Notes for GEOCAL 2006 talk

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Intro

Title

I'm presenting here a joint work with PAM. We are building a game semantic model of linear logic where **both** games and strategies are possibly **non-alternating**. In contrast to mainstream game semantics which is usually alternating. The main difficulty here is to provide a **satisfactory notion of innocence** in this new framework. The usual notion of innocence relies on the notion of view which requires alternating plays to work (in its usual formulation). The first point we should address is: *what is a satisfactory notion of innocence?*

Implementations of conjunction

Let's first briefly recall what is the usual (HO-style) alternating model and why the **wish** to explore a non-alternating setting arises naturally. Here, formulas are interpreted by arenas, which are trees corresponding to the syntax of the formula and proofs are interpreted as strategies which are made of plays – that is explorations of the tree of the formula starting from the root. Usually, plays are represented as pointed strings. I've figured here a play in one possible **implementation** of the conjuction function which computes its result by first looking at its first argument and then its second argument.

Describe the example: environment – query the arguments

Similarly, right conjuction.

However, we would like to have a **continuum** between those two implementation i.e. have a strategy whose behaviour is in between those implementations. And actually, the parallel implementation of the conjuction arises naturally in computer science. The parallel implementation would query both its arguments, then wait for both answers and then give the result. This play is **non-alternating**! Here since we want to consider **only sequences** of atomic moves, we represent the situtation where both arguments are queried at the same time by the two stratgies $q_1 \cdot q_2$ / $q_2 \cdot q_1$ and the information that we can have "everything in between" (that is we can "deform" one play into the other within the strategy).

...in transition system

Let us shift from this presentation (which is particular to game semantics) to the more usual presentation in transition systems. Arenas will be presented as a graph with an initial position and strategies will be set of plays – that is set of paths in those graphs starting from the initial position. Here, by **position** we mean a vertex of the graph of the game.

Describe the plays.

The parallel implementation of the conjuction can query its arguments in both orders (left-right / right-left) and even **simultaneously**. The point is that the environment is asynchronous and thus does not make a difference. So from an interactive point of view, they are like the strategy which would ask really simultaneously the value of both of its arguments. It is interesting to distinguish between a strategy that can incidently play both $q_1 \cdot q_2$ and $q_2 \cdot q_1$ and the strategy which can really play the moves q_1 and q_2 in parallel. We therefore add a 2-dimensional information: we represent the case $q_1 \| q_2$ by adding an **homotopy tile**. Our graphs will be equipped with an equivalence relation between paths and two paths will be homotopic if intuitively one can be obtained from the other by permuting independent moves. This enables us to imagine a connection between our work and the study of geometric properties of spaces modulo homotopy [Eric Goubalt, Marco Grandis, Emmanuel Haucourt] (geometry of concurrency). Indeed, playing two moves a and b in parallel can be represented as playing $a \cdot b$ or $b \cdot a$ (the extremal sequentializations) or "something in between". This "something is between" can be thought as the ability to deform continuously $a \cdot b$ into $b \cdot a$ within the strategy.

Non-alternation in LL

Formulas in linear logic are alternating only after focalization. For instance, this formula contains two positive connectives and a proof of it will start with two Player moves corresponding to the two connectives (it won't be alternating). We are interested in studying semantically the **process of focalization**. Thus, we need to consider at the same time alternating and non-alternating games.

1 What is innocence?

In order to propose a definition of innocence in our non-alternating setting, it is good to think about what is innocence in the usual alternating setting. We select here **three properties** of innocence in alternating games which we believe are the fundamental properties of innocence: they characterize the interactive behaviour of proofs. Then later on, we will try to keep those three fundamental properties in our non-alternating games.

Order

An arena is a tree and is threfore a **partial order on the moves**. By full abstraction, innocent strategies correspond to Böhm trees and therefore to partial orders too. As a partial order, they refine the order of their type (they are more sequential).

Positionality

Explain. Two plays with **homotopic histories** (modulo permutation of independent moves) have the same future.

Relationality

Explain halting positions. We can recover innocent strategies from their sets of halting positions.

The operation which to a game associates its **positions** (the vertices of the graph) and to an innocent strategy associates its set of halting position can be extended as a functor from the game model of innocent strategies to the relational model of linar logic. *Commutative diagram.* Functoriality has first been achieved [Martin Hyland, Andrea Schalk] but the novelty here is that the functor is **striong** monoidal.

If σ is a strategy on $A \to B$, a position of σ is a pair made of a position of A and a position of B. The set of halting positions of σ can therefore be seen as a relation between the positions of A and the positions of B. **Functoriality** of the operation $(-)^\circ$ tells us that dynamic composition (the usual composition of strategies) corresponds to the static composition (the relational composition on positions).

Aim

We now propose a tentative definition of innocence in our non-alternating setting, capturing the **behaviours** of proofs as non-alternating strategies. Our definition is based on a series of diagrammatic axioms – that is, we impose some local commutation of moves in the strategies – from which we deduce the three fundamental properties of innocence.

2 The Cube Property

Describe the order. Every partial order, generates a graph, the *positional graph*, whose nodes are the downward closed subsets of the order. We go from a position x to a position y by adding an element a to x. Paths starting from $*$ are linearizations of the order. If we think of the order as an arena, the positional graph will be the graph of the underlying game. If we think of the order as a Böhm tree, the paths in the graph starting from $*$ will be the plays in the innocent strategy.

Observation: the Cube Property is satisfied in such graphs. Intuition: the three moves are pairwise independents.

The converse is also true: to every pointed graph satisfying the Cube Property, we can associate an order which generates the graph by the method we described.

Braids

By Poincaré duality [John Baez] we get Yang-Baxter (braids). *Draw the duality.* If we orient this equation in one way or the other, we get a confluent rewriting system [Yves Lafont].

Union

Suppose that $s \cdot s'$ and $t \cdot t'$ are two homotopic paths. We can go from $s \cdot s'$ to $t \cdot t'$ by a "chaotic" sequence of permutation tiles. If we apply our rewriting system the we obtain a plane square making the whole diagram looking like a "big version of the cube". This computes in fact the pushout of s and t (in the category of prefixes of $s \cdot s'$ modulo homotopy).

In fact, the prefixes of a path f modulo homotopy are a partial order. The construction that we have just shown, enables us to compute the union of two paths (and dually we can compute the intersection of two paths) in this category. Moreover, this order category can be shown to be a distributive lattice.

By Birkhoff theorem, it is therefore generated by an order on its prime elements, which in that case are the moves. Any homotopy class $[f]$ is characterized by a partial order on the moves. A path g is homotopic to f if and only if it is a linearization of this order.

3 From sequentiality to positionality

By **sequentiality** here, we mean the definition of a strategy as a set of sequences.

Let us recap our setting. **Games** = 2-dimentional graphs (that is graphs equipped with an equivalence relation on paths) + an initial position. The graphs are required to satisfy the Cube Property: they are locally generated by a partial order which corresponds to a *generalized syntax for formulas*. A strategy is, as usual, a non-empty set of plays which is closed under prefix. We now show how to impose further properties on those strategies in order for them to satisfy the three properties mentionned before, which characterize **innocence**.

We want our strategies to be positional which means that they should be the set of paths starting from the initial from position of some **subgraph** of the game. We moreover want this subgraph to be somehow characterized by a **partial order** on the moves. We therefore impose some further properties on our graphs.

- The Cube Property (now expressed on set of plays instead of graphs).
- Two moves compatible in a game are compatible inside the strategy

(in terms of rewriting theory, a strategy is as locally confluent as the underlying game).

- Extension property: if $s \cdot m \cdot n$ is a path in σ that I can extent by a path u and the path $s \cdot n \cdot m$ (obtained from $s \cdot m \cdot n$ by permutation of the independent moves m and n) is still in σ, then I can also extend the path $s \cdot n \cdot m$ by u and the result is still in σ. *The future of a strategy depends on its past but modulo homotopy* (that is modulo permutation of independent moves).

We can show that strategies statisfying those three diagrammatic conditions are **positional**. They are the sets of plays of a subgraph of the game and we can even characterize the subgraphs generating those strategies: the subgraphs satisfying the two dual conditions of preservation of compatibility.

4 From positionality to relationality

The previous conditions enables us to extend the two first properties of innocence to non-alternating strategies. Now we would like to recover relationality.

The **halting positions** are the positions where the strategy has noting to play (it is either waiting for an Opponent move or cannot play anything at all anymore). Relationality means that innocent strategies should be characterized by those complete positions (we should be able to recover an innocent strategy from its set of halting positions). Another way to see this is that innocent strategies are **closure operators** which to a position x associates the least halting position containing x. When two strategies are interacting, they start from the initial position $*$ and alternatively complete the position until they reach an agreement. This is reminiscent of previous work [Paul-André Melliès, Samson Abramsky].

In order for this to be possible, our strategies should have some **étiquette**. That is they should behave well according to basic principles of asynchrony in order for the composition to be possible.

- Courtesy: a strategy should be willing to delay what is has to play in order to let the other strategy play during an interaction. $\approx$ let the other people go through the door before yourself.

- Receptivity: a strategy is always listening to its environment.

- As usual in games semantics, our strategies should also be deterministic. In our asynchronous setting, this is expressed by the fact that Player move should be compatible with other moves: a Player move cannot change the current slice (in LL terms).

We can define a notion of **composition** on our strategies as usual by *parallel composition and hiding*. Relationality also says that this "dynamic" composition should correspond to the "static" composition (the relational composition on halting positions). It can be observed that it does not immediately works. The composition can go wrong for two reasons: there can either be a *livelock* (the strategies start an infinite chattering) or a *deadlock* ("please go first — no *you* please go first — ..."). This is solved by adding **payoffs** on paths [Martin Hyland, Luke Ong, 92, *fairness*]. You either win or loose at a position. By payoff on infinite positions we can avoid livelocks and by payoff on finite positions, we can force a player to play something and thus avoid deadlocks. With those conditions, the operation which to an innocent strategy associates its set of halting positions becomes a *strong monoidal functor* between our game model of non-alternating games and strategies and the usual relational model of LL.

5 Further work

One reassuring fact is that we can **recover** the usual alternating game model of innocent strategies as a subcategory of our model by restricting to alternating games and strategies which satisfy some properties (a Player move is justified by at most one Opponent move).

We can construct a game model in which every connective is interpreted as a move. A categorical structure arises naturally: it is a lax and unbiased category. This means that you have n-ary tensors. We can therefore represent interaction by **clusters** of moves. This gives therefore a framework to study *focalization*.

Generalize this work to **concurrent languages** (π-calculus, concurrent algol/ML, etc.).