a *differential*

$$m_1 \circ m_1 \quad = \quad 0$$

2. $n = 2$: m_1 is a derivation for the *product* m_2

$$m_1 \circ m_2 \quad = \quad m_2 \circ (m_1 \otimes \mathrm{id}_A) + m_2 \circ (\mathrm{id}_A \otimes m_1)$$

3. $n = 3$: associativity defect of m_2 is the border of the *associator* m_3

$$m_2 \circ (\mathrm{id}_A \otimes m_2 - m_2 \otimes \mathrm{id}_A) \quad = \quad m_1 \circ m_3 + m_3 \circ (m_1 \otimes \mathrm{id}_A^{\otimes 2} + \mathrm{id}_1 \otimes m_1 \otimes \mathrm{id}_A + \mathrm{id}_A^{\otimes 2} \otimes m_1)$$

4. $n = 4$: m_4 is the *pentagonator* and measures the failure of m_3 to satisfy MacLane's pentagon.

Definition 113. A **nilpotent** A_∞**-coalgebra** is a graded vector space C equipped with a differential

$$\Delta \quad : \quad T^c(s^{-1}\overline{C}) \quad \to \quad T^c(s^{-1}\overline{C})$$

($|\Delta| = -1$ and $\Delta^2 = 0$). This map is determined by maps

$$\Delta^n \quad : \quad C \quad \to \quad C^{\otimes n}$$

for $n \geq 1$ of degree $|\Delta^n| = 2 - n$ satisfying relations

$$\sum_{p+q+r=n} (-1)^{p+qr} (\mathrm{id}_C^{\otimes p} \otimes \Delta^q \otimes \mathrm{id}^{\delta r}) \circ \Delta^{p+1+q} = 0$$

The cobar-twisting-bar correspondence extends to A_∞ as follows [Kel05]. Given a (dg) coalgebra C and an augmented A_∞-algebra A the complex $\mathrm{Hom}(C, A)$ is an A_∞-algebra with

$$\star_n(f_1, \ldots, f_n) \quad = \quad \mu_n \circ (f_1 \otimes \ldots \otimes f_n) \circ \Delta^n$$

where Δ^n is the iterated coproduct. An ∞**-twisting morphism** is $\tau : C \to A$ such that

$$\sum_{n \geq 0} \star_n(\tau, \ldots, \tau) \quad = \quad 0$$

and we write $\mathrm{Tw}_\infty(C, A)$ for the space of twisting morphisms from C to A. The functor

$$\begin{array}{rcl} \mathbf{Coalg} & \to & \mathbf{Set} \\ C & \mapsto & \mathrm{Tw}_\infty(C, A) \end{array}$$

is representable and we write $B_\infty A$ for a representative: it is $T^c(sA)$ endowed with the coderivation whose post-composition by the projection $B_\infty A \twoheadrightarrow sA$ has components $\star_n : (sA)^{\otimes n} \to sA$ for $n \geq 1$.

Remark 114. A similar construction can be performed in the case where C is an ∞-coalgebra and A is a dg algebra for the cobar construction.

Anick resolution can be recasted, at least for monomial algebras, in the above setting as follows. This is mostly inspired of [DK09, DK13]. Consider a (non-homogeneous) monomial presentation of an algebra $A(V, R)$ with $V = \Bbbk X$. Consider the vector space

$$O_n^{(k)} \quad = \quad \Bbbk \{u \otimes u_1 \wedge \ldots \wedge u_n\}$$

with $|u| = k$ and the u_i are subwords of u in R, i.e.

$$u_i \quad = \quad u_i' \otimes r_i \otimes u_i'' \quad \in \quad TV \otimes R \otimes TV$$

i.e. elements of the free TV-bimodule over R, with

$$u_i' r u_i'' = u$$

We write

$$O_n = \bigoplus_k O_n^{(k)}$$

We define a differential by

$$d(u \otimes u_1 \wedge \ldots \wedge u_n) = \sum_{i=1}^{n} (-1)^{i-1} u \otimes u_1 \wedge \ldots \wedge \hat{u}_i \wedge \ldots \wedge u_n$$

and a product by

$$(u \otimes u_1 \wedge \ldots \wedge u_m) \cdot (v \otimes v_1 \wedge \ldots \wedge v_n) = uv \otimes u_1 v \wedge \ldots \wedge u_m v \wedge u v_1 \wedge \ldots \wedge u v_n$$

This makes O a dg-algebra.

Definition 115. Given an augmented algebra A, the space of **indecomposables** is

$$\mathrm{indec}(A) = \overline{A}/(\overline{A})^2$$

An element $a \neq 1$ is thus indecomposable when $a = bc$ implies $b = 1$ or $c = 1$.

Lemma 116. *The algebra O is free over its indecomposable elements.*

Proposition 117. *The dg-algebra O is a free resolution of the algebra A.*

In fact, we have

$$O = \Omega_\infty C$$

for some ∞-coalgebra C: C is the free ∞-coalgebra

We now build a resolution of A by free right A-modules

$$\ldots \xrightarrow{d_2} C_1 \otimes 1 \xrightarrow{d_1} C_0 \otimes A \xrightarrow{d_0} A \xrightarrow{\varepsilon} \Bbbk \to 0$$

We define $d_0(x \otimes 1) = x$

References

[Ani86] David J Anick. On the homology of associative algebras. *Transactions of the American Mathematical Society*, 296(2):641–659, 1986.

[Ber98] Roland Berger. Weakly confluent quadratic algebras. *Algebras and Representation*, 1(3):189–213, 1998.

[DK09] Vladimir Dotsenko and Anton Khoroshkin. Free resolutions via gröbner bases. *arXiv preprint arXiv:0912.4895*, 2009.

[DK13] Vladimir Dotsenko and Anton Khoroshkin. Quillen homology for operads via gröbner bases. *Documenta Mathematica*, 18:707–747, 2013.

[ERT94] David Eisenbud, Alyson Reeves, and Burt Totaro. Initial ideals, Veronese subrings, and rates of algebras. *Advances in Mathematics*, 109(2):168–187, 1994.

[FL91] Ralf Fröberg and Clas Löfwall. On Hilbert series for commutative and noncommutative graded algebras. *Journal of pure and Applied Algebra*, 76(1):33–38, 1991.

[Frö99] Ralph Fröberg. Koszul algebras. *Lecture notes in pure and applied mathematics*, pages 337–350, 1999.

[Kel05] Bernhard Keller. A-infinity algebras, modules and functor categories. *arXiv preprint math/0510508*, 2005.

[Kob90] Yuji Kobayashi. Complete rewriting systems and homology of monoid algebras. *Journal of Pure and Applied Algebra*, 65(3):263–275, 1990.

[Krä] Ulrich Krähmer. Notes on Koszul algebras.

[Lod98] Jean-Louis Loday. *Cyclic homology*, volume 301. Springer, 1998.

[LP91] Yves Lafont and Alain Prouté. Church-Rosser property and homology of monoids. *Mathematical structures in computer science*, 1(03):297–326, 1991.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346. Springer, 2012.

[Man87] Yu I Manin. Some remarks on Koszul algebras and quantum groups. *Ann. Inst. Fourier (Grenoble)*, 37(4):191–205, 1987.

[MLM95] Saunders Mac Lane and Saunders MacLane. *Homology*, volume 114. Springer, 1995.

[Squ87] Craig C Squier. Word problems and a homological finiteness condition for monoids. *J. Pure Appl. Algebra*, 49(1-2):201–217, 1987.

[TVdB96] John Tate and Michel Van den Bergh. Homological properties of Sklyanin algebras. *Inventiones mathematicae*, 124(1-3):619–648, 1996.