Non-Alternating Innocence

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(joint work with Paul-André Melliès)
Alternating game semantics

Left and
Alternating game semantics

Right and
Non-alternating game semantics

Parallel and
Non-alternating game semantics

Parallel and
Non-alternating game semantics

Parallel and

\[
\begin{array}{c}
B \Rightarrow B \Rightarrow B \\
V & q & V \\
q & V & q \\
\end{array}
\]
Asynchrony: Non-alternating strategies

Left and
Asynchrony: Non-alternating strategies

Right and
Asynchrony: Non-alternating strategies

Parallel and
Asynchrony: Non-alternating strategies

Parallel and
Asynchrony: Non-alternating strategies

Parallel and
Asynchrony: Non-alternating strategies

Parallel and
Formulas are inherently non-alternating

Each connective $\otimes$ and $\oplus$ is performed by a Player move
Part I

What is innocence [in alternating games]?
Innocent strategies are partial orders

In alternating games:

\[
\text{arena} = \text{formula} = \text{partial order}
\]

\[
\text{innocent strategy} = \text{Böhm tree} = \text{partial order}
\]

Every Böhm tree refines its formula
Innocent strategies are positional

In alternating games:

**Positionality of Innocence** [Melliès 2004]

Suppose that \( \sigma \) is innocent, and that \( s \in \sigma \) and \( t \in \sigma \),

\[
s \sim t \quad \text{and} \quad s \cdot u \in \sigma \quad \text{implies} \quad t \cdot u \in \sigma
\]
In innocent strategies are relational

In alternating games:

The set of halting positions of a strategy $\sigma$ is defined as

$$\sigma^\circ = \{ x \mid \exists s \in \sigma, \ s: * \rightarrow x \}$$

**Relationality of Innocence** [Melliès 2004]
Every innocent strategy $\sigma$ is characterized by the set $\sigma^\circ$. 
Innocent strategies are relational

A strong monoidal functor \((—)^°\) from games to relations.

\[
\begin{align*}
\text{Games} & \rightarrow \text{Rel} \\
A & \mapsto A^° \\
\sigma & \mapsto \sigma^°
\end{align*}
\]

\[
\sigma \quad \text{strategies}
\]

\[
\pi \quad \text{proofs}
\]

\[
\sigma^° \quad \text{relations}
\]

\[
(\sigma \otimes \tau)^° = \sigma^° \otimes \tau^°
\]
Positions as relations

To every strategy $\sigma : A \rightarrow B$, we associate a relation on $A^\circ \rightarrow B^\circ$

$$\sigma^\circ = \{(x, y) \in A^\circ \times B^\circ \mid \exists s : \ast \rightarrow (x, y) \in \sigma\}$$

Functoriality

$$(\sigma; \tau)^\circ = \sigma^\circ; \tau^\circ$$

dynamic composition \hspace{2cm} static composition
The aim of this talk

A tentative definition of innocence in non-alternating games

*Methodology:* extend the three properties by diagrammatic methods.
Part II

Homotopy classes are partial orders
Every partial order generates a 2-graph
Every partial order generates a 2-graph
Every partial order generates a 2-graph
Every partial order generates a 2-graph
Every partial order generates a 2-graph
Every partial order generates a 2-graph
The Cube Property
The Cube Property

1: $m \parallel n$

2: $m \parallel o$

3: $n \parallel o$
Conversely...

Let us consider a 2-graph satisfying the Cube Property.
Poincaré Duality: from Cubes to Braids

Yang-Baxter equations as a confluent 3-dimensional Rewriting System
Unions and intersections as normal forms
Structure of the prefixes

Consequence
The prefixes of a path $f$ modulo homotopy form a distributive lattice.
Every homotopy class is a partial order

Every path $f$ generates a partial order $\llbracket f \rrbracket$ on its set of moves, such that

$$g \sim f \iff g \text{ is a linearization of } \llbracket f \rrbracket.$$ 

An embarassingly simple notion of homotopy!
Part III

From sequentiality to positionality
Definition of asynchronous game

An asynchronous game is a 2-graph satisfying the Cube Property.

A vertex $*$ is chosen as initial position of the game.
The sequential definition of a strategy

A strategy is a set of paths

\[
\ast \xrightarrow{m_1} x_1 \xrightarrow{m_2} x_2 \cdots x_{k-1} \xrightarrow{m_k} x_k
\]

which is

- non-empty,
- closed under prefix.

The traditional definition of a strategy in game semantics.
Positionality

Definition
A strategy is **positional** when it is the set of paths

\[ \ast \xrightarrow{m_1} x_1 \xrightarrow{m_2} x_2 \cdots x_{k-1} \xrightarrow{m_k} x_k \]

of a subgraph of the 2-graph.

Same definition as previously.
From sequentiality to positionality

When is a sequential strategy positional?
Three properties: The Cube
Three properties: Preservation of Compatibility

Preservation of compatibility

\[
\sigma \ni n \downarrow \sim m \downarrow \ni \in \sigma \implies \sigma \ni n \downarrow \sim m \downarrow \ni \in \sigma
\]
Three properties: Extension

Extension property

\[ s \cdot m \cdot n \in \sigma \]
\[ s \cdot n \cdot m \in \sigma \]
\[ s \cdot m \cdot n \cdot u \in \sigma \]
\[ s \cdot n \cdot m \cdot u \in \sigma \]
Dynamic positionality

**Theorem**
An innocent strategy is a subgraph of the graph of the game which satisfies

\[
\sigma \ni n \uparrow \sim m \downarrow m \uparrow \sim \downarrow n \in \sigma \Rightarrow \sigma \ni n \uparrow \sim m \downarrow m \uparrow \sim \downarrow n \in \sigma
\]

and

\[
\sigma \ni n \uparrow \sim m \downarrow m \uparrow \sim \downarrow n \in \sigma \Rightarrow \sigma \ni n \uparrow \sim m \downarrow m \uparrow \sim \downarrow n \in \sigma
\]
Part IV

From positionality to relationality
Halting positions

The set of **halting positions** of a strategy \( \sigma \) is defined as

\[
\sigma^0 = \{ x \mid \forall s : * \rightarrow x \in \sigma, \forall m \in M, \ s \cdot m \in \sigma \Rightarrow \lambda(m) = P \}
\]

**halting position** = the strategy has nothing left to play
Relationality

Strategies are characterized by their halting positions: we can recover \( \sigma \) from \( \sigma^\circ \).

\[
\text{strategy} = \text{closure operator}
\]
Definition of asynchronous strategy

- **Courteous**: for every Player move \( m \),

\[
\sigma \ni m \sim n \implies \sigma \ni n \sim m \ni s \ni * \ni \sigma \ni m \sim n \ni s \ni *
\]

- **Receptive**: for every Opponent move \( m \)

\[
\sigma \ni m \ni s \ni * \ni \sigma \ni m \ni s \ni *
\]
Definition of deterministic strategy

- for every Player move $m$

\[ \sigma \ni m \implies \sigma \ni n \in \sigma \]

\[ \sigma \ni m \implies \sigma \ni n \in \sigma \]
Functoriality of relationality

Functoriality:

$$(\sigma; \tau)^\circ = \sigma^\circ; \tau^\circ$$

Livellocks/deadlocks avoided by adding payoff on paths.

We get a faithful strong monoidal functor from Games to Rel.
Part V

Further work
Recovering alternating innocence

The subcategory of alternating innocent strategies:

- games are alternating
- for every Opponent moves $m$ and $n$, and Player move $o$,
Summary

Four interactive paradigms:

1. small steps (sequential)
2. big steps (sequential by clusters of moves)
3. dynamic positionality (closure operators)
4. static positionality (halting positions)
What’s next?

• Construct a model of Linear Logic in which every connective is interpreted by a move, based on a lax and unbiased monoidal category with $n$-ary tensor products:

$$(A_1 \otimes \cdots \otimes A_n)$$

and a 2-categorical notion of cartesian product.

• Reconstruct semantically focalization and correctness criteria.

$$(A_1 \otimes \cdots \otimes A_k) \otimes (A_{k+1} \otimes \cdots \otimes A_n) \quad \Rightarrow \quad (A_1 \otimes \cdots \otimes A_n)$$

• Exhibit truly concurrent models of concurrent languages.