Asynchronous innocence

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Game theoretical semantics

• We study logic from a dynamic point of view: interaction
• We try to recover the syntax from the semantics
• We want our model to be
  • general
  • concurrent
  • natural / elegant
• Disclaimer: this is a work in progress
Main features

- Games played on graphs (Mazurkiewicz traces)
- A diagrammatic characterization of innocence
- A *positional* characterization of innocent strategies
- A general model of interaction
Road map

1. Pointer games
2. Asynchronous games + sequential strategies
3. Asynchronous games + concurrent strategies
Usual (pointer) games

- A *game* is played on an arena

  \[ B : \quad q \xrightarrow{T} F \quad B : \quad q_1 \xrightarrow{T_1} F_1 \]

- A *strategy* is a set of plays

  \[ B \Rightarrow B : \quad q_2 \xrightarrow{T_2} F_2 \quad B \Rightarrow B : \quad q_1 \xrightarrow{T_1} F_1 \]
- Playing with the rear-view mirror.

\[(B \Rightarrow B) \Rightarrow B\]
Towards a more general framework
Games on event structures

- Games are now played event structures: $(M, \preceq)$

**Figure:** $\mathcal{B} \otimes \mathcal{B}$: trees vs dags

- + a polarization function: $\lambda : M \rightarrow \{-1, +1\}$
Towards a more general framework

Positions

- Position: finite compatible downward closed subset of $M$
- Positional graph $G(M)$:
  - positions: $x, y, \ldots$ of $M$
  - arrows: $x \xrightarrow{m} x + \{m\}$

\[ \begin{array}{c}
M_3 \\
\downarrow \\
M_1 & \quad & \quad & M_2 \\
\downarrow & \quad & \quad & \downarrow \\
(M, \preceq) & \quad & \quad & G(M)
\end{array} \]
Diagrammatic innocence: backward consistency

\[ \sigma \ni \]

\[ \Rightarrow \]

\[ \in \sigma \]
Diagrammatic innocence: forward consistency

\[ \sigma \ni m_2 \nearrow \searrow m_1 \searrow n_2 \nearrow n_1 \Rightarrow n_2 \nearrow \searrow n_1 \searrow m_2 \nearrow m_1 \ni \sigma \ni s \uparrow \uparrow \uparrow \uparrow \Rightarrow n_2 \nearrow \searrow n_1 \searrow m_2 \nearrow m_1 \ni \sigma \ni s \uparrow \uparrow \uparrow \uparrow \]
A strategy is positional when for every two plays $s_1, s_2 : \ast \rightarrow x$

$$s_1 \in \sigma \text{ and } s_2 \in \sigma \text{ and } s_1 \cdot t \in \sigma \implies s_2 \cdot t \in \sigma$$
Innocent strategies are positional

- A strategy $\sigma$ is characterized by $\sigma^\circ$
- Composition can be seen as a relational composition on positions:
  $$(\sigma; \tau)^\circ = \sigma^\circ; \tau^\circ$$
  ($\rightarrow$ monoidal functor to $\text{Rel}$)
Our work

- We now play on asynchronous graphs (instead of event structures).
- No more O/P alternation.
- We seek connections with existing models.
Asynchronous graphs

Let $M$ be a set of moves.

**Definition**
An *asynchronous graph* is a graph whose edges are labeled by moves and which satisfies:

- **linearity**: at most one occurrence of a move in a path
- **determinism**:

$$w \xrightarrow{m} v \xrightarrow{m} w' \Rightarrow w = w'$$

(and the dual)
Towards true concurrency: homotopy
Towards true concurrency: homotopy

What is really meaningful is not the precise order of the moves but what moves were played in the path.

We will work with $[G]$ which is the free category generated by $G$ whose arrows are quotiented by $\sim_G$. 
A diagrammatical characterization of innocence

Definition
An asynchronous graph is called *innocent* when it is:

- *stable / costable* (cube property):

\[
\begin{align*}
\text{cube property:} \quad n & \rightarrow m \\
\text{stable / costable:} \quad n & \rightarrow o
\end{align*}
\]

\[
\begin{align*}
\text{cube property:} \quad o & \rightarrow m \\
\text{stable / costable:} \quad o & \rightarrow n
\end{align*}
\]
Residual techniques

- Arrows in $[\mathcal{G}]$ are epi and mono

$$h_1 \cdot f \cdot h_2 \sim_{\mathcal{G}} h_1 \cdot g \cdot h_2 \Rightarrow f \sim_{\mathcal{G}} g$$
The lattice of factorizations of a path

- A factorization of a path:

- The category of factorizations:

- Ordering factorizations: \((f_1, f_2) \preceq (g_1, g_2)\) iff \(f_1\) is a prefix of \(g_1\) modulo \(\sim_G\)

- Property: \(f \preceq g\) iff every move in \(f\) is also a move in \(g\)

- Property: the factorizations of a path is a distributive lattice
The lattice of factorizations of a path

- Intersection of two paths
The lattice of factorizations of a path

• Intersection of two paths

⇒ structure of *distributive lattice* of the prefixes of a paths (ordered by inclusion of the set of moves)
Strategies

Take one distinguished vertex ∗ in a costable asynchronous graph.

Definition

A strategy is a set of paths starting from ∗, closed under prefix.

Definition

An innocent strategy is a strategy which is:

- deterministic
- stable / costable
- closed under local union

\[ n \rightarrow n \sim G \rightarrow m \rightarrow n \rightarrow n \sim G \rightarrow m \rightarrow s \rightarrow s \in \sigma \]
Positionality

- An innocent strategy $\sigma$ is a subgraph of $\mathcal{G}$ which is innocent.
Polarizing games!

- Now, we polarize the moves: $\lambda : M \to \{-1, +1\}$
- A game: $A = (M_A, G_A, \lambda_A)$
- Some more conditions are now required to hold for innocent strategies

$\sim G$

$\sigma \ni O \triangleright \triangleright \triangleright \triangleright \triangleright \triangleright \triangleright \triangleright \triangleright \triangleright \sim G \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \nearrow \in \sigma$

$\approx$ every O-move points to the preceding move in a view
Complete positions

- A position is said to be *complete* when no more player move can be played.
- $\sigma^\circ$: complete positions
Positionality

• An innocent strategy $\sigma$ is characterized by its complete positions $\sigma^\circ$
• If we add a payoff condition on strategies, we then have
  \[
  (\sigma; \tau)^\circ = \sigma^\circ; \tau^\circ
  \]
  (≈ acyclicity criterion on nets?)
• And now, two innocent strategies compose
Recovering other game models

asynchronous innocent games

alternation on strategies

sequential games [P.-A. Melliès]

alternation on games

L-nets [P.-L. Curien, C. Faggian, F. Maurel]
What’s next

- Characterizing the usual game models in our framework: sequential games, L-nets, ...
- Full completeness
- Pinch holes in the homotopy to have models of concurrent languages: CCS, π-calculus, ...