DIHOMOTOPY AND THE CUBE PROPERTY

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joint work with
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Concurrent programs can be interpreted as directed spaces, but methods in algebraic topology have been devised for non-directed spaces.

Dihomotopy and homotopy coincide for common programs!
General idea

Concurrent programs can be interpreted as directed spaces, but methods in algebraic topology have been devised for non-directed spaces.

Dihomotopy and homotopy coincide for common programs!

Here, I will focus on some algebraic and topological aspects.
PART I

CUBICAL SEMANTICS
OF
CONCURRENT PROGRAMS
Commutation of actions concurrent programs

In concurrent programs, some actions do commute

\[ x := 5 \parallel y := 9 \]

in the sense that their order do not matter
Commutation of actions concurrent programs

In concurrent programs, some actions do not commute

\[ \text{x := 5} \parallel \text{x := 9} \]

in the sense that their order does matter

In fact, the resulting \( x \) could even be different from 5 and 9!
Mutexes

In order to prevent incompatible actions from running in parallel, one uses **mutexes**, which are **resources** on which two actions are available

- $P_a$: *take* the resource $a$
- $V_a$: *release* the resource $a$

and implementation

- guarantees that a resource has been taken at most once at any moment,
- forbids releasing a resource which as not been taken.
Mutexes

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and implementation

- guarantees that a resource has been taken at most once at any moment,
- forbids releasing a resource which as not been taken.

Our earlier program should be rewritten as

$$P_a; x:=5; V_a \parallel P_a; x:=9; V_a$$
Mutexes

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\[ P_a ; x := 5 ; V_a \parallel P_a ; x := 9 ; V_a \]

Possible executions are
Mutexes

Our earlier program should be rewritten as

\[ P_a; x:=5; V_a \parallel P_a; x:=9; V_a \]

Possible executions are

\[
\begin{align*}
P_a & \rightarrow x:=5 \rightarrow P_a \rightarrow x:=9 \rightarrow P_a \\
V_a & \rightarrow x:=5 \rightarrow V_a \\
P_a & \rightarrow x:=9 \rightarrow P_a \\
V_a & \rightarrow x:=9 \rightarrow V_a \\
P_a & \rightarrow x:=5 \rightarrow P_a \\
V_a & \rightarrow x:=5 \rightarrow V_a
\end{align*}
\]
Concurrent programs

We consider **concurrent programs** defined by

\[ p \ ::= \ A \mid p ; p \mid p + p \mid p \parallel p \mid p^* \mid P_a \mid V_a \]

where

- **A** an *action* (e.g. \( x := 5 \))
- **\( p ; q \)** do \( p \) then \( q \)
- **\( p + q \)** do \( p \) or \( q \) (if / then / else)
- **\( p^* \)** repeat \( p \) (while)
- **\( P_a \)** take mutex \( a \)
- **\( V_a \)** release mutex \( a \)
A **cubical graph** $C$ consists of

- a set $C_0$ of *vertices*
- a set $C_1$ of *edges*
- source and target maps $\partial^-, \partial^+: C_1 \to C_0$
- a set $C_2$ of *squares*
- source and target maps $\partial^-, \partial^+, \partial^-, \partial^+: C_2 \to C_1$
- a transposition $\tau: C_2 \to C_2$

satisfying axioms so that
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- a transposition $\tau: C_2 \to C_2$

satisfying axioms so that

We sometimes add **labels** on edges.
We write

\[ A \xRightarrow{} B' \xRightarrow{} A' \xLeftarrow{} B \]

or \[ A \cdot B \diamond B' \cdot A' \]

to indicate that there exists a square \( \alpha \) with

\[
\partial^{-}_0 (\alpha) = A \quad \partial^{+}_1 (\alpha) = B \quad \ldots
\]
Cubical graph associated to a program

To every program $p$ we can associate a cubical graph $C_p$, together with *beginning* vertex $b_p$ and *end* vertex $e_p$, by induction:

- **A:**

\[ C_A = b_A \xrightarrow{A} e_A \]
Cubical graph associated to a program

To every program $p$ we can associate a cubical graph $C_p$, together with beginning vertex $b_p$ and end vertex $e_p$, by induction:

- **$A$:**
  \[
  C_A = b_A \xrightarrow{A} e_A
  \]

- **$P_a$:**
  \[
  C_{P_a} = b_{P_a} \xrightarrow{P_a} e_{P_a}
  \]

- **$V_a$:**
  \[
  C_{V_a} = b_{V_a} \xrightarrow{V_a} e_{V_a}
  \]
Cubical graph associated to a program

To every program $p$ we can associate a cubical graph $C_p$, together with $beginning$ vertex $b_p$ and $end$ vertex $e_p$, by induction:

$\triangleright p ; q :$

\[
\begin{align*}
  b_p & \quad \cdots \quad C_p \quad \cdots \quad e_p \\
  b_q & \quad \cdots \quad C_q \quad \cdots \quad e_q
\end{align*}
\]
Cubical graph associated to a program

To every program $p$ we can associate a cubical graph $C_p$, together with beginning vertex $b_p$ and end vertex $e_p$, by induction:

- $p ; q$:

$$C_{p; q} = b_p \quad C_p \quad e_p \quad b_q \quad C_q \quad e_q$$
Cubical graph associated to a program

To every program $p$ we can associate a cubical graph $C_p$, together with *beginning* vertex $b_p$ and *end* vertex $e_p$, by induction:

- $p \mathbin; q$:

$$C_{p \mathbin; q} = b_p \mathbin C_p e_p \mathbin b_q C_q e_q$$

- $p + q$:

$$C_{p + q} = b_{p + q} \mathbin C_p e_p = e_q = e_{p + q}$$
Cubical graph associated to a program

To every program $p$ we can associate a cubical graph $C_p$, together with *beginning* vertex $b_p$ and *end* vertex $e_p$, by induction:

- $p^*$:
  
  $$p^* = b_{p^*} = e_p \quad C_p \quad b_p,$$

![Diagram of a cubical graph with vertices $b_p$ and $e_p$ and edges labeled $\varepsilon$.](image)
Cubical graph associated to a program

To every program $p$ we can associate a cubical graph $C_p$, together with *beginning* vertex $b_p$ and *end* vertex $e_p$, by induction:

- $p^*$:

  $$p^* = b_{p^*} = e_p$$

- $p \parallel q$:

  $$C_{p \parallel q} = C_p \otimes C_q$$
The **tensor product** $C \otimes D$ of two cubical graphs $C$ and $D$ has

- vertices: $(C \otimes D)_0 = C_0 \times D_0$
- edges: $(C \otimes D)_1 = (C_1 \times D_0) \sqcup (C_0 \times D_1)$
- squares are of the form

\[
\begin{array}{c}
(x,y) \\
(f,y) \quad \quad \quad \quad \quad \quad (x,g) \\
\Downarrow \quad \quad \quad \quad \quad \quad \Downarrow \\
(x',y) \quad \quad \quad \quad \quad \quad \alpha \quad \quad \quad \quad \quad \quad \quad (x,y') \\
\Downarrow \quad \quad \quad \quad \quad \quad \Downarrow \\
(x',g) \quad \quad \quad \quad \quad \quad (f,y') \\
\Downarrow \quad \quad \quad \quad \quad \quad \Downarrow \\
(x',y') \quad \quad \quad \quad \quad \quad (x',y')
\end{array}
\]

for $f : x \to x'$ in $C$ and $g : y \to y'$ in $D$. 

\[10/56\]
Tensor product of cubical graphs

For instance:

\[
\begin{align*}
&\quad P_a \xrightarrow{x:=5} V_a \quad \odot \quad P_a \xrightarrow{x:=9} V_a \\
= \\
&\quad \begin{array}{c}
\begin{array}{c}
P_a \\
V_a
\end{array} \quad \begin{array}{c}
P_a \\
V_a
\end{array} \\
\begin{array}{c}
P_a \\
V_a
\end{array} \quad \begin{array}{c}
P_a \\
V_a
\end{array} \\
\begin{array}{c}
P_a \\
V_a
\end{array} \quad \begin{array}{c}
P_a \\
V_a
\end{array}
\end{array}
\end{align*}
\]
Definition
The cubical semantics $\tilde{C}_p$ of a program $p$ is the cubical graph obtained from $C_p$ by removing vertices (as well as adjacent vertices and squares) which are forbidden because some resource is taken more than once.
Cubical semantics

Definition
The **cubical semantics** $\tilde{C}_p$ of a program $p$ is the cubical graph obtained from $C_p$ by removing vertices (as well as adjacent vertices and squares) which are **forbidden** because some resource is taken more than once.

Remark
This supposes that the resource consumption is unambiguously defined for a vertex. A program for which this is the case is called **conservative**, e.g. not
Proposition

Paths in $\tilde{\mathcal{C}}_p$ starting from $b_p$ are in bijection with executions of the program $p$.

$$\tilde{\mathcal{C}}(P_a ; x:=5 ; V_a) \parallel (P_a ; x:=9 ; V_a) =$$

Diagram:

```
  +-------------------+   +-------------------+
  | Pa                |   | Pa                |
  +-------------------+   +-------------------+
  | x:=5              |   | x:=9              |
  +-------------------+   +-------------------+
  | V_a               |   | V_a               |
  +-------------------+   +-------------------+
  | Pa                |   | Pa                |
  +-------------------+   +-------------------+
  | x:=9              |   | x:=5              |
  +-------------------+   +-------------------+
  | V_a               |   | V_a               |
  +-------------------+   +-------------------+
```
Homotopy between paths

Definition
The **homotopy** relation $\sim$ between paths is the smallest congruence such that $A \cdot B \sim B' \cdot A'$ whenever $A \cdot B \diamond B' \cdot A'$:

\[ A \rightarrow A' \quad \diamond \quad B' \rightarrow B \]

Proposition
For "reasonable" programs, two homotopic executions lead to the same state.
It seems interesting to study the space of paths up to homotopy.
Definition
The **homotopy** relation $\sim$ between paths is the smallest congruence such that $A \cdot B \sim B' \cdot A'$ whenever $A \cdot B \bowtie B' \cdot A'$:

![Diagram showing homotopy between paths]

Proposition
*For “reasonable” programs, two homotopic executions lead to the same state.*
Homotopy between paths

Definition

The **homotopy** relation $\sim$ between paths is the smallest congruence such that $A \cdot B \sim B' \cdot A'$ whenever $A \cdot B \bowtie B' \cdot A'$:

![Diagram](image)

Proposition

For “reasonable” programs, two homotopic executions lead to the same state.

It seems interesting to study the space of paths up to homotopy.
PART II

HOMOTOPY VS DIHOMOTOPY
In classical topology paths are not *directed*: given a path $p : I \to X$ we also have a reverse path $\overline{p} : I \to X$ defined by

$$\overline{p}(t) = p(1 - t)$$

and most constructions in algebraic topology depend on this (the fundamental *group*, etc.)
Path direction

In classical topology paths are not *directed*: given a path $p : I \to X$ we also have a reverse path $\bar{p} : I \to X$ defined by

$$\bar{p}(t) = p(1 - t)$$

and most constructions in algebraic topology depend on this (the fundamental *group*, etc.)

On the contrary our paths must follow the directions indicated by arrows.

How can we compare the two?
We call a **dipath** what we have been calling a path, i.e. a sequence of composable arrows:

$$A \rightarrow B \rightarrow C$$ or $$A \cdot B \cdot C$$
Dipaths

We call a **dipath** what we have been calling a path, i.e. a sequence of composable arrows:

\[
\begin{array}{c}
A \rightarrow \\
B \rightarrow \\
C \\
\end{array}
\]

or

\[
A \cdot B \cdot C
\]

We call a **path** a sequence of *possibly reversed* composable arrows:

\[
\begin{array}{c}
A \rightarrow \\
B \leftarrow \\
C \\
\end{array}
\]

or

\[
A \cdot \overline{B} \cdot C
\]
We call **dihomotopy** between paths, the smallest congruence \( \sim \) such that for every square

\[
\begin{array}{c}
A \\
\Downarrow \\
B
\end{array}
\begin{array}{c}
A' \\
\Downarrow \\
B'
\end{array}
\]

we have

\[
A \cdot B \sim B' \cdot A' \\
\bar{A} \cdot B' \sim B \cdot \bar{A}' \\
\bar{B} \cdot \bar{A} \sim \bar{A}' \cdot \bar{B}'
\]
Dihomotopy

We call **dihomotopy** between paths, the smallest congruence \(\sim\) such that for every square

\[
\begin{array}{c}
\vdots \\
A & \sim & B' \\
\vdots \\
B & \sim & A'
\end{array}
\]

we have

\[
A \cdot B \sim B' \cdot A' \\
\overline{A} \cdot B' \sim B \cdot \overline{A'} \\
\overline{B} \cdot \overline{A} \sim \overline{A'} \cdot \overline{B'}
\]

**Remark**

A path dihomotopic to a dipath is necessarily a dipath.
The **homotopy** relation on paths $\sim$ is the smallest congruence containing dihomotopy and such that for every edge $x \xrightarrow{A} y$

we have

$$\text{id}_x \sim A \cdot \overline{A} \quad \overline{A} \cdot A \sim \text{id}_y$$
The **homotopy** relation on paths $\sim$ is the smallest congruence containing dihomotopy and such that for every edge $x \xrightarrow{A} y$

we have

$$\text{id}_x \sim A \cdot \bar{A} \quad \bar{A} \cdot A \sim \text{id}_y$$

**Remark**

Clearly $f \overset{\sim}{\leftrightarrow} g$ implies $f \sim g$, but converse is *not* generally true.
Consider the following “matchbox”:

where every square is filled excepting the top one:

\[ A_1 \cdot B_4 \triangleleft B_1 \cdot A_4 \]
Consider the following “matchbox”:

We have

\[ A_1 \cdot B_4 \sim B_1 \cdot A_4 \quad \text{but not} \quad A_1 \cdot B_4 \leftrightarrow B_1 \cdot A_4 \]
Homotopy vs dihomotopy

This example cannot be obtained as the semantics of a program!
Homotopy vs dihomotopy

\[ A_1 \cdot B_4 \sim C_1 \cdot \overline{C_1} \cdot A_1 \cdot B_4 \]
Homotopy vs dihomotopy

\[ A_1 \cdot B_4 \sim C_1 \cdot \overline{C_1} \cdot A_1 \cdot B_4 \]
\[ \sim C_1 \cdot A_2 \cdot \overline{C_4} \cdot B_4 \]
Homotopy vs dihomotopy

This example cannot be obtained as the semantics of a program!
Homotopy vs dihomotopy

This example cannot be obtained as the semantics of a program!
Homotopy vs dihomotopy

This example cannot be obtained as the semantics of a program!
Homotopy vs dihomotopy

This example cannot be obtained as the semantics of a program!

\[
\begin{align*}
A_1 \cdot B_4 & \sim C_1 \cdot \overline{C_1} \cdot A_1 \cdot B_4 \\
& \sim C_1 \cdot A_2 \cdot \overline{C_4} \cdot B_4 \\
& \sim C_1 \cdot A_2 \cdot B_3 \cdot \overline{C_3} \\
& \sim C_1 \cdot B_2 \cdot A_3 \cdot \overline{C_3} \\
& \sim B_1 \cdot C_2 \cdot A_3 \cdot \overline{C_3} \\
& \sim B_1 \cdot A_4 \cdot C_3 \cdot \overline{C_3}
\end{align*}
\]
Homotopy vs dihomotopy

This example cannot be obtained as the semantics of a program!
Homotopy vs dihomotopy

This example cannot be obtained as the semantics of a program!
Binary conflicts

In a situation such as

\[ P_a \parallel P_a \parallel A = \]

the vertex \( x \) is forbidden (and has to be removed).
Binary conflicts

In a situation such as

$$P_a \parallel P_a \parallel A$$

the vertex $x$ is forbidden (and has to be removed).

In this case, the vertex $y$ has to be removed too, because $A \neq V_a$!
The cube property

Semantics of programs satisfy the **cube property**:

\[
\begin{array}{c}
A \quad C' \\
\downarrow \quad \downarrow \\
B \quad B' \\
\downarrow \quad \downarrow \\
C \quad A'
\end{array}
\quad \iff 
\quad \begin{array}{c}
A \quad C' \\
\downarrow \quad \downarrow \\
B \quad B' \\
\downarrow \quad \downarrow \\
C \quad A'
\end{array}
\]
The cube property

Semantics of programs satisfy the cube property:

\[
\begin{array}{cccc}
A & C' & \Leftrightarrow & A' \\
\downarrow & \downarrow & & \downarrow \\
B & B' & & B' \\
\downarrow & \downarrow & & \downarrow \\
C & A' & & A' \\
\end{array}
\]

and other more minor properties, e.g.

\[
\begin{array}{cccc}
A & B' & & A' \\
\downarrow & \downarrow & & \downarrow \\
B & A' & & A' \\
\end{array}
\]

implies \( A' = A'' \) and \( B' = B'' \).
Theorem

In a cubical graph satisfying the cube property, two dipaths are dihomotopic if and only if they are homotopic.
PART III

PRESENTING THE FUNDAMENTAL CATEGORY AND GROUPOID
Fundamental groupoid and category

To every cubical graph $C$, we can associate

1. a **fundamental groupoid** $\Pi_1(C)$ of vertices and paths up to homotopy,

2. a **fundamental category** $\vec{\Pi}_1(C)$ of vertices and $d$ipaths up to $d$ihomotopy.
To every cubical graph $C$, we can associate

1. a **fundamental groupoid** $\Pi_1(C)$ of vertices and paths up to homotopy,

2. a **fundamental category** $\vec{\Pi}_1(C)$ of vertices and *di*paths up to *di*homotopy.

Notice that previous theorem can be reformulated as

**Theorem**

*If $C$ satisfies the cube property, then the inclusion functor*

$$\vec{\Pi}_1(C) \hookrightarrow \Pi_1(C)$$

*is faithful.*
The fundamental 2-category

In order to study the relationships between the two categories, we introduce:

**Definition**

The **fundamental 2-category** $\tilde{\Pi}_2(C)$ is the 2-category whose

- 0-cells are vertices of $C$,
- 1-cells are paths in $C$,
- 2-cells are generated by

\[ \gamma^A, B_{B', A'} : A \cdot B \Rightarrow B' \cdot A' \]

whenever

\[ \eta_A : \text{id}_x \Rightarrow A \cdot \overline{A} \quad \varepsilon_A : \overline{A} \cdot A \Rightarrow \text{id}_y \]

for

\[ x \xrightarrow{A} y \]

- quotiented by relations on 2-cells
- horizontal composition is concatenation of paths
Towards a proof

Notice that

- two paths $f, g$ are *homotopic* if and only if there is a 2-cell

  $$
  \alpha : f \Rightarrow g
  $$

- the paths $f, g$ are *dihomotopic* if and only if there is such a 2-cell constructed without generators $\eta_A$ and $\varepsilon_A$:

  $$
  \eta_A : \text{id}_x \Rightarrow A \cdot \overline{A} \quad \varepsilon_A : \overline{A} \cdot A \Rightarrow \text{id}_y
  $$
Towards a proof

Notice that

- two paths $f, g$ are *homotopic* if and only if there is a 2-cell

$$\alpha : f \Rightarrow g$$

- the paths $f, g$ are *dihomotopic* if and only if there is such a 2-cell constructed without generators $\eta_A$ and $\varepsilon_A$:

$$\eta_A : \text{id}_x \Rightarrow A \cdot \overline{A} \quad \quad \varepsilon_A : \overline{A} \cdot A \Rightarrow \text{id}_y$$

**Remark**

Notice that this does not depend on the relations on 2-cells.
Towards a proof

Notice that

- two paths $f, g$ are *homotopic* if and only if there is a 2-cell
  \[ \alpha : f \Rightarrow g \]

- the paths $f, g$ are *dihomotopic* if and only if there is such a 2-cell constructed without generators $\eta_A$ and $\varepsilon_A$:
  \[ \eta_A : \text{id}_x \Rightarrow A \cdot \overline{A} \quad \varepsilon_A : \overline{A} \cdot A \Rightarrow \text{id}_y \]

**Theorem**

Any 2-cell $\alpha : f \Rightarrow g$ between $f$ and $g$ is equal to one without the bad generators (with the right relations!).
For the 2-cells I will use the string-diagrammatic notation:

For $\gamma_{B',A'}^{A,B}$ and

for $\eta_A$ and $\varepsilon_A$. 

String diagrams
We relations on 2-cells so that

\[ \gamma_{B',A'}^{A,B} \] acts like a symmetry:

\[ A B \]
\[ B' A' \]
\[ A B \]
\[ B' A' \] =
\[ A B \]
\[ A B \]
\[ A B \]
\[ A B \]

\[ A B C \]
\[ B' A' C' \]
\[ A'' B'' C'' \]
\[ A B C \]
\[ B' A' C' \]
\[ A'' B'' C'' \] =
\[ A B C \]
\[ B' A' C' \]
\[ A'' B'' C'' \]
\[ A B C \]
\[ B' A' C' \]
\[ A'' B'' C'' \]
Relations on 2-cells

We relations on 2-cells so that

- $\eta_A$ and $\varepsilon_A$ act as (co)units of an adjunction:

\[
\begin{align*}
\begin{array}{c}
A \\
A \\
\end{array}
\begin{array}{c}
\overline{A} \\
\overline{A} \\
\end{array}
= 
\begin{array}{c}
A \\
A \\
\end{array}
\begin{array}{c}
\overline{A} \\
\overline{A} \\
\end{array}
\end{align*}
\]

and

\[
\begin{align*}
\begin{array}{c}
A \\
A \\
\overline{A} \\
\end{array}
\begin{array}{c}
\overline{A} \\
\overline{A} \\
\end{array}
= 
\begin{array}{c}
A \\
A \\
\overline{A} \\
\end{array}
\begin{array}{c}
\overline{A} \\
\overline{A} \\
\end{array}
\end{align*}
\]
Relations on 2-cells

We relations on 2-cells so that

- the two are “naturally” compatible:

\[
B \quad A' \quad \overline{A'}
\]
\[
A \quad B' \quad \overline{A}
\]

\[
\overline{A'} \quad A' \quad \overline{A'}
\]
\[
A \quad \overline{A} \quad \overline{A}
\]

+ dual and symmetric relations
Some other relations are derivable:
Well-definedness

Notice that “not every diagram makes sense”: if we cannot commute some actions for instance.

**Lemma**

*If the left member of a relation is well-defined then the right member too.*
Well-definedness

Notice that “not every diagram makes sense”: if we cannot commute some actions for instance.

**Lemma**

*If the left member of a relation is well-defined then the right member too.*

**Proof.**

This is where we use our properties on the cubical graph:
A rewriting system

We can turn our relations into a rewriting system (from left to right), e.g.

Conjecture

The rewriting system is convergent, thus normal forms are canonical representatives of equivalence classes.
A proof for our theorem

Suppose given a 2-cell between dipaths $\alpha : f \Rightarrow g$. This 2-cell is equal to a normal form, so we suppose that we are in this case.

**Proposition**

*The 2-cell $\alpha$ does not contain $\eta_A$ or $\varepsilon_A$ generators.*
A proof for our theorem

Suppose given a 2-cell between paths \( \alpha : f \Rightarrow g \). This 2-cell is equal to a normal form, so we suppose that we are in this case.

**Proposition**

*The 2-cell \( \alpha \) does not contain \( \eta_A \) or \( \varepsilon_A \) generators.*

**Proof.**

Suppose that it “contains”

\[
\varepsilon_A : \overline{A} \cdot A \Rightarrow \text{id}_x
\]

i.e.

\[
\alpha = \psi \circ (\text{id}_f \cdot \varepsilon_A \cdot \text{id}_g) \circ \phi
\]
What can $\phi$ be?

\[
\begin{array}{c}
\ldots \\
\alpha \\
\ldots \\
\end{array}
\quad = \quad
\begin{array}{c}
\ldots \\
\begin{array}{c}
A \\
\cup \\
A \\
\end{array}
\phi \\
\psi \\
\ldots \\
\end{array}
\]
A proof for our theorem

What can $\phi$ be?

- Notice that $\phi$ cannot be an identity, otherwise $\alpha$ would contain $\overline{A}$ in its source (a reversed edge), which would not be a dipath.
A proof for our theorem

What can $\phi$ be?

Notice that $\phi$ cannot be an identity, otherwise $\alpha$ would contain $\overline{A}$ in its source (a reversed edge), which would not be a dipath.

Thus $\phi$ is thus of the form

where $\rho$ is a generator.
We then proceed on case analysis on $\rho$ and its position, keeping in mind that $\alpha$ must be in normal form. For instance, if $\rho = \gamma$,

- in a case such as

we can use the exchange law to “put the $\gamma$ down in the $\psi$” and reason by induction on $\phi'$. 
A proof for our theorem

We then proceed on case analysis on \( \rho \) and its position, keeping in mind that \( \alpha \) must be in normal form. For instance, if \( \rho = \gamma \),

- in a case such as

\[
\begin{array}{c}
\cdots \\
\phi' \\
\cdots \\
\psi \\
\cdots
\end{array}
\]

we can use the exchange law to “put the \( \gamma \) down in the \( \psi \)” and reason by induction on \( \phi' \).
A proof for our theorem

We then proceed on case analysis on $\rho$ and its position, keeping in mind that $\alpha$ must be in normal form. For instance, if $\rho = \gamma$,

- the following cannot happen

\[
\begin{array}{c}
\cdots \\
\phi' \\
\cdots \\
\psi \\
\cdots \\
\end{array}
\]

otherwise $\alpha$ would not be normal.
We then proceed on case analysis on $\rho$ and its position, keeping in mind that $\alpha$ must be in normal form. For instance, if $\rho = \gamma$,

- we can show that $\alpha$ is of the form

![Diagram]

and thus the morphism would contain $\bar{A}$ (a reversed transition in its source).
A real proof

Showing that the rewriting system is convergent is difficult:

- there is an infinite number of critical pairs even though there is a finite number of rules (they can however be grouped in a finite number of families),
- there is an awful lot of cases to be checked.
A real proof

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- there is an infinite number of critical pairs even though there is a finite number of rules (they can however be grouped in a finite number of families),
- there is an awful lot of cases to be checked.

In practice, we only need a representative (not necessarily unique), which can be defined by hand, and the proof goes on roughly as indicated before. So we actually have a proof here.
Notes on the axioms

In the category \textbf{Vect} we have bijections

\[
\begin{align*}
A \otimes B & \rightarrow C \\
A & \rightarrow C \otimes B^* \\
A & \rightarrow B \otimes C \\
B^* \otimes A & \rightarrow C
\end{align*}
\]
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In particular, consider the morphisms associated to $\text{id}_A : A \to A$,

\[
\eta : k \to A \otimes A^* \\
\varepsilon : A^* \otimes A \to k
\]

Together with the symmetry $\gamma : A \otimes A \to A \otimes A$, these satisfy the axioms before, i.e. these correspond to cubical graph with one vertex, one edge and one square.
Notes on the axioms

In the category \textbf{Vect} we have bijections

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A \otimes B & \to C \\
\frac{A \to C \otimes B^*}{B^* \otimes A} & \to C
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In particular, consider the morphisms associated to $\text{id}_A : A \to A$,

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To be precise, we also have to satisfy the axiom

\[
\begin{align*}
(\text{dim}A) \text{id}_k & = \text{tr}(\text{id}_A) = A \circlearrowleft A = \text{id}_k
\end{align*}
\]

i.e. $\text{dim}A = 1$.  

PART IV

UNIVERSAL DISCOVERING

(or not)
The universal covering

**Definition**
A map $p : \tilde{X} \to X$ is a **covering** when every point $x \in X$ admits an open neighborhood $U$ such that $p^{-1}(U)$ is a disjoint union of open sets homeomorphic to $U$.
The universal covering

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A covering is **universal** when it is “most general” (most unfolded).

**Theorem**
The universal discovery can be constructed as the space of homotopy classes $[f]$ of paths $f$ with origin $x_0 \in X$. 
The universal directed covering

A directed topological space $X$ is a space equipped with a coherent set $dX$ of directed paths.
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A covering is **universal** when it is “most general”.
Theorem
Consider a precubical set $C$ satisfying the cube property. Consider its geometric realization $|C|$ (as a directed space) and a point $x_0 \in |C|$. The subspace of the universal dicovering reachable from $x_0$ can be constructed as the space of dihomotopy classes $[f]$ of directed paths $f$ with origin $x_0 \in X$.
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Constructing the universal dicovering

**Theorem**
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The subspace of the universal dicovering reachable from $x_0$ can be constructed as the space of dihomotopy classes $[f]$ of directed paths $f$ with origin $x_0 \in X$.

It can be seen as the traditional covering together with the “inherited direction”.
Constructing the universal dicovering

In the general case, the two do not coincide and the characterization can be taken as a definition [Fajstrup&Rosický10].

For instance, consider the “matchbox” again:

- Topologically is it $S^2$, so identity is the only covering.
- However, there are two non-dihomotopic paths from $x_0$ to $y$. 
PART V

GEOMETRIC POINT OF VIEW

(or not)
Precubical sets

A cubical graph consists of

- 0-cubes: vertices
- 1-cubes: edges
- 2-cubes: squares

There is a well-known generalization of this to any dimension:

**Definition**

A **precubical set** $C$ is a family $(C_n)_{n \in \mathbb{N}}$, the elements of $C_n$ being called $n$-cubes, together with suitable face maps

$$\partial_i^-, \partial_i^+ : C_{n+1} \to C_n$$

with $0 \leq i < n$. 
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with $0 \leq i < n$.

In the following, a cubical graph $C$ will be seen a precubical set with $C_n$, for $n > 2$, being the set of all possible hollow $n$-cubes in $C$. 

**Precubical sets**
Precubical sets as presheaves

There is a category $\square$, whose objects are integers, such that precubical sets are presheaves over it:

$$\text{PCSet} \cong \square$$
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We can thus use the classical abstract machinery in order to construct a geometric realization $|C| \in \text{Top}$: we “glue” geometric $n$-cubes according to the “plan” provided by faces in $C$. 
Precubical sets as presheaves

There is a category $\Box$, whose objects are integers, such that precubical sets are presheaves over it:

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We can thus use the classical abstract machinery in order to construct a geometric realization $|C| \in \text{Top}$: we “glue” geometric $n$-cubes according to the “plan” provided by faces in $C$.

The geometric realization can also be performed in $\text{Met}$, the category of metric spaces!
Geometric realization in metric spaces

A realization in metric spaces is desirable.

- We want to have a notion of length of paths (corresponding to the duration of an execution).
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- We want to have a notion of length of paths (corresponding to the duration of an execution).
- The category of metric spaces is not cocomplete: we have to take a variant of metric spaces.
- We would also like to encode the time direction in the metric.
Generalizing metric spaces

Definition

A **metric space** is a space $X$ equipped with a metric $d : X \times X \to [0, \infty]$ such that, given $x, y, z \in X$,

1. **point equality**: $d(x, x) = 0$
2. **triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$
3. **finite distances**: $d(x, y) < \infty$
4. **separation**: $d(x, y) = 0$ implies $x = y$
5. **symmetry**: $d(x, y) = d(y, x)$

We consider contracting maps $f : X \to Y$:

$$d_Y(f(x), f(x')) \leq d_X(x, x')$$

Unfortunately, the resulting category is not cocomplete!
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Intuitively, $X + Y$ should be such that

$$d(x, y) = \infty$$

for $x \in X$ and $y \in Y.$
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Consider the relation $\approx$ on $X$ identifying a family of points $(x_i)_{i \in \mathbb{N}}$ such that $d(x_i, y) = 1/i$ for some $y$

Intuitively, in $X/ \approx$, we should have $d([x_i], [y]) = 0$. 
Generalizing metric spaces

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We can encode direction in the distance!

$$d(x, y) = \bigwedge \left\{ \rho - \theta \mid x = e^{i2\pi\theta}, y = e^{i2\pi\rho}, \rho \geq \theta \right\}$$
Generalized metric spaces

Definition (Lawvere)

A **generalized metric space** is a space $X$ equipped with a metric $d : X \times X \to [0, \infty]$ such that, given $x, y, z \in X$,

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2. triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

The category $\text{GMet}$ enjoys the following:

- the category $\text{GMet}$ is complete and cocomplete,
- the forgetful functor $\text{GMet} \to \text{Set}$ has left and right adjoints,
- the forgetful functor $\text{GMet} \to \text{Top}$ preserves finite (co)limits.
Directed metric realization

We write $\vec{I}$ for the **directed interval** $[0, 1]$ equipped with

$$d(x, y) = \begin{cases} 
    y - x & \text{if } y \geq x \\
    \infty & \text{if } y < x
\end{cases}$$
Directed metric realization

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\]

The product \( \vec{I}^n \) is equipped with

\[
d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = d(x_1, y_1) \lor \ldots \lor d(x_n, y_n)
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The geometric realization of a precubical set \( C \) is

\[
|C| = \int_{n \in \Box} C_n \cdot \vec{I}^n
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Directed metric realization

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The geometric realization of a precubical set $C$ is

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Proposition

For finite-dimensional precubical sets, geometric realization commutes with forgetful functor $\text{GMet} \rightarrow \text{Top}$ and produces geodesic length spaces.
Colimits in \textbf{GMet} vs \textbf{Top}

Colimits in \textbf{GMet} do not necessarily coincide with those in \textbf{Top}.
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Write $I_n$ for the space $I = [0, 1]$ equipped with $d_n(x, y) = |y - x|/n$. 
Colimits in **GMet** do not necessarily coincide with those in **Top**.

Write $l_n$ for the space $l = [0, 1]$ equipped with $d_n(x, y) = |y - x|/n$.

Consider the colimit

$$l_{\infty} = \coprod_{n \in \mathbb{N}} l_n / \cong$$

where $\cong$ identifies 0 (resp. 1) in various $l_n$. 
Colimits in **GMet** do not necessarily coincide with those in **Top**.

Write \( l_n \) for the space \( l = [0, 1] \) equipped with \( d_n(x, y) = |y - x|/n \).

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\[
I_\infty = \coprod_{n \in \mathbb{N}} I_n/ \approx
\]

where \( \approx \) identifies 0 (resp. 1) in various \( l_n \).

We have \( d(0, 1) = 0 \) and therefore the points 0 and 1 are not separated in \( I_\infty \) (see Bridson & Haefliger).
Geometric realization in metric spaces works well.

Moreover, the resulting spaces are non-positively curved.
Interestingly, the cube property was used by Gromov to characterize non-positively curved spaces.
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A locally $\text{CAT}(0)$ space is called non-positively curved (NPC).
Interestingly, the cube property was used by Gromov to characterize non-positively curved spaces.

- A **geodesic triangle** $\Delta(x, y, z)$ in a metric space $X$ consists of three points $x, y, z$ and geodesics joining any pairs.
- A **comparison triangle** for a geodesic triangle $\Delta(x, y, z)$ consists of an isometry $-_\Delta: \Delta(x, y, z) \to \mathbb{R}^2$ whose image $\Delta(x, y, z)$ is a geodesic triangle.
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- A **geodesic triangle** $\triangle(x, y, z)$ in a metric space $X$ consists of three points $x$, $y$, $z$ and geodesics joining any pairs.
- A **comparison triangle** for a geodesic triangle $\triangle(x, y, z)$ consists of an isometry $\sim : \triangle(x, y, z) \to \mathbb{R}^2$ whose image $\triangle(x, y, z)$ is a geodesic triangle.

**Definition**
A geodesic space is **CAT(0)** if for every geodesic triangle $\triangle(x, y, z)$, there exists a comparison triangle $\triangle(x, y, z)$ such that for every points $p, q \in \triangle(x, y, z)$, we have $d(p, q) \leq d_{\mathbb{R}^2}(p, q)$. 
CAT(0) spaces

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A locally CAT(0) space is called **non-positively curved** (NPC).
Reformulating ("flag links") in our setting (and omitting minor details):

**Theorem (Gromov)**

The geometric realization of a precubical set is NPC if and only iff it satisfies the cube property.
Gromov’s theorem

Reformulating ("flag links") in our setting (and omitting minor details):

**Theorem (Gromov)**

_The geometric realization of a precubical set is NPC if and only if it satisfies the cube property._

Such a space is locally uniquely geodesic. In particular, directed paths are local geodesics:

an analogue of the _least action principle_

Moreover, it enjoys many nice properties (e.g. Greedy normal forms for paths, universal cover is CAT(0), fundamental group is automatic, …).
Consider

\[ P_a \parallel P_a \parallel P_a \]

whose realization of geometric semantics is

A small example

a mutex

NPC
Consider

\[ P_a \parallel P_a \parallel P_a \]

whose realization of geometric semantics is

- a mutex
- NPC

- a of arity 2
- not NPC
CONCLUSION
Going further

For a cubical graph satisfying the cube property:

- universal **dicovering** has a simple definition,
- its unfolding corresponds to the configuration space of an **event structure** (Chepoi, Ardilla et al., …)
- its trace space can be computed thanks to (traditional) **homology**
- metric geometric realization is **non-positively curved** (= locally CAT(0))

Also:

- **Relations** on 2-cells are meaningful?
- Variants for **n-semaphores**, etc.
- Links with motion planning (Ghrist et al.)
- Links with geometric group theory (Dehornoy, …)