Hypercubical manifolds in homotopy type theory

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Abstract

Homotopy type theory is a logical setting in which one can perform geometric constructions and proofs in a synthetic way. Namely, types can be interpreted as spaces up to homotopy, and proofs as homotopy invariant constructions. In this context, we introduce a type which corresponds to the *hypercubical manifold*, a space first introduced by Poincaré in 1895. Its importance stems from the fact that it provides an approximation of the group Q of quaternionic units, in the sense of being the first step of a cellular resolution of Q. In order to ensure the validity of our definition, we show that it satisfies the expected property: it is the homotopy quotient of the sphere S^3 by the expected action of Q. This is non-trivial and requires performing subtle combinatorial computations based on the flattening lemma, thus illustrating the effective nature of homotopy type theory. Finally, based on the previous construction, we introduce new higher-dimensional generalizations of this manifold, which correspond to better cellular approximations of Q, converging toward a delooping of Q.

Introduction

Homology [11] is a family of abelian groups defined on spaces, such as topological spaces or simplicial sets, which are invariant under homotopy equivalence, i.e. under deformation of such spaces. After their introduction, it was wondered how precise these invariants where and, in particular, whether they would characterize spheres. It turns out that the answer is negative: there are spaces, called *homology spheres*, which have the same homology groups as spheres, but are not homotopy equivalent to them. Defining such examples has lead Poincaré to study a family of spaces obtained by gluing faces of platonic solids [15].

In particular, the *hypercubical manifold* K consists of

(A) a 3-dimensional cube where each pair of opposite squares have been identified after a quarter-turn rotation.

Nice modern presentations of this space can be found in [7, 17]. Its fundamental group can be computed to be the quaternion group Q and its universal cover to be the 3-sphere S^3. It is well known that we can recover any (nice enough) space K as the quotient of its universal cover under the canonical action of its fundamental group [8, Proposition 1.40]: this means that we have an alternative definition of K is as

(B) a quotient of the 3-sphere S^3 under a free action of the quaternion group Q.

Defining the hypercubical manifold in homotopy type theory. Over recent years, a variant of dependent type theory, homotopy type theory [18], has been introduced. In this setting, types can be interpreted as homotopy types, i.e. spaces up to deformation, thus allowing for performing constructions on spaces in a synthetic way. Advanced versions of this type theory (such as the one implemented in cubical Agda [19]) feature higher inductive types (HITs) which allow higher constructors in the definition of types, corresponding to attaching cells like in CW-complexes. Our goal in this paper is to define the hypercubical manifold in homotopy type theory and show that it satisfies the expected properties: in particular, we will show the equivalence of the two above definitions within the theory.

The hypercubical manifold can easily be constructed as a HIT [9] by generalizing the well-known constructions for simple spaces such as the torus [19], which essentially consists in translating the standard description (A) of the space as a CW-complex. However, constructing K as a quotient of the 3-sphere by an action of Q, following the second definition (B), seems difficult to achieve, because there is no direct way of defining such an action. Namely, we would need to have a *coherent* description of it, that is as a map $\phi : \mathrm{B}\,Q \to \mathcal{U}$ where $\mathrm{B}\,Q$ is a type representing internally the group Q (this is called a *delooping* of Q) and $\mathcal{U}$ is the universe of all types, whose elements can be thought here as spaces. Classical definitions of $\mathrm{B}\,Q$ (as a type of Q-torsors, or as an Eilenberg-Maclane space [10]) have an universal property that only allows eliminating to groupoids. However, there is no groupoid subtype of the universe containing S^3, because S^3 is not n-truncated for any n, and thus we cannot resort to this universal property to construct the map $\phi : \mathrm{B}\,Q \to \mathcal{U}$.

Here, we explain how we can still manage to define the coherent action of Q on S^3 in homotopy type theory, from a map $K \to \mathrm{B}\,Q$ whose fiber is S^3, and we show that the resulting homotopy quotient is equivalent to the definition of K as a HIT. This thus implies that the latter definition is "correct": it is a quotient of S^3 by Q as expected. This construction uses two main ingredients: the action-fibration duality in order to construct the map ϕ, and the flattening lemma in order to compute its fibers.

Higher hypercubical manifolds. This work also suggests useful generalizations of the hypercubical manifold. Namely, there are also free actions of Q on spheres S^{4n-1} for any natural number n, giving rise to higher hypercubical manifolds K^n. We will show that those can also be defined in homotopy type theory, and that they allow constructing a "resolution" of Q, i.e. a family of maps $K^n \to \mathrm{B}\,Q$ which behave more and more as equivalences as n increase, in the sense that they are $(4n-2)$-connected. Compared to resolutions obtained by other traditional methods (such as those based on rewriting), this resolution is quite "small" and thus more amenable to computations.

Plan of the paper. We begin by recalling basic notations and constructions in homotopy type theory (section 1), as well as the traditional definition of the hypercubical manifold in topological spaces (section 2). The main novel contents of the paper can be found in section 3, where we define the hypercubical manifold in homotopy type theory, and show that the fiber of the canonical map $K \to \mathrm{B}\,Q$ is S^3, by iteratively using the flattening lemma on skeleta of K.

We then deduce all expected properties from this fiber sequence. Finally, we introduce and motivate higher dimensional generalizations of the hypercubical manifold (in section 4) and conclude.

1 Homotopy type theory

The goal of this section is to fix notations and recall the main tools of homotopy type theory used throughout the paper. A proper introduction to the topic can be found in the reference book [18].

1.1 Elementary constructions

We write $\mathcal{U}$ for the universe of all types (to be precise, we should need a hierarchy of Grothendieck universes, but we will not detail this as it plays no significant role here). Given a type $A : \mathcal{U}$ and a type family $B : A \to \mathcal{U}$, we write $\Pi A.B$, or $\Pi(x : A).B\,x$, or $(x : A) \to B\,x$ for the associated dependent product type: its inhabitants are dependent functions and we write $\lambda x.t$ for the function which to x (of type A) associates the term t (of type $B\,x$). As usual, we write $A \to B$ for the type of non-dependent functions (here, we suppose that B is a type). Similarly, we write $\Sigma A.B$ or $\Sigma(x : A).B\,x$ for the type of dependent sums, whose elements are dependent pairs $\langle t, u \rangle$ consisting of a term t (of type A) and a term u (of type $B\,t$). The two dependent projections are respectively written $\pi : \Sigma A.B \to A$ and $\pi' : (x : \Sigma AB) \to B\,(\pi\,x)$. Given two type families $B : A \to \mathcal{U}$ and $B' : A' \to \mathcal{U}$, and maps $f : A \to A'$ and $g : (x : A) \to B\,x \to B'\,(f\,x)$, we write $\Sigma f.g : \Sigma A.B \to \Sigma A'.B'$ for the canonically induced map. Given a natural number n, we still write n for the type with n elements, and even allow ourselves to use the comprehension notation such as $\{a, b, c\}$ to denote the type 3 where the three elements are called a, b and c. In particular, we denote by 0 (resp. 1) the initial (resp. terminal) type.

We write $t \equiv u$ to indicate that two terms t and u are definitionally equal. Homotopy type theory also features, for every elements t and u of common type A, a type $t = u$ of propositional equalities between t and u, also called paths because of their topological interpretation. We write $\mathrm{refl}_x : x = x$ for the trivial path on a point $x : A$. Given paths $p : x = y$ and $q : y = z$, we write $p \cdot q : x = z$ for the path obtained by concatenation, and $\overline{p} : y = x$ for the symmetric of p. Given a type family $P : A \to \mathcal{U}$ and a path $p : t = u$ in A, we write $P^{\to} : P\,t \to P\,u$ for the canonical function induced by transport [18, Lemma 2.3.1]. Given a function $f : A \to B$ and a path $p : t = u$ in A, we write $f^{=}(p) : f(t) = f(u)$ (or even sometimes $f(p)$) for the canonical path in B [18, Lemma 2.2.1] which can be thought of as being obtained by applying f pointwise to p. All constructions in homotopy type theory are compatible with identities; in particular, all (co)limits are homotopy ones.

A *pointed type* consists of a type A together with an element of A, which we often denote by $\star$, and a pointed map is a function preserving the distinguished element. Given a pointed type A, we write $\Omega\,A \equiv (\star = \star)$ for the *loop space* of A.

Given a type A, we write $\|A\|_{-1}$ (resp. $\|A\|_0$, resp. $\|A\|_1$) for its propositional (resp. set, resp. groupoid) truncation, and $|-|_i : A \to \|A\|_i$ for the canonical map. A type is *connected* when it satisfies $\Sigma(x : A).\Pi(y : A).\|x = y\|_{-1}$.

1.2 Fibrations

Any type family $F : A \to \mathcal{U}$ can be thought of as a fibration with A as base space, whose fiber at $x : A$ is simply $F\,x$. The total space of such a fibration is $\Sigma A.F$, which comes equipped with a map

$$\pi : \Sigma A.F \to A$$

onto the base space given by the first projection. Conversely, any map $f : X \to A$ exhibiting X as a type over A induces a type family $\mathrm{fib}_f : A \to \mathcal{U}$ which to any element $y : A$ associates its *homotopy fiber* defined as

$$\mathrm{fib}_f\, y \quad \equiv \quad \Sigma(x : A).(y = f\,x)$$

which corresponds to the pullback

$$\begin{tikzcd} \mathrm{fib}_f\, y \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "y"] \\ B \arrow[r, "f"'] & A \end{tikzcd}$$

These constructions provides an equivalence between the two points of views on fibrations, as type families indexed by A or as types over A. This is known as the Grothendieck duality [18, Theorem 4.8.3]:

Proposition 1 (Grothendieck duality)**.** *For any type $A : \mathcal{U}$, we have an equivalence between type families and types over A*

$$(A \to \mathcal{U}) \quad \simeq \quad (\Sigma(B : \mathcal{U}).B \to A)$$

We say that a pair of composable maps

$$F \xrightarrow{i} B \xrightarrow{f} A$$

with A pointed, is a *fiber sequence* when $F = \mathrm{fib}_f \star$ and i is the canonical map given by the above pullback. This indicates that f can be thought of as a fibration with all the fibers being F (when A is connected all fibers are indeed merely equivalent to F). Formally, fiber sequences play a role analogous to short exact sequences in abelian categories.

For reasons indicated above, given a type family $F : A \to \mathcal{U}$, it is often desirable to compute its total space $\Sigma A.F$. When A is constructed as a colimit, this total space can also be obtained as a similar colimit: this is called the *flattening lemma*. For pushouts, which is the variant we will use here, this can be formulated as follows:

Lemma 2 (Flattening lemma for pushouts)**.** *Consider a pushout square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "i"] \\ B \arrow[r, "j"'] \arrow[ur, phantom, "\ulcorner", very near end] & A \sqcup_X B \end{tikzcd}$$

with $p : (x : X) \to i \circ f(x) = j \circ g(x)$ *witnessing for its commutativity, and a type family*

$$F : A \sqcup_X B \to \mathcal{U}$$

Then the following square of total spaces is also a pushout:

$$\begin{tikzcd}
\Sigma X.(F \circ i \circ f) X \arrow[r, "\Sigma f.(\lambda_\,.\,\mathrm{id})"] \arrow[d, "\Sigma g.e"'] & \Sigma A.(F \circ i) \arrow[d, "\Sigma i.(\lambda_\,.\,\mathrm{id})"] \\
\Sigma B.(F \circ j) \arrow[r, "\Sigma j.(\lambda_\,.\,\mathrm{id})"'] & \ulcorner\, \Sigma(A \sqcup_X B).F
\end{tikzcd}$$

with

$$\begin{aligned} e : (x : F) \to F \circ i \circ f(x) &\to F \circ j \circ g(x) \\ x\, y &\mapsto F^{\to}(p\, x)\, y \end{aligned}$$

being the map canonically induced by p.

A similar results holds when A is constructed as a coequalizer [18, Lemma 6.12.2].

1.3 Groups and actions

A group G is a set together with a multiplication and unit satisfying the usual axioms. A *delooping* of G is a pointed connected space $\mathrm{B}\,G$ together with an isomorphism of groups $\Omega\,\mathrm{B}\,G \simeq G$ (where the group structure on $\Omega\,\mathrm{B}\,G$ is the one induced by concatenation of paths). A type with this property can be shown to exist and be unique, so that we can talk about the delooping of a group, and, in fact, the loop space provides an equivalence between pointed connected groupoids and groups [1]. We have the following recurrence principle:

Proposition 3. *Given a group* G*, a pointed groupoid* X *and a group morphism* $f : G \to \Omega\,X$*, there exists a unique pointed map* $\tilde{f} : \mathrm{B}\,G \to X$ *such that the composite* $G \longrightarrow \Omega\,\mathrm{B}\,G \xrightarrow{\Omega\,\tilde{f}} \Omega\,X$ *is* f.

A map $\phi : \mathrm{B}\,G \to \mathcal{U}$ can be thought of as an *action* of G on the type $X \equiv \phi(\star)$. Namely, any $a \in G$ corresponds to a path $a : \star = \star$ in $\mathrm{B}\,G$, and thus induces an endomorphism $a^{\to} : X \to X$. In the case where X is a set, this can be shown to correspond to the traditional notion of action. More generally, a map $\mathrm{B}\,G \to \mathcal{U}$ can be seen as an action of G on a higher type, in a coherent way [1]. By the Grothendieck duality (proposition 1),

Proposition 4. *A fiber sequence*

$$X \longrightarrow A \xrightarrow{\;f\;} \mathrm{B}\,G$$

corresponds to a coherent action of G *on* X *with* A *being the homotopy quotient* $X /\!\!/ G$.

Namely, in the above proposition, the action is obtained as the map $\mathrm{fib}_f : \mathrm{B}\,G \to \mathcal{U}$, whose total space is $A = \Sigma\,\mathrm{B}\,G.\,\mathrm{fib}_f$ which is, by definition, the homotopy quotient of X by the action of G.

2 Topological definition of the hypercubical manifold

Before turning to our implementation in homotopy type theory, we first recall the traditional construction of the hypercubical manifold, based on a free action of the quaternion group on a 3-sphere.

2.1 The quaternion group

Consider the real 4-dimensional vector space $\mathbb{H}$ generated by four basis vectors 1, i, j, and k. This space is an algebra under the Hamilton product, with 1 as neutral element, and given on basis elements by $i^2 = j^2 = k^2 = -1$, $ij = k$, $ji = -k$, $jk = i$, $kj = -i$, $ki = j$ and $ik = -j$. The *quaternion group* is the 8-element group consisting of $1, i, j, k$ and their opposites. This group is easily shown to have the presentation

$$Q = \left\langle e, i, j, k \;\middle|\; i^2 = e, j^2 = e, k^2 = e, ijk = e, e^2 = 1 \right\rangle \tag{1}$$

where e denotes -1. The following alternative presentation will turn out to be useful later in the paper.

Lemma 5. *The quaternion group admits the presentation*

$$Q = \langle i, j \mid i = jij, j = iji \rangle \tag{2}$$

Proof. First, we show that the group admits the presentation

$$Q = \left\langle i, j \;\middle|\; i^4 = 1, i^2 = j^2, iji = j \right\rangle \tag{3}$$

We namely have the following sequence of equivalent presentations

$$\begin{aligned}
&Q = \left\langle e, i, j, k \;\middle|\; i^2 = e, j^2 = e, k^2 = e, ijk = e, e^2 = 1 \right\rangle \\
&Q = \left\langle i, j, k \;\middle|\; j^2 = i^2, k^2 = i^2, ijk = i^2, i^4 = 1 \right\rangle && \text{by } e = i^2 \\
&Q = \left\langle i, j, k \;\middle|\; i^2 = j^2, k^2 = i^2, ijk = i^2, i^4 = 1, ij = k \right\rangle && ij = k \text{ derivable} \\
&Q = \left\langle i, j \;\middle|\; i^2 = j^2, (ij)^2 = i^2, ijij = i^2, i^4 = 1 \right\rangle && \text{by } k = ij
\end{aligned}$$

Above $ij = k$ is derivable because we have $ijk = i^2$, and thus $ijk^4 = i^2k^3$ by multiplication by k^3, and thus $ij = k$ by $k^4 = i^4 = 1$ and $i^2k^2 = i^4 = 1$. The relation $(ij)^2 = i^2$ is now redundant, so that we can remove it, and then, from the relation $ijij = i^2 = j^2$, we can derive the equivalent relation $iji = j$ by dividing by j, and we obtain (3).

Now the presentation (3) can be shown to be equivalent to (2). Namely, from (3), we can derive the missing relation $i = jij$ by

$$\begin{aligned}
jij &= iji^2j && \text{by } j = iji \\
&= ij^4 && \text{by } i^2 = j^2 \\
&= i && \text{by } j^4 = i^4 = 1
\end{aligned}$$

Conversely, from (2) we can derive the relations of (3) since $i^2 = jiji = j^2$, and $j = iji = jij^2i = ji^4$ which implies $1 = i^4$ by division by j. □

2.2 The hypercubical manifold

We write $C = \{t1 + xi + yj + zk \mid t, x, y, z \in [-1, 1]\} \subseteq \mathbb{H}$ for the *unit cube* in $\mathbb{H}$, and $\partial C \subseteq C$ for its boundary (consisting of points such that at least one of the coordinates is either -1 or 1). Note that this space is homotopic to the 3-sphere S^3. There is a natural action of Q on C obtained by left multiplication. This action is not free because it fixes 0, but it become so when restricted to ∂C. We write

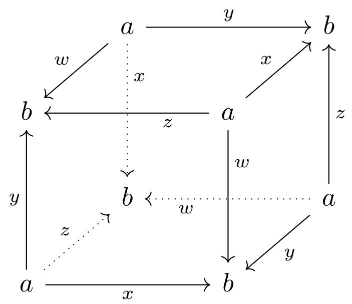
$$K \quad \equiv \quad \partial C/Q$$

for the quotient space of $\partial C \simeq \mathrm{S}^3$ under this action, which we call *hypercubical manifold*. It is not obvious from this description that it is indeed a manifold (as opposed to a topological space), but this will not play a role here. Since the action is free, and properly discontinuous because Q is finite, by [8, Proposition 1.40] the quotient map $\partial C \to K$ is covering with Q as fiber, i.e. we have a fiber sequence

$$Q \longrightarrow \partial C \longrightarrow K$$

from which we deduce that the fundamental group of K is Q [8, Theorem 4.41].

The original definition of K in [15, p. 66, section 13, troisième exemple] is of a quite different nature, the equivalence with the previous one being explained in [7]. Indeed the hypercubical manifold can also be defined as a 3-dimensional cube, whose opposite faces are identified after a quarter-turn rotation:



(4)

It can thus be described as a CW-complex consisting of 2 0-cells (a and b), 4 1-cells (x, y, z and w), 3 2-cells (α, β and γ, the front, right and top faces)

$$\begin{tikzcd} b & a \arrow[l, "z"'] \arrow[d, "w"] \\ a \arrow[u, "y"] \arrow[r, "x"'] \arrow[ur, phantom, "\alpha"] & b \end{tikzcd} \qquad \begin{tikzcd} b & a \arrow[l, "w"'] \arrow[d, "x"] \\ a \arrow[u, "y"] \arrow[r, "z"'] \arrow[ur, phantom, "\beta"] & b \end{tikzcd} \qquad \begin{tikzcd} b & a \arrow[l, "w"'] \arrow[d, "y"] \\ a \arrow[u, "z"] \arrow[r, "x"'] \arrow[ur, phantom, "\gamma"] & b \end{tikzcd} \tag{5}$$

and 1 3-cell (A, the interior of the cube). This definition allows for a direct computation of the fundamental group $\pi_1(K)$. We first contract w to an identity (thus identifying a and b) to obtain the presentation

$$\pi_1(K) = \langle x, y, z \mid x = zy, y = xz, z = yx \rangle$$

where the three relations correspond to the 2-cells (5). By removing z (which is equal to yx), renaming x to i and y to j, we finally recover the presentation of lemma 5 of the quaternion group.

2.3 Higher hypercubical manifolds

We now briefly mention higher-dimensional generalizations of the hypercubical manifold. Those arose quite naturally form our work, but to the best of our knowledge, they have not been already considered in the literature. Writing K^1 for the hypercubical manifold, we have that K^1 is a "good approximation" of Q up to dimension 2, in the sense that we have

$$\pi_0(K^1) = \pi_0(\mathrm{B}\,Q) = 1 \qquad \pi_1(K^1) = \pi_1(\mathrm{B}\,Q) = Q \qquad \pi_2(K^1) = \pi_2(\mathrm{B}\,Q) = 1$$

More generally, we would like to have cellular spaces K^n which coincide with Q up to dimension $f(n)$ where f is a strictly increasing function on natural numbers, thus constructing a resolution of Q. In fact, the construction of K^1 generalizes as follows.

Given $n \in \mathbb{N}$, we can consider the unit cube $C^n \subseteq \mathbb{H}^n$. We write ∂C^n for its boundary, which is homotopic to the $(4n-1)$-sphere S^{4n-1}. For similar reasons as previously, we have an action of Q on C^n, which restricts to a free action on S^{4n-1}: we have that S^{4n-1} can be expressed as the join of n copies of S^3, and Q acts independently on each of those 3-spheres as explained in previous section. We define

$$K^n \quad \equiv \quad \mathrm{S}^{4n-1}/Q$$

as the quotient of the sphere under this action. We thus have the fiber sequence

$$Q \longrightarrow \mathrm{S}^{4n-1} \longrightarrow K^n$$

which is of degree $8 \equiv |Q|$, from which we deduce that $\pi_i(K^n) = \pi_i(\mathrm{B}\,Q)$ for $0 \leq i \leq 4n-2$, by using the induced long exact sequence in homotopy [8, Theorem 4.41].

There is a canonical inclusion $\mathbb{H}^n \to \mathbb{H}^{n+1}$ (obtained by adding 0 as last coordinate) which induces an inclusion $\partial C^n \to \partial C^{n+1}$, and thus a map $K^n \to K^{n+1}$. By taking the inductive limit, we obtain a space K^∞ which has the same homotopy groups as $\mathrm{B}\,Q$ (because homotopy groups commute with inductive limits [12, Chapter 9, Section 4]) and is thus homotopy equivalent to $\mathrm{B}\,Q$ by [8, Theorem 1B.8]. However, this model has the advantage of being a CW-complex which can be explicitly described.

3 The hypercubical manifold in homotopy type theory

We will now show that we can perform the previous constructions in the setting of homotopy type theory, which requires more than a simple direct translation. It is clear that we can use higher inductive types (HITs) to give a direct definition of K as a type consisting of two elements, four identities, three 2-identities and one 3-identity [9]. However, it is not obvious at all that it comes from a quotient of S^3 by an action of Q. Defining such an action amounts to define a map $\phi : \mathrm{B}\,Q \to \mathcal{U}$ such that $\phi(\star) = \mathrm{S}^3$, but the recurrence principle for $\mathrm{B}\,Q$ given in proposition 3 only allows eliminating toward a groupoid, whereas there is no subtype of the universe which is a groupoid and contains S^3 (for instance, we know that $\pi_3(\mathrm{S}^3) = \mathbb{Z}$, see [18, Theorem 8.6.17] and, in fact, S^3 has an infinite number of non-trivial homotopy groups). Instead, since we know that the homotopy quotient of the action encoded by ϕ is K, ϕ corresponds to a fibration $\mathrm{S}^3 \to K \to \mathrm{B}\,Q$ under the action-fibration duality of proposition 4, which turns out to be much easier to define. Note that the map $\mathrm{S}^3 \to K$ is the one considered in section 2.2, so that we will have 8 n-cells in the fiber over each n-cell in the base. This is encoded by the following long fiber sequence combining the previous ones:

$$Q \longrightarrow \mathrm{S}^3 \longrightarrow K \xrightarrow{\phi} \mathrm{B}\,Q$$

We will proceed in two steps. We first define a map $\phi : K \to \mathrm{B}\,Q$ by using the elimination principle of K due to its definition as a HIT. Then, we show that its fiber $\mathrm{fib}_\phi(\star)$ (sometimes simply written fib_ϕ) is precisely S^3 by using the flattening lemma. More precisely, writing K_k for the k-skeleton of K, we reason by recurrence on k, and compute the fiber of the restriction $K_k \to \mathrm{B}\,Q$ to be the k-skeleton of S^3, or more precisely a cubical model of S^3.

3.1 The hypercubical manifold

We are now ready to define the type which will be of central interest here:

Definition 6. The hypercubical manifold K is the higher inductive type generated by

— 2 elements a and b,

— 4 identities $x, y, z, w : a = b$,

— 3 identities between identities as in (5):

$$\alpha : y \cdot \overline{z} = x \cdot \overline{w} \qquad \beta : y \cdot \overline{w} = z \cdot \overline{x} \qquad \gamma : z \cdot \overline{w} = x \cdot \overline{y}$$

— 1 identity between identities between identities as in (4).

We will see in proposition 11 that its fundamental group is Q. This is expected from the topological constructions (see section 2.2), but showing this directly is not easy (although it should be doable by using the van Kampen theorem [18, Section 8.7]).

Lemma 7. *The type K is connected.*

Proof. We need to show that there merely exists a path from a to x for any $x : K$. By using the induction principle of K, and because the goal is a proposition, it amounts to find a path $a = a$ and a path $a = b$. We can use refl_a and x respectively. □

3.2 Definition of the action

We now want to define the map $\phi : K \to \mathrm{B}\,Q$. In order to come up with its definition, we can start from the computation of section 2.2 which identifies the fundamental group of K to be Q (an alternative, more abstract, starting point will also be given in section 3.8). We see that x corresponds to i, y corresponds to j, the relation $z = yx$ imposes that z should correspond to $ji = -k$, and w was contracted and should thus correspond to 1. This suggests the following definition of the map ϕ, using the elimination principle of K.

Definition 8. We define a morphism $\phi : K \to \mathrm{B}\,Q$ on 0-cells by

$$\phi(a) \equiv \phi(b) \equiv \star$$

and on 1-cells by

$$\phi(x) \equiv i \qquad \phi(y) \equiv j \qquad \phi(z) \equiv -k \qquad \phi(w) \equiv 1$$

The 2-cells are sent to 2-cells coming from the following relations which are known to hold in Q:

$$\phi(\alpha) \equiv (i = jk) \qquad \phi(\beta) \equiv (j = ki) \qquad \phi(\gamma) \equiv (k = ij)$$

On 3-cells, ϕ is unambiguously defined because $\mathrm{B}\,Q$ is a groupoid.

For $k \in \mathbb{N}$, we define K_k to be the k-skeleton of K, i.e. K_k is defined as the HIT with the same generating i-cells for $i \leq k$ (and none for $i > k$). We write $\phi_k : K_k \to \mathrm{B}\,Q$ for the map obtained as the composite $K_k \hookrightarrow K \xrightarrow{\phi} \mathrm{B}\,Q$. Our aim is now to compute the homotopy fiber fib_{ϕ_k}, by recurrence on k, using the flattening lemma in order to ultimately show that $\mathrm{fib}_\phi \equiv \mathrm{fib}_{\phi_3} = \mathrm{S}^3$ in theorem 10.

3.3 Fiber of the 0-skeleton

The 0-skeleton K_0 is a space consisting of two points a and b, and the induced map $\phi_0 : K_0 \to \mathrm{B}\,Q$ is the constant map with $\star$ as image. Its fiber is

$$\begin{aligned} \mathrm{fib}_{\phi_0} &\equiv \Sigma(x : K_0).(\star = \phi_0(x)) \\ &= (\star = \phi(a)) \sqcup (\star = \phi(b)) \\ &= \{a, b\} \times (\star = \star) \\ &= \{a, b\} \times Q \end{aligned}$$

so that we have a fiber sequence

$$\{a, b\} \times Q \xrightarrow{\pi} K_0 \xrightarrow{\phi_0} \mathrm{B}\,Q$$

3.4 Fiber of the 1-skeleton

The 1-skeleton K_1 is the complex pictured in (4), with two 0-cells a and b, and four 1-cells x, y, z and w. It can be obtained from K_0 as the coequalizer

$$\begin{tikzcd} \{x, y, z, w\} \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\tau"'] & K_0 \arrow[r, dotted, "\iota_1"] & K_1 \end{tikzcd}$$

where the maps σ and τ respectively send an edge to its source and target, i.e. they are the constant maps respectively equal to a and b.

Our aim is now to computer the fiber of $\phi_1 : K_1 \to \mathrm{B}\,Q$. We consider the type family

$$\begin{aligned} F_1 : K_1 &\to \mathcal{U} \\ x &\mapsto (\star = \phi_1(x)) \end{aligned}$$

This map is defined so that we have

$$\mathrm{fib}_{\phi_1} \equiv \Sigma(x : K_1).F_1(x)$$

which can be computed using the flattening lemma for coequalizers (see section 1.2 and [18, Section 6.12]) to be the coequalizer

$$\Sigma\{x, y, z, w\}.(F_1 \circ \iota_1 \circ \sigma) \underset{\Sigma\tau.e}{\overset{\Sigma\sigma.\lambda\_\ \mathrm{id}}{\rightrightarrows}} \Sigma K_0.(F_1 \circ \iota_1) \xrightarrow{\Sigma\iota_1.\lambda\_\ \mathrm{id}} \Sigma K_1.F_1 \tag{6}$$

Note that $F_1(\iota_1(\sigma(x))) \equiv (\star = \star) = Q$ and similarly for y, z and w, so that this can be rewritten as

$$\{x, y, z, w\} \times Q \underset{\Sigma\tau.e}{\overset{\Sigma\sigma.\lambda\_\ \mathrm{id}}{\rightrightarrows}} K_0 \times Q \xrightarrow{\Sigma\iota_1.\lambda\_\ \mathrm{id}} \mathrm{fib}_{\phi_1} \tag{7}$$

In (6) above, e is defined as

$$\begin{aligned} e : \{x, y, z, w\} &\to F_1 \circ \iota_1 \circ \sigma(p) \to F_1 \circ \iota_1 \circ \tau(p) \\ p\ q &\mapsto F_1^{\rightarrow}\, p\, q \end{aligned}$$

By [18, Theorem 2.11.4], we thus have $e\,p\,q = q \cdot \phi^{=}\, p$. Equivalently, in (7), we can thus define

$$\begin{aligned} e : \{x, y, z, w\} &\to Q \to Q \\ p\ q &\mapsto q \times \phi^{=} p \end{aligned}$$

The coequalizer (7) can be interpreted as a description of fib_{ϕ_1} as a 1-dimensional cell complex, with $K_0 \equiv \{a, b\} \times Q$ as 0-cells and $\{x, y, z, w\} \times Q$ as 1-cells, the source and target of a 1-cell $(p, q) : \{x, y, z, w\} \times Q$ being respectively (a, q) and $(b, q \times \phi^{=}(p))$. This complex is pictured in figure 1. Here, we write ai (resp. ai^{-}) for (a, i) (resp. $(a, -i)$) and similarly for other cells, so that the general form of a 1-cell is

$$aq \xrightarrow{pq} b(q \times \phi^{=}(p))$$

We can observe that this is the 1-skeleton of a tessseract (a 4-dimensional cube), which is homotopic to S^3 as expected.

Remark 9. The group Q being generated by i and j, it admits the Cayley C graph shown on figure 2. This graph describes a higher inductive type (with the vertices as generators and edges as identities) which can be obtained as the fiber of the canonical map $B\{i, j\}^* \to \mathrm{B}\,Q$, where $\{i, j\}^*$ is the free group on two generators, see [6]. Similarly, we have a Cayley graph C' associated to $\{x, y, z\}$ as generators (where x, y and z respectively correspond to i, j and $-k$). We have a map $f : K_1 \to \mathrm{B}\{x, y, z\}^*$ which expresses that the target can be obtained from the source by contracting the 1-cell w to a point. We also have a map $g : \mathrm{B}\{i, j\}^* \to \mathrm{B}\{x, y, z\}^*$ which is the canonical inclusion, respectively sending x and y to i and j. Those maps make the two triangles on the right

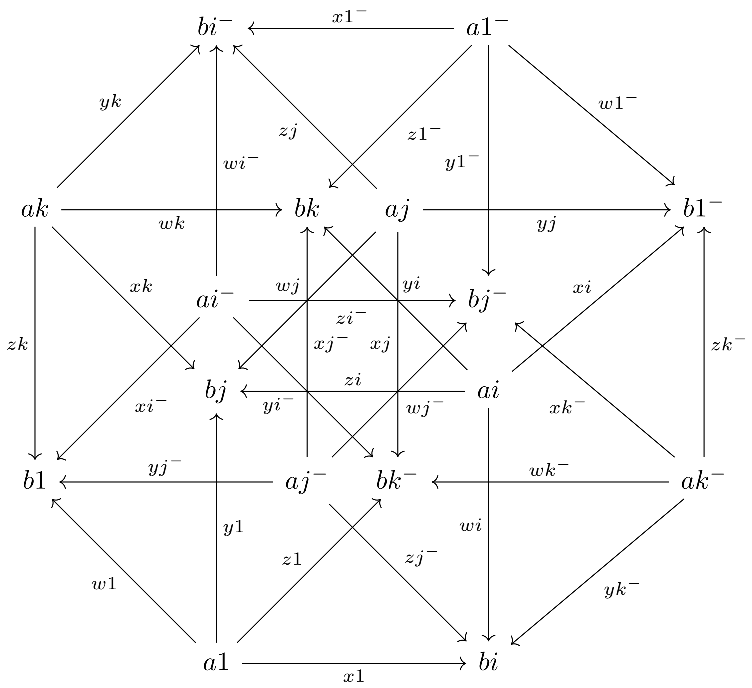


Figure 1: The fiber of ϕ_1 (the 1-skeleton of a cubical model of S^3).

commute:

$$\begin{tikzcd}
\mathrm{fib}_{\phi_1} \arrow[r] \arrow[d, two heads, dotted, "f'"'] & K_1 \arrow[d, two heads, "f"'] \arrow[dr, "\phi_1"] & \\
C' \arrow[r] & \mathrm{B}\{x,y,z\}^* \arrow[r] & \mathrm{B}\,Q \\
C \arrow[r] \arrow[u, hook, dotted, "g'"] & \mathrm{B}\{i,j\}^* \arrow[u, hook, "g"] \arrow[ur] &
\end{tikzcd}$$

In other words, those maps express that $\mathrm{B}\{i,j\}^*$ can be obtained as a subquotient of K_1 (quotienting w and forgetting z). By the universal property of C' which is a fiber, we have similar maps f' and g' as shown above between the fibers, expressing that C can be obtained as a subquotient of fib_{ϕ_1}, by quotienting all 1-cells of the from (w,q) with $q \in Q$ (and thus identifying all 0-cells (b,q) with (a,q)) and forgetting all the 1-cells of the form (z,q) with $q \in Q$. It can namely be checked that if we perform those operations on K_1 pictured in figure 1 we recover the traditional Cayley graph pictured in figure 2.

3.5 Fiber of the 2-skeleton

We write $\square$ for the HIT corresponding to an empty square, generated by four 0-cells v_0, v_1, v_2, v_3 and four 1-cells e_0, e_1, e_2, e_3 attached as in the following

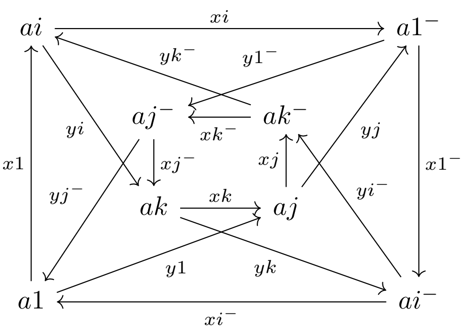


Figure 2: Cayley graph of Q generated by i and j.

picture:

$$
\begin{tikzcd}
v_2 & v_3 \arrow[l, "e_3"'] \arrow[d, "e_2"] \\
v_0 \arrow[u, "e_1"] \arrow[r, "e_0"'] & v_1
\end{tikzcd}
$$

We also write $\blacksquare$ for the HIT corresponding to a filled square, obtained from the previous one by attaching a 2-cell as expected. There is an induced canonical inclusion $\square \hookrightarrow \blacksquare$. The type $\blacksquare$ is contractible, and we will consider a particular equivalence $\blacksquare \simeq 1$ consisting of the terminal map $\tau : \blacksquare \to 1$ and the map $\upsilon : 1 \to \blacksquare$ pointing at v_0. Since those form an equivalence, their composite is the identity on $\blacksquare$, which is witnessed by a map $\varepsilon : (x : \blacksquare) \to v_0 = x$. Up to deformation (being an equivalence is a property), we can consider a particular definition for ε, which is obtained by induction on $\blacksquare$, and satisfies

$$
\varepsilon(v_0) \equiv \mathrm{refl} \qquad \varepsilon(v_1) \equiv e_0 \qquad \varepsilon(v_2) \equiv e_1 \qquad \varepsilon(v_3) \equiv e_0 \cdot \overline{e_2}
$$

definitionally (on 1- and 2-cells, this is defined arbitrarily, using the fact that the filled square is contractible). This will slightly simplify computations in the following.

The 2-skeleton K_2 can be obtained from K_1 as the pushout on the left

$$
\begin{tikzcd}
\{\alpha,\beta,\gamma\} \times \square \arrow[r, hook, "\iota"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\tilde{e}"] & \{\alpha,\beta,\gamma\} \times \blacksquare \arrow[d, "\tilde{\rho}"] \\
K_1 \arrow[r, hook, "\iota_2"'] & K_2
\end{tikzcd}
\qquad
\begin{tikzcd}
\{\alpha,\beta,\gamma\} \times \square \arrow[r, hook, "\pi"] \arrow[d, "\psi"'] \arrow[dr, phantom, "e"] & \{\alpha,\beta,\gamma\} \arrow[d, "\rho"] \\
K_1 \arrow[r, hook, "\iota_2"'] & K_2
\end{tikzcd}
$$

where the vertical map ψ sends each formal cell to its boundary, as pictured in (5), and the horizontal map ι is the identity on the first component and the canonical inclusion of the boundary of the square. We write $\tilde{e} : \tilde{\rho} \circ \iota = \iota_2 \circ \psi$ for the canonical map witnessing for the commutation of the square. By composing the upper-right corner with the equivalence $\blacksquare \simeq 1$ described above, we obtain the pushout figured on the right above where $\pi : \{\alpha,\beta,\gamma\} \times \square \to \{\alpha,\beta,\gamma\}$ is the first projection $\pi \equiv (\mathrm{id}_{\{\alpha,\beta,\gamma\}} \times \tau) \circ \iota$ and $\rho : \{\alpha,\beta,\gamma\} \to K_2$, defined as $\rho \equiv \tilde{\rho} \circ (\mathrm{id}_{\{\alpha,\beta,\gamma\}} \times \upsilon)$ sends α to the corner of the square α in K_2, corresponding to v_0, i.e. a, (and similarly for β and γ). Hence ρ is constant map equal to a. More interestingly, the family of equalities $\tilde{e}$ is transformed into the family $e : \rho \circ \pi = \iota_2 \circ \psi$ obtained as the composite $e = \tilde{\varepsilon} \cdot \tilde{e}$ where $\tilde{\varepsilon} : \rho \circ \pi \equiv \tilde{\rho} \circ (\mathrm{id}_{\{\alpha,\beta,\gamma\}} \times (\upsilon \circ \tau)) \circ \iota = \tilde{\rho} \circ \iota$ is the equality canonically induced by ε.

As previously, we consider the type family $F_2 : K_2 \to \mathcal{U}$ defined by

$$F_2(x) \equiv (\star = \phi_2(x))$$

The flattening lemma for pushouts (lemma 2) ensures that $\Sigma K_2.F_2$ can be computed as the pushout

$$\begin{tikzcd}
\Sigma \{\alpha,\beta,\gamma\} \Box.(F_2 \circ \rho \circ \pi) \arrow[r, "\Sigma\pi.\,\mathrm{id}"] \arrow[d, "\Sigma\psi.E"'] & \Sigma \{\alpha,\beta,\gamma\}.(F_2 \circ \rho) \arrow[d, "\Sigma\rho.\lambda_.\,\mathrm{id}"] \\
\Sigma K_1.(F_2 \circ \iota_2) \arrow[r, "\Sigma\iota_2.\lambda_\ \mathrm{id}"'] & \Sigma K_2.F_2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{8}$$

The morphism

$$E : ((\omega, s) : \{\alpha,\beta,\gamma\} \Box) \to F_2 \circ \rho \circ \pi(\omega, s) \to F_2 \circ \iota_2 \circ \psi(\omega, s)$$

is defined as follows on $(\omega, s) : \{\alpha,\beta,\gamma\} \times \Box$. We have a path

$$e(\omega, s) : \rho \circ \pi(\omega, s) = \iota_2 \circ \psi(\omega, s)$$

which, by applying F_2, induces a path

$$F_2^{=}(e(\omega, s)) : F_2 \circ \rho \circ \pi(\omega, s) = F_2 \circ \iota_2 \circ \psi(\omega, s)$$

which, by transport, induces the desired map

$$(F_2^{=}(e(\omega, s)))^{\to} : F_2 \circ \rho \circ \pi(\omega, s) \to F_2 \circ \iota_2 \circ \psi(\omega, s)$$

Now, observe that for any $\omega : \{\alpha,\beta,\gamma\}$, we have

$$F_2 \circ \rho(\omega) = F_2(a) = (\star = \phi_2(a)) = (\star = \star) = Q$$

because ρ is the constant map at $a : HM_2$, and thus the upper-right corner of the pushout (8) is

$$\Sigma \{\alpha,\beta,\gamma\} F_2 \circ \rho(\omega) = \{\alpha,\beta,\gamma\} \times Q$$

the upper-left corner can be transformed similarly as $\Sigma(\{\alpha,\beta,\gamma\} \Box).Q$, which is equivalent to $\{\alpha,\beta,\gamma\} \times Q \times \Box$. Finally, the lower left corner

$$\Sigma K_1.(F_2 \circ \iota_2) \equiv \Sigma K_1.F_1$$

is precisely the fiber fib_{ϕ_1} computed in previous section. The pushout (8) can thus be slightly simplified as the pushout

$$\begin{tikzcd}
\{\alpha,\beta,\gamma\} \times Q \times \Box \arrow[r, "\pi"] \arrow[d, "\tilde{\psi}"'] & \{\alpha,\beta,\gamma\} \times Q \arrow[d, "\Sigma\rho.\lambda_.\,\mathrm{id}"] \\
\mathrm{fib}_{\phi_1} \arrow[r, "\Sigma\iota_2.\lambda_\ \mathrm{id}"'] & \Sigma K_2.F_2 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

The map $\tilde{\psi}$ is defined, for $\omega : \{\alpha, \beta, \gamma\}$, $q : Q$ and $s : \square$, by

$$\tilde{\psi}(\omega, q, s) \quad = \quad (\psi(\omega, s), (F_2^{=}(e(\omega, s)))^{\to} q) \quad = \quad (\psi(\omega, s), q \cdot \phi_2^{=}(e(\omega, s)))$$

The values of this map can be computed explicitly as we now explain. We fix $\omega \equiv \alpha$ (computing the images for β and γ is similar), and compute the image depending on s. When s is a vertex, we have:

— for v_0, the image is (a, q),

— for v_1, the image is $(b, q \cdot \phi_2(x)) \equiv (b, q \cdot i)$ because x is the path in α corresponding to $e_0 \equiv \varepsilon(v_1)$,

— for v_2, the image is $(b, q \cdot \phi_2(y)) \equiv (b, q \cdot j)$ because y is the path in α corresponding to $e_1 \equiv \varepsilon(v_2)$,

— for v_3, the image is $(a, q \cdot \phi_2(x) \cdot \phi_2(\overline{w})) \equiv (a, q \cdot i)$ because $x \cdot \overline{w}$ is the path in α corresponding to $e_0 \cdot \overline{e_2} \equiv \varepsilon(v_3)$.

The image of an edge $e_i : v_j \to v_k$ is the edge whose first component is the edge in α corresponding to e_i and the second component is the element of Q corresponding to the first component of the image of its source v_j. The squares α, β and γ are thus respectively sent to the following squares in fib_{ϕ_1}:

$$\begin{tikzcd}
(b, q \cdot j) & (a, q \cdot i) \arrow[l, "{(z,q\cdot i)}"'] \arrow[d, "{(w,q\cdot i)}"] \\
(a, q) \arrow[u, "{(y,q)}"] \arrow[r, "{(x,q)}"'] & (b, q \cdot i)
\end{tikzcd}
\qquad
\begin{tikzcd}
(b, q \cdot j) & (a, q \cdot k^- \cdot i^-) \arrow[l, "{(w,q\cdot j)}"'] \arrow[d, "{(x,q\cdot j)}"] \\
(a, q) \arrow[u, "{(y,q)}"] \arrow[r, "{(z,q)}"'] & (b, q \cdot k^-)
\end{tikzcd}$$

and

$$\begin{tikzcd}
(b, q \cdot k^-) & (a, q \cdot i \cdot j^-) \arrow[l, "{(w,q\cdot k^-)}"'] \arrow[d, "{(y,q\cdot k^-)}"] \\
(a, q) \arrow[u, "{(z,q)}"] \arrow[r, "{(x,q)}"'] & (b, q \cdot i)
\end{tikzcd}$$

We respectively write (α, q), (β, q) and (γ, q) for those cells. Finally, the previous pushout states that fib_{ϕ_2} can be obtained from fib_{ϕ_1} by attaching cells along those boundaries, for any $q : Q$.

3.6 Fiber of the 3-skeleton

The computation of the 3-skeleton can be performed similarly, as we now explain (with less details, the computations being similar). We write ⧉ for the standard cube which is the HIT pictured on the left

$$\begin{tikzcd}
& v_{-++} \arrow[rr, "v_{.++}"] \arrow[dl, "v_{-+.}"'] \arrow[dd, dotted, "v_{-.+}"] & & v_{+++} \\
v_{-+-} & & v_{++-} \arrow[ur, "v_{++.}"] \arrow[ll, "v_{.+-}"] \arrow[dd, "v_{+.-}"] & \\
& v_{--+} & & v_{+-+} \arrow[uu, "v_{+.+}"'] \arrow[ll, dotted, "v_{.-+}"] \arrow[dl, "v_{+-.}"] \\
v_{---} \arrow[uu, "v_{-.-}"] \arrow[ur, dotted, "v_{--.}"] \arrow[rr, "v_{.--}"'] & & v_{+--} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& a \arrow[rr, "y"] \arrow[dl, "w"'] \arrow[dd, dotted, "x"] \arrow[dr, phantom, "\gamma"] & & b \\
b & & a \arrow[ur, "x"] \arrow[ll, "z"] \arrow[dd, "w"] \arrow[r, phantom, "\beta"] & \\
& b \arrow[u, phantom, "\alpha"] & & a \arrow[uu, "z"'] \arrow[ll, dotted, "w"] \arrow[dl, "y"] \\
a \arrow[uu, "y"] \arrow[ur, dotted, "z"] \arrow[rr, "x"'] & & b &
\end{tikzcd}$$

we also write $\blacksquare$ for the variant with a 3-cell filling the interior of the cube. We have a map $\chi : \square \to K_2$ sending the standard cube (on the left) to the expected cube in K_2 (on the right above), so that we have a pushout as on the left:

$$
\begin{tikzcd}
\square \arrow[r] \arrow[d, "\chi"'] & \blacksquare \arrow[d] \\
K_2 \arrow[r, "\iota_3"'] & K \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\square \arrow[r, "\tau"] \arrow[d, "\chi"'] & 1 \arrow[d, "\rho"] \\
K_2 \arrow[r, "\iota_3"'] & K \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

Since the type $\blacksquare$ is contractible, the pushout is equivalent to the one on the right, where the right vertical map is pointing to a (the image of v_{---}).

Consider the type family $F : K \to \mathcal{U}$ defined by $F(x) \equiv (\star = \phi(x))$. By the flattening lemma for pushouts, we have a pushout as on the left, which can equivalently be computed as on the right:

$$
\begin{tikzcd}
\Sigma\square.(F \circ \rho \circ \tau) \arrow[r] \arrow[d] & \Sigma 1.(F \circ \rho) \arrow[d] \\
\Sigma K_2.(F \circ \iota_3) \arrow[r] & \Sigma K.F \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\square \times Q \arrow[r] \arrow[d, "\tilde{\chi}"'] & Q \arrow[d] \\
\mathrm{fib}_{\phi_2} \arrow[r] & \mathrm{fib}_{\phi} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

thus finally, providing us with an explicit construction for fib_ϕ. The map $\tilde{\chi}$ can be computed in a similar way as above and sends for $q : Q$ the cube $\square$ to

$$
\begin{tikzcd}
& (a, q \cdot j) \arrow[rr, "{(y,q\cdot j)}"] \arrow[dl, "{(w,q\cdot j)}"'] \arrow[dd, dotted, "{(x,q\cdot j)}"] \arrow[dr, phantom, "{(\gamma, q \cdot i)}"] & & (b, q \cdot 1^-) \\
(b, q \cdot j) & & (a, q \cdot i) \arrow[ll, "{(z,q\cdot i)}"] \arrow[ur, "{(x,q\cdot i)}"] \arrow[dd, "{(w,q\cdot i)}"] \arrow[dr, phantom, "{(\beta, q \cdot k^-)}"] \arrow[dl, phantom, "{(\alpha, q)}"] & \\
& (b, q \cdot k^-) & & (a, q \cdot k^-) \arrow[ll, dotted, "{(w,q\cdot k^-)}"] \arrow[uu, "{(z,q\cdot k^-)}"'] \arrow[dl, "{(y,q\cdot k^-)}"] \\
(a, q) \arrow[uu, "{(y,q)}"] \arrow[ur, dotted, "{(z,q)}"] \arrow[rr, "{(x,q)}"'] & & (b, q \cdot i) &
\end{tikzcd}
$$

Finally, this provides us with a description of fib_ϕ as a HIT consisting of cubical cells (16 0-cells, 32 1-cells, 24 2-cells, 8 3-cells) which are easily checked to form an empty 4-cube, which is homotopy equivalent to S^3.

3.7 The fundamental fiber sequence

The previous computation can be summarized by the following fiber sequence.

Theorem 10. *We have a* fundamental fiber sequence

$$\mathrm{S}^3 \longrightarrow K \xrightarrow{\ \phi\ } \mathrm{B}\,Q$$

and thus a coherent action of Q on S^3 whose homotopy quotient is K.

This thus justifies considering that our first definition of K as a HIT is the hypercubical manifold (up to homotopy).

As a consequence we have the following:

Proposition 11. *The fundamental group of the hypercubical manifold is* $\pi_1(K) = Q$.

Proof. By [18, Theorem 8.4.6], the fiber sequence theorem 10 induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_1(\mathrm{S}^3) \longrightarrow \pi_1(K) \xrightarrow{\pi_1\phi} \pi_1(\mathrm{B}\,Q) \longrightarrow \pi_0(\mathrm{S}^3) \longrightarrow \cdots$$

which simplifies as

$$\cdots \longrightarrow 1 \longrightarrow \pi_1(K) \xrightarrow{\pi_1\phi} Q \longrightarrow 1 \longrightarrow \cdots$$

thus showing the desired isomorphism of groups. □

As another consequence, we have

Lemma 12. *The map* ϕ *is 2-connected.*

Proof. By theorem 10, we have $\|\, \mathrm{fib}_\phi \,\|_2 = \|\, \mathrm{S}^3 \,\|_2 = 1$ since S^3 is 2-connected [18, Corollary 8.2.2]. □

3.8 Recovering the Galois fibration

Taking groupoid truncations induces a commuting square

$$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d, "|-|_1"'] & \mathrm{B}\,Q \arrow[d, "|-|_1"] \\ \|K\|_1 \arrow[r, "\|\phi\|_1"'] & \|\mathrm{B}\,Q\|_1 \end{tikzcd}$$

where the vertical map on the right is an equivalence because, by definition, $\mathrm{B}\,Q$ is a groupoid. Moreover, since ϕ is 1-connected (by lemma 12), the horizontal map at the bottom is an isomorphism [18, Lemma 7.5.14]. Thus, up to an automorphism of the codomain, the $\phi : K \to \mathrm{B}\,Q$ coincides with the map $|-|_1 : K \to \|K\|_1$, which is known as the *Galois fibration* [14]:

Lemma 13. *We have a commuting triangle*

$$\begin{tikzcd} & K \arrow[dl, "|-|_0"'] \arrow[dr, "\phi"] & \\ \|K\|_0 \arrow[rr, "\sim"'] & & \mathrm{B}\,Q \end{tikzcd}$$

where the bottom map is an equivalence.

Let us briefly explain the importance of this fibration. Given a pointed connected type A, we have that $\|A\|_1$ is a connected groupoid whose fundamental group is $\pi_1(A)$ by [18, Theorem 7.3.12]. The fiber of the truncation map $\|-\|_1 : A \to \|A\|_1$ is $\Sigma(x : A).(|\star|_1 = |x|_1)$ which, by [18, Theorem 7.3.12] again, coincides with the universal cover of A which is

$$\tilde{A} \quad \equiv \quad \Sigma(x : A).\|\star = x\|_0$$

along with the first projection $\pi : \tilde{A} \to A$ as universal covering map. In other words, we have a fiber sequence

$$\tilde{A} \xrightarrow{\pi} A \xrightarrow{|-|_1} \mathrm{B}\,\pi_1(A)$$

which, under the action-fibration duality (proposition 4), corresponds to the canonical action of $\pi_1(A)$ on the universal cover. The fiber sequence of theorem 10 is essentially of this form, with $A \equiv K$ so we have shown:

Theorem 14. *The universal cover of the hypercubical manifold is* S^3.

Note that we have defined the map $\phi : K \to \mathrm{B}\,Q$ in definition 8 by making an educated guess, but we could have actually computed it from the map $|-|_0 : K \to \|K\|_0$.

4 Higher dimensional hypercubical manifolds

We now construct higher dimensional variants of the hypercubical manifold: for every natural number n, we define a type K^n equipped with a canonical map $\phi^n : K^n \to \mathrm{B}\,Q$. In particular, for $n = 1$, we recover the previous constructions: $K^1 \equiv K$ and $\phi^1 \equiv \phi$. Those will be such that there is a fiber sequence

$$\mathrm{S}^{4n-1} \longrightarrow K^n \xrightarrow{\phi^n} \mathrm{B}\,Q$$

exhibiting the fact that K^n is a quotient of S^{4n-1} by an action of Q, and therefore the map ϕ^n will be $(4n-2)$-connected. In this sense, the types K^n provide better and better approximations of $\mathrm{B}\,Q$ as n increases.

In order to define those in type theory, we should draw inspiration from the topological definitions recalled in section 2.3. For instance, K^2 is obtained as a quotient of S^7 by Q. Recalling that S^7 can be constructed as a join $\mathrm{S}^7 = \mathrm{S}^3 * \mathrm{S}^3$ of two copies of S^3, the action of Q on S^7 can also be decomposed as a form of join of the action of Q on S^3 used to define K. This suggests defining ϕ^2 as a join $\phi^2 \equiv \phi * \phi$, and similarly for ϕ^n with arbitrary n (we will recall below how the join operation translate in type theory). This construction is reminiscent of the one of real projective spaces [5] and lens spaces [13], which are also obtained as joins of suitable maps to the delooping of their fundamental group.

4.1 The join operation in type theory

Given two types A and B, their *join* $A * B$, see [18, Section 6.8], is the pushout of the product projections:

$$\begin{tikzcd} A \times B \arrow[r, "\pi'"] \arrow[d, "\pi"'] & B \arrow[d, dotted, "\iota'"] \\ A \arrow[r, dotted, "\iota"'] \arrow[ur, phantom, "\ulcorner" very near end] & A * B \end{tikzcd}$$

In type theory, the type $A * B$ can be described as the HIT with $A \to A * B$ and $B \to A * B$ as base constructors and $(a, b) : A \times B \to A * B$ as a higher constructor adding paths $a = b$ indexed by pairs $(a, b) : A \times B$.

The operation $\mathrm{S}^0 * -$ is the suspension operation so that $\mathrm{S}^0 * \mathrm{S}^n = \mathrm{S}^{n+1}$. The join operation is associative, commutative and admits 0 as neutral element [2, Section 1.8], and we thus have the fact that $\mathrm{S}^n * \mathrm{S}^m = \mathrm{S}^{n+m-1}$. It can be shown that taking joins makes types more connected in the following sense [3, Proposition 3]:

Proposition 15. *If A is m-connected and B is n-connected, $A*B$ is $(m+n+2)$-connected.*

We can generalize the join construction on pairs of morphisms with common codomain as follows. Given maps $f : A \to C$ and $g : B \to C$, their *join* $f * g$ is the universal map obtain from the pushout of the dependent projections π and π' from their pullback $A \times_C B$:

$$\begin{tikzcd}
A \times_C B \arrow[r, "\pi'"] \arrow[d, "\pi"'] & B \arrow[d, "\iota'"] \arrow[ddr, bend left, "g"] & \\
A \arrow[r, "\iota"'] \arrow[drr, bend right, "f"'] & A *_C B \arrow[dr, dashed, "f*g"] & \\
& & C
\end{tikzcd}$$

The source $A *_C B$ can be thought of as a "dependent join over C" in the sense that we have $A *_1 B = A * B$: we recover the join operation on types by considering terminal maps. An important result about the join operation is the following [16, Theorem 2.3.15]:

Proposition 16. *Given morphism $f : A \to C$ and $g : B \to C$, for any $x : C$ we have an equivalence*

$$\mathrm{fib}_{f*g}\, x = (\mathrm{fib}_f\, x) * (\mathrm{fib}_g\, x)$$

Note that, as a direct consequence, we have

Proposition 17. *Given two fiber sequences*

$$A_1 \longrightarrow B_1 \xrightarrow{f_1} C \qquad\qquad A_2 \longrightarrow B_2 \xrightarrow{f_2} C$$

their join

$$A_1 * A_2 \longrightarrow B_1 *_C B_2 \xrightarrow{f_1 * f_2} C$$

is a fiber sequence.

Now consider two maps $\phi_i : A_i \to \mathrm{B}\,G$ for $i \in \{1, 2\}$ and some group G: those maps can be understood as actions of G on $F_i \equiv \mathrm{fib}_{\phi_i}$ by action-fibration duality. Their join $\phi_1 * \phi_2$ corresponds to the action of G on $F_1 * F_2$ obtained by letting G act as ϕ_i on F_i seen as a subtype of $F_1 * F_2$ through the canonical map $F_i \to F_1 * F_2$ (given by the definition as a HIT), and extended "by continuity" on other points.

Given a map $f : A \to B$, we write f^n for the iterated join of n copies of f. Combining propositions 15 and 16, we have that f^n is more and more connected as n increases. Taking the inductive limit as n goes to infinity, it can be shown that these maps converge toward the inclusion $\mathrm{im}(f) \hookrightarrow B$ [16, Theorem 4.2.13].

4.2 Higher-dimensional hypercubical manifolds in type theory

We define the n-th higher hypercubical manifold as the type

$$K^n \equiv K *_{\mathrm{B}\,Q} \ldots *_{\mathrm{B}\,Q} K$$

as the iterated join of n copies of K, and

$$\phi^n \equiv \phi * \ldots * \phi : K^n \to \mathrm{B}\,Q$$

as the corresponding universal map. By joining the fundamental fiber sequence of theorem 10 using proposition 17,

Proposition 18. *We have, for every $n \in \mathbb{N}$, a fiber sequence*

$$\mathrm{S}^{4n-1} \longrightarrow K^n \xrightarrow{\phi^n} \mathrm{B}\,Q$$

As a direct consequence of the previous fiber sequence, and the fact that S^n is $(n-1)$-connected [18, Corollary 8.2.2], we have

Proposition 19. *The canonical map $\phi^n : K^n \to \mathrm{B}\,Q$ is $(4n-2)$-connected.*

By taking the inductive limit as explained above, we obtain a delooping K^∞ of Q:

Proposition 20. *We have a fiber sequence*

$$1 \longrightarrow K^\infty \xrightarrow{\phi^\infty} \mathrm{B}\,Q$$

and thus ϕ^∞ is an equivalence.

Proof. The fact that ϕ^∞ is an equivalence follows form the fact that it merely has a point and $\mathrm{B}\,Q$ is connected. $\square$

Remark 21. Note that the previous construction would have worked starting from any map $\phi : A \to \mathrm{B}\,Q$ (as long as A is inhabited), the pointing map $1 \to \mathrm{B}\,Q$ for instance.

5 Conclusion

We have defined the hypercubical manifold in homotopy type theory and shown the relevance of our construction by exhibiting the resulting type K as a quotient of the sphere S^3 under the canonical action of the fundamental group Q. We have also introduced higher dimensional variants of those spaces.

As a direct application, it would be interesting to use K^n in order to compute the cohomology groups of Q in low dimensions (up to $4n-2$), by applying the construction of [4] to perform cellular cohomology synthetically in homotopy type theory. With integral coefficients, all those groups are already known, and we should for instance be able to recover $H^2(Q)$.

We believe that the methodology introduced here is very general and should be applicable in order to define types corresponding to various well-known spaces, including those considered by Poincaré in [15]. In particular, we plan to investigate a definition of the homology sphere, which is defined in a similar way although the combinatorics is more involved: it can be described as cellular complex with 5 0-cells, 10 1-cells, 6 2-cells and one 3-cell, its fundamental group has 120 elements and the fiber of the fundamental fibration is a model of S^3 with 600 0-cells, 1200 1-cells, 720 2-cells and 120 3-cells.

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