SYMMETRIES

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This is a small survey on the use of symmetry in:

- algebra: Galois theory
- geometry: deck transformations

CLASSICAL GALOIS THEORY

Starting point

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Otherwise said, we have two automorphisms of $\mathbb{Q}(i, \sqrt{5})$ fixing $\mathbb{Q}$:

$$\begin{array}{rcl} i & \mapsto & -i \\ \sqrt{5} & \mapsto & \sqrt{5} \end{array} \qquad\qquad \begin{array}{rcl} i & \mapsto & i \\ \sqrt{5} & \mapsto & -\sqrt{5} \end{array}$$

Intermediate fields

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Given a field extension L/K, we write

$$L /\!\!/ K$$

for the poset of **intermediate extensions** $K \subseteq M \subseteq L$, the order being $\subseteq$.

Classical Galois theory

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We also have a poset

$$\mathrm{Aut}(L/\!\!/K)$$

of subgroups of $\mathrm{Aut}(L/K)$ ordered by $\subseteq$.

The Galois correspondence

Theorem

Given a field extension L/K, there is an adjunction

$$\mathrm{Aut}(L/\!\!/K) \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (L/\!\!/K)^{\mathrm{op}}$$

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- G takes $K \subseteq M \subseteq L$ to the subgroup of $\mathrm{Aut}(L/K)$:

$$GM \quad = \quad \{f : L \to L \mid f \text{ fixes } M\}$$

- F takes a subgroup $A \subseteq \mathrm{Aut}(L/K)$ to

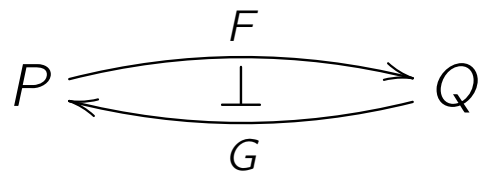
$$FA \quad = \quad \{x \in K \mid A \text{ fixes } x\}$$

We can check that they are functors such that

$$\frac{FA \supseteq B}{A \subseteq GB}$$

Galois correspondences

An adjunction between two posets



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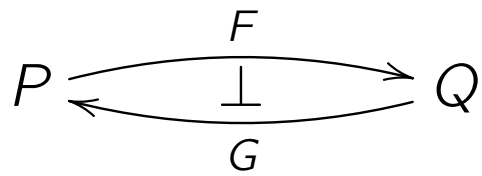

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In this case $T = G \circ F$ is a **closure operator**:

- *extensive*: $x \leq T(x)$
- *increasing*: $x \leq y$ implies $T(x) \leq T(y)$
- *idempotent*: $T(T(x)) = T(x)$

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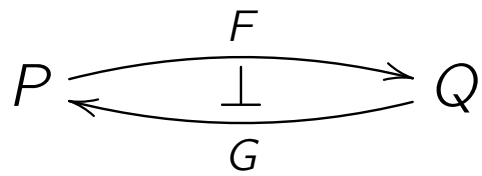

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We write $P^* \subseteq P$ for the set of its **fixpoints**:

$$P^* \quad = \quad \{x \in P \mid G \circ F(x) = x\}$$

and similarly for $Q^* \subseteq Q$.

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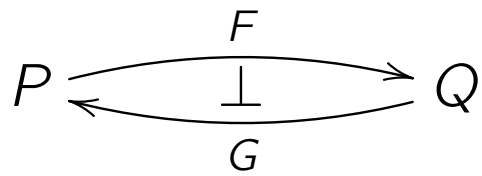

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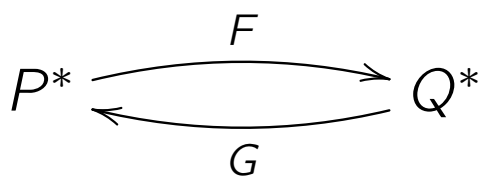
Proposition

We have an induced bijection



Generality

Note that up to now, the fact that we consider fields was used nowhere: it would equally work with groups, rings, or whatever you like...

The fundamental theorem

In the case of

$$\mathrm{Aut}(L/\!\!/K) \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (L/\!\!/K)^{\mathrm{op}}$$

the correspondence is a bijection if (and only if) L/K is an extension which is *finite*, *separable* and *normal*.

In the general case, F is injective and G is surjective.

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Example

$$\begin{aligned}[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] &= [\mathbb{Q}(\sqrt{2})(\sqrt{3}) : \mathbb{Q}(\sqrt{2})] \times [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] \\ &= 2 \times 2 = 4\end{aligned}$$

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- **Galois** when algebraic, normal and separable.

Why we need normal

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We have

$$G \circ F(\mathbb{Q}(\sqrt[3]{2})) \quad = \quad G \circ F(\mathbb{Q})$$

Let's go through this slowly...

Generated extensions

Given an extension L/K and $A \subseteq L$ a subset, we write $K(A)$ for the **extension generated** by A, which can be described as

- the intersection of all extensions of K containing A,
- the subfield of L whose elements are of the form

$$(P/Q)(a_1, \ldots, a_n)$$

where P/Q is a rational fraction in $K(X_1, \ldots, X_n)$.

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Proposition

A simple extension $K(a)$ is either of the form

- $K(X)$ if a is transcendental over K,
- $K[X]/I$ if a is algebraic over K where I is the ideal

$$I \quad = \quad \{P \in K[X] \mid P(a) = 0\}$$

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Remark

I is the kernel of the evaluation at a (therefore an ideal) whose target is the field K, therefore it is a maximal ideal and $K[X]/I$ is a field.

Simple extensions

In fact, extensions are usually simple:

Theorem (Primitive element theorem)

If L/K is a separable extension of finite degree then it is simple.

Simple extensions

The ring $K[X]$ is always a PID, therefore

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$\underline{P}$ divides any polynomial Q such that $Q(a) = 0$.

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Remark

$\underline{P}$ is the only irreducible polynomial with a as root
(up to a non-zero multiplicative constant).

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Remark

This means that if we add a root of P to K,
we necessarily add all of them.

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Proposition

Given $K(a)/K$ a simple extension one has

- $[K(a) : K] = \infty$ if a is transcendental,
- $[K(a) : K] = \deg(P)$ where P is the minimal polynomial of a otherwise.

In fact, a basis of $K(a)/K$ consists of the a^i with $0 \leq i < \deg(P)$.

Finite extensions

Lemma

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Remark

Every finite extension is algebraic, but an algebraic extension can be infinite. For instance, the set $\mathbb{A}$ of **algebraic numbers**, i.e. those which are algebraic over $\mathbb{Q}$:

$$\mathbb{A} \quad = \quad \{x \in \mathbb{C} \mid P(x) = 0 \text{ for some } P \in \mathbb{Q}[X]\}$$

Ruler and compass

Consider the points you can construct in $\mathbb{R}^2$, starting from two points, by

1. drawing a straight line through two points,
2. drawing circles centered at a point and going through another point,

and taking points at intersections of those.

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Proposition

The elements x_{i+1} and y_{i+1} are zeros of polynomials of degree one or two in K_i. Therefore, $[K_i : K]$ is a power of two.

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- We begin with $p_0 = (0, 0)$ and $p_1 = (1, 0)$.
- We should be able to construct $(\sqrt{\pi}, 0)$ and therefore $(\pi, 0)$.
- This would imply that $[\mathbb{Q}(\pi) : \mathbb{Q}]$ is a power of two, but we know that π is not algebraic over $\mathbb{Q}$.
- Contradiction.

For the rest, we should rely on subtler invariants than size...

We should take *symmetries* in account!

General idea

Suppose that we want to make a simple algebraic extension $K(a)/K$, where P is the minimal polynomial of a.

We may not be able to distinguish between a and another root of P: more precisely, there will be an action of the Galois group.

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Lemma

For any $P \in K[X]$, the splitting field exists and is unique up to isomorphism.

Proof.

Iteratively formally add roots of irreducible factors Q of P to K, i.e. take $K[x]/(Q)$, until P splits. □

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In characteristic 0, every irreducible polynomial is separable.

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When K is separable, a finite extension L/K is normal if and only if L is the splitting field for some polynomial over K.

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Proposition (???)

A finite extension L/K is separated iff the action of $\mathrm{Aut}(L/K)$ on the conjuguates of an element of K is faithful.

The fundamental theorem

Theorem (Galois)

When L/K is a finite, separable and normal extension, we have an isomorphism

$$\mathrm{Aut}(L/\!\!/K) \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (L/\!\!/K)^{\mathrm{op}}$$

Back to Galois

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The group $\mathrm{Aut}(L/K)$ can thus be seen as a group of permutation of roots of P (this was Galois' original definition).

An example

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Therefore

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Its degree is

$$[K : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2})][\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 2 \times 4 = 8$$

(minimal polynomials are respectively, $X^2 - 1$ and $X^4 - 2$).

An example

The elements $f \in \mathrm{Aut}(K/\mathbb{Q})$ have to satisfy

$$f(i) \in \{i, -i\} \qquad\qquad f(\sqrt[4]{2}) \in \left\{\sqrt[4]{2}, i\sqrt[4]{2}, -\sqrt[4]{2}, -i\sqrt[4]{2}\right\}$$

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We can even work out a presentation with generators r, s

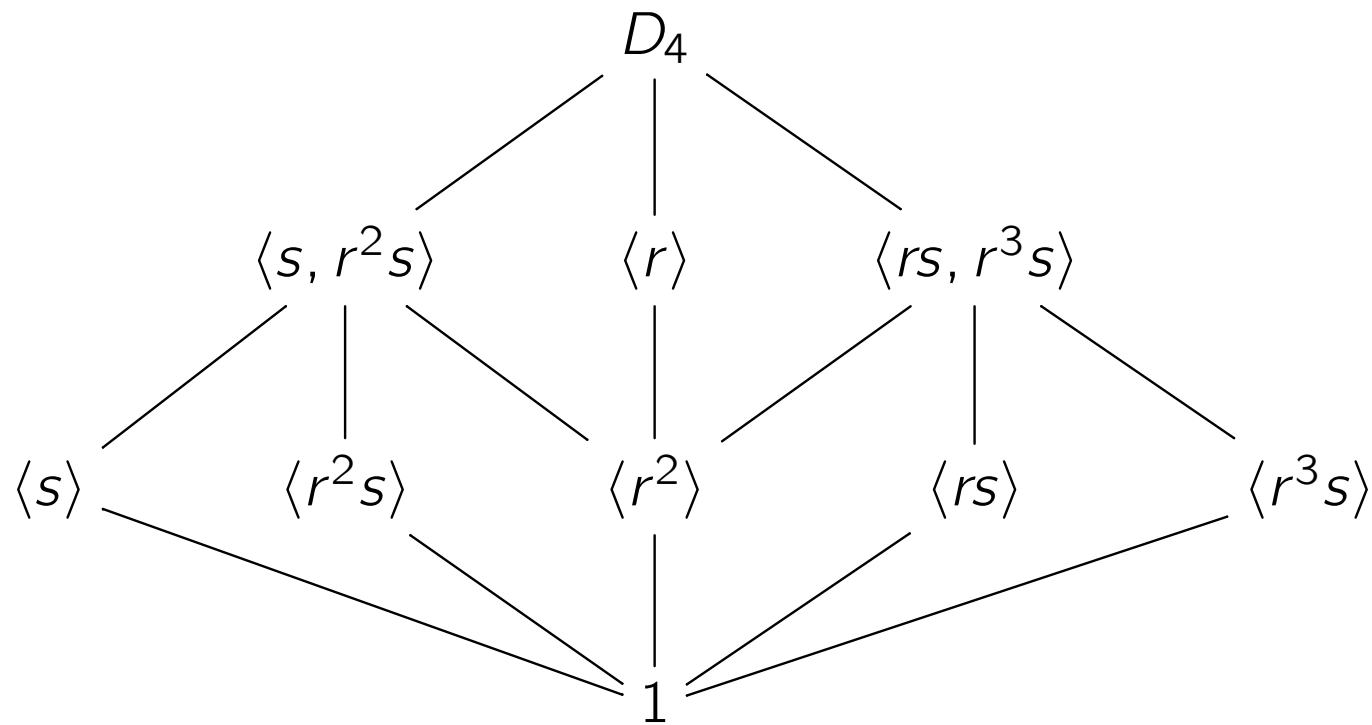
$$r(i) = i \qquad r(\sqrt[4]{2}) = i\sqrt[4]{2} \qquad s(i) = -i \qquad s(\sqrt[4]{2}) = \sqrt[4]{2}$$

namely

$$\mathrm{Aut}(K/\mathbb{Q}) \quad = \quad \langle r, s \mid r^4 = 1, s^2 = 1, srsr = 1\rangle \quad = \quad D_4$$

An example

The subgroups of $\mathrm{Aut}(K/Q) = D_4$ are



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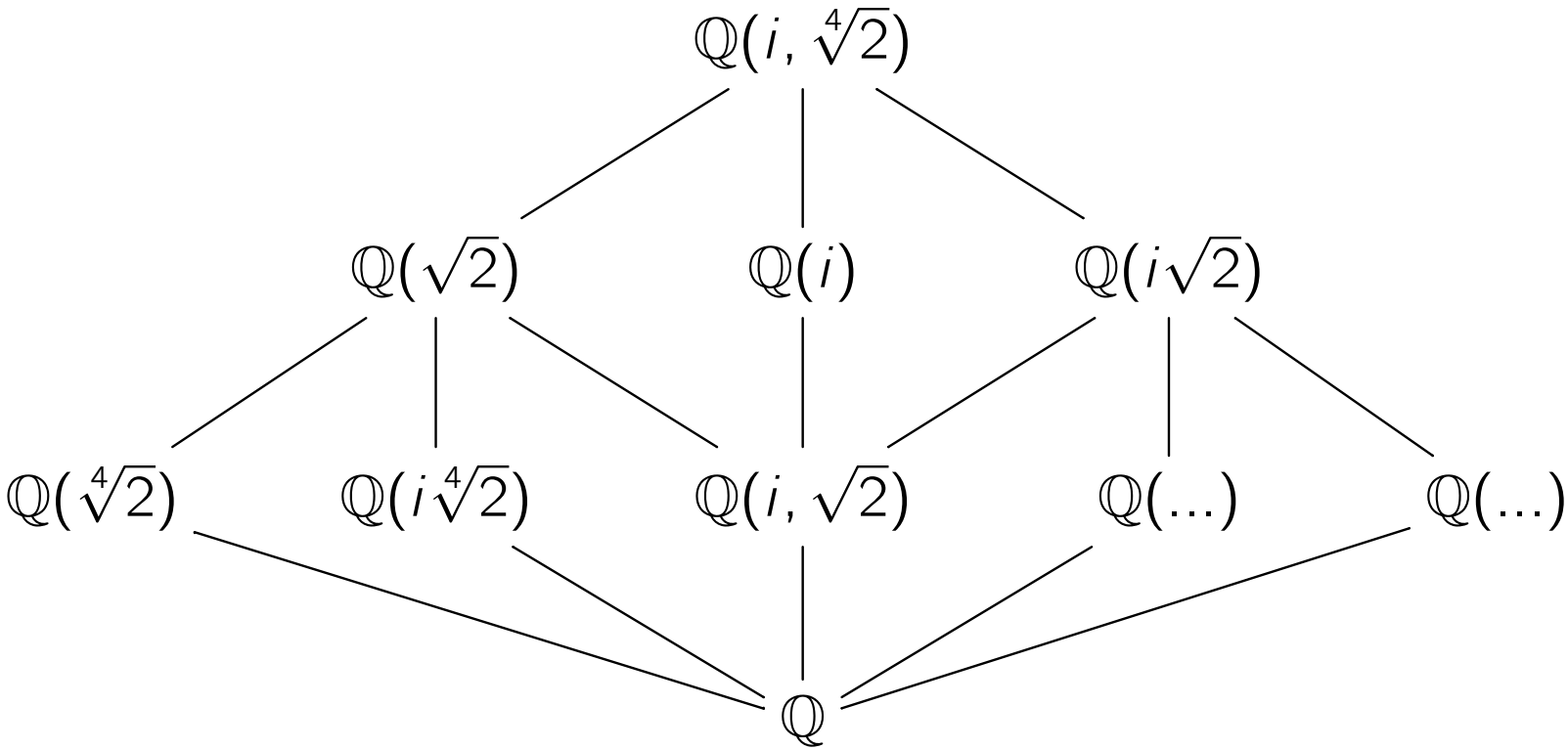
The intermediate extensions of K/Q are



Radical extensions

An extension L/K is **radical** when $L = K(a_1, \ldots, a_k)$ with

$$a_i^{n_i} \quad \in \quad K(a_1, \ldots, a_{i-1})$$

for some $n_i \in \mathbb{N}$, i.e. L can be obtained from K by adjoining a sequence of n_i-th roots.

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A polynomial $P \in K[X]$ is **solvable by radicals** when its splitting field is an intermediate field of a radical extension L/K.

Theorem

An separable extension L/K is radical if and only if $\mathrm{Aut}(L/K)$ *is solvable.*

Solvable groups

A group G is **solvable** if there exists subgroups

$$1 = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n = G$$

such that

- G_i is a normal subgroup of G_{i+1},
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The relation "being a normal subgroup" is not transitive.

Proposition

The symmetric groups S_n are solvable only for $n < 5$.

Idea of the proof

Lemma

Write L for the splitting field of $P = X^p - 1$ with p prime. The Galois group $\mathrm{Aut}(L/K)$ is abelian.

Proof.

- Since $P' = pX^{p-1}$, P and P' have not zeros in common, therefore P has only simple zeros.
- Thus the group of roots of P is cyclic of order p.
- Writing a for a root, $L = K(a)$ and $f \in \mathrm{Aut}(L/K)$ is determined by $f(a)$ which should be in $\{a^i\}$.
- Writing f_i for the automorphism such that $f_i(a) = a^i$, we have $f_j \circ f_i(a) = f_i \circ f_j(a) = a^{ij}$ and therefore any two elements of $\mathrm{Aut}(L/K)$ commute. □

Idea of the proof

Lemma

Given a field K in which $X^n - 1$ splits, $a \in K$, and L the splitting field for $X^n - a$ over K, then $\mathrm{Aut}(L/K)$ is abelian.

Proof.

- Given b such that $b^n = a$, the roots of $X^n - a$ are of the form $u^i b$ with u a root of $X^n - 1$.
- Automorphisms are of the form f_i with $f_i(b) = u^i b$ and

$$f_j \circ f_i(b) = u^{i+j} b$$

they thus commute. □

Idea of the proof

Theorem

If a separable extension L/K is radical then $\mathrm{Aut}(L/K)$ *is solvable.*

Proof.

Given a radical extension L/K:

- we can suppose that we only take roots of prime powers,
- we take the normal closure of L,
- the splitting field of $X^n - a$ splits $X^n - 1$,
- we apply previous lemmas. □

An insoluble polynomial

Consider $P = X^5 - 6X + 3$ over $\mathbb{Q}$.

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Its Galois group is S_5, which is not solvable.

Therefore, P is not solvable by radicals.

Primitives

Note that this result is really due to the fact that our primitive for computing are **radicals**, i.e. roots of

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For instance, an **ultraradical** is the real solution of

$$X^5 + X - a$$

Every quintic is solvable with radicals and ultraradicals.

Can we use symmetry to show that some tasks cannot be implemented?

The Galois task

A **task** is a polynomial P for which we want to find a root.

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A **process** is a program consisting of a loop which iteratively

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Can every task be solved by a process?

Where can we find symmetries?

Sources of symmetry:

- high-level programming languages can manipulate memory locations, but the implementation guarantees that the behavior will not depend on the chosen locations:

  invariance under action of the symmetric group on memory!

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- there can be a symmetry between the various inputs of a programs.

COVERING SPACES

Covering maps

A continuous map $p : E \to B$ between topological spaces is **covering** when every point $x \in B$ has an neighborhood U such that

$$p^{-1}(U) \quad \cong \quad \coprod_{i \in I} U$$

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for some set I.

The set $p^{-1}(x)$ is called the **fiber** over x.

Pointed covers

A *pointed space* (X, x) is a space X together with $x \in X$.

A *pointed morphism* $f : (X, x) \to (Y, y)$ is a morphism $f : X \to Y$ such that $f(x) = y$.

We write $\mathbf{Top}_\bullet$ for the resulting category.

A *pointed covering* is a pointed morphism which is also covering.

The universal cover

Given a pointed space (B, b), consider the full subcategory of

$$\mathbf{Top}_\bullet/(B, b)$$

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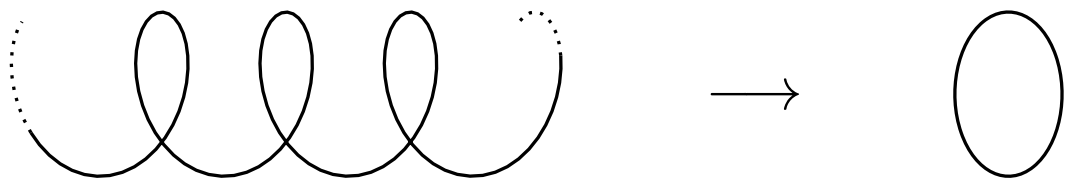
A **universal cover** is an initial object in this category.

Remark

It is not hard to show that morphisms of the above category are covering.

The universal cover

When B is "reasonable" (connected, locally path-connected and semilocally simply connected) the universal cover exists and can be described as the space whose points are homotopy classes of paths in B originating in b.



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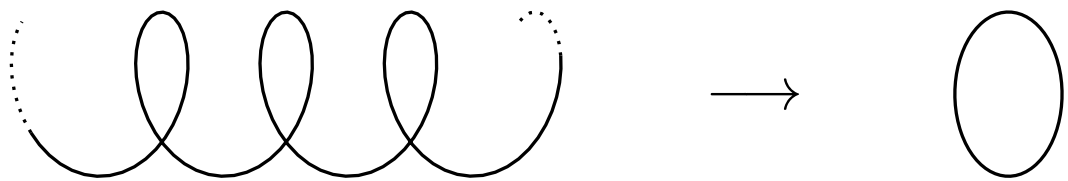

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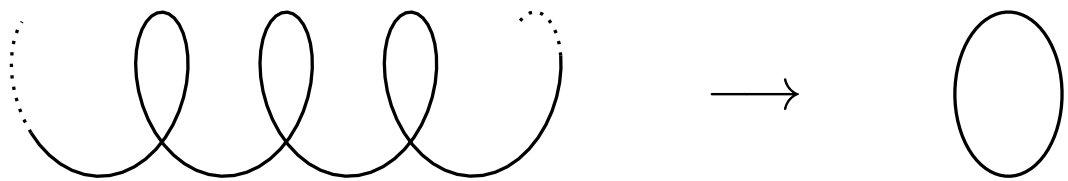

This construction does not depend on the choice of b in its connected component.

It can be characterized as the *simply connected* pointed cover of (B, b).

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We will be in this case in the following and forget about the base point (otherwise consider connected components).

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We will also suppose that covering spaces we consider are connected.

Deck transformations

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We write $\mathrm{Aut}(p)$ for the **deck group**.

The fundamental theorem

Given an universal cover $p : (\tilde{X}, \tilde{x}) \to (X, x)$, there is an isomorphism

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There is a bijective correspondence between

- subgroups of $\pi_1(X, x)$,
- coverings of (X, x).

$$\mathrm{Aut}(p) \underset{G}{\overset{F}{\rightleftarrows}} (\mathbf{Cov}(X))^{\mathrm{op}}$$

Let's detail this...

The fundamental groupoid

Given a space X, its **fundamental groupoid** $\Pi_1(X)$ is the category whose objects are points in X and morphisms

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X being connected, for every points x, y, there exists a morphism $f : x \twoheadrightarrow y$. It induces a group isomorphism

$$\begin{array}{ccc} \pi_1(X, x) & \to & \pi_1(X, y) \\ g & \mapsto & f \circ g \circ f^{-1} \end{array}$$

The fundamental groupoid

This construction is functorial

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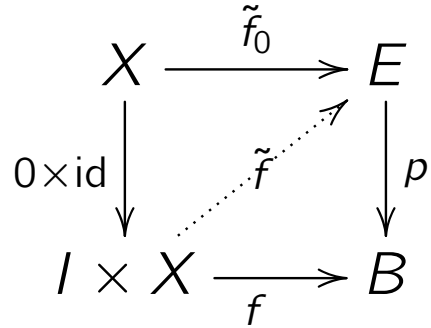
and thus a group morphism

$$f_* \quad : \quad \pi_1(X, x) \quad \to \quad \pi_1(Y, f(x))$$

The homotopy lifting property

Proposition

Given a covering $p : E \to B$, a homotopy $f : X \times I \to B$ and a lifting $\tilde{f}_0 : X \to E$, there exists a homotopy $\tilde{f} : X \times I \to E$ such that



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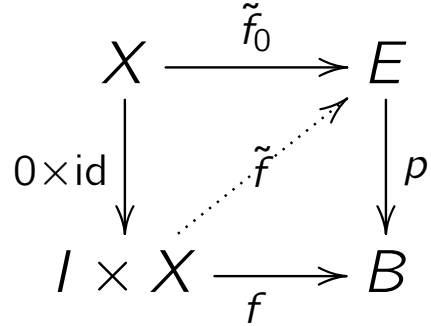

For instance,

- with $X = \{*\}$, we get the **path lifting property**.
- with $X = I$, we can lift homotopies between paths,
- etc.

Faithfulness

The induced functor is always faithful:

$$p_* \quad : \quad \Pi_1(E) \quad \to \quad \Pi_1(B)$$

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We can thus see $\pi_1(E, e)$ as a subgroup of $\pi_1(B, b)$.

The action of $\pi_1(X)$

By the path lifting property, every path

$$f \quad : \quad x \quad \twoheadrightarrow \quad y$$

induces a function

$$p^{-1}(x) \quad \to \quad p^{-1}(y)$$

sending $\tilde{x} \in p^{-1}(x)$ to the endpoint $\tilde{y}$ of the path

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By the homotopy lifting property, two homotopic paths give rise to the same function.

The action of $\pi_1(X)$

We thus get a functor

$$p^* \quad : \quad \Pi_1(X) \quad \to \quad \mathbf{Set}$$

such that

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and for $f : x \twoheadrightarrow y$ the function

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Since X is connected, we obtain for instance that any two fibers are isomorphic: their cardinal is called the **degree** of the cover.

Lifting morphisms

An automorphism $g \in \mathrm{Aut}(p)$ gives rise to an isomorphism on the set $p^{-1}(x)$, i.e. we have a group morphism

$$\mathrm{Aut}(p) \quad \to \quad \mathrm{Iso}(p^{-1}(x))$$

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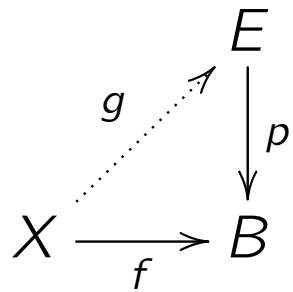
In fact, we will see that such an automorphism is determined by the image of one element of $p^{-1}(x)$.

Lifting morphisms

Theorem

Given a covering $p : E \to B$, a continuous $f : X \to B$, $x \in X$ and $e \in p^{-1}(f(x))$,



there exists g such that $p \circ g = f$ and $g(x) = e$ if and only if

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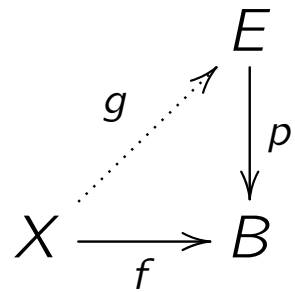

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Proof.

Given a path $x \twoheadrightarrow y$ in X, its image by f has a lifting $\tilde{x} \twoheadrightarrow \tilde{y}$ under p and we set $g(x) = \tilde{y}$. The condition ensure that this does not depend on the path. □

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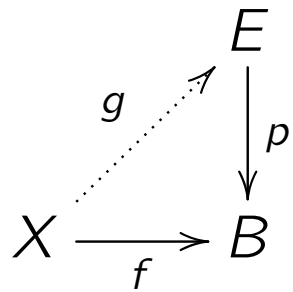

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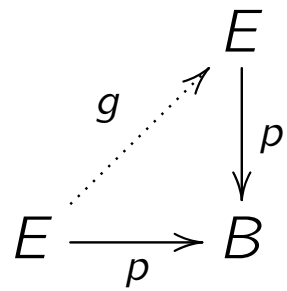
Remark

When X is simply connected, the condition is always satisfied!

Lifting morphisms

We can apply the theorem to



and deduce that, given $x \in E$, a p-automorphism g is uniquely determined by the image $g(x)$, and any $y \in E$ such that

$$p_*(\pi_1(E, x)) \quad = \quad p_*(\pi_1(E, y))$$

is possible as value for $g(x)$.

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We thus have

$$\mathrm{Aut}(p) \quad \cong \quad \pi_1(X, x)$$

Theorem

There is a bijective correspondence between

- *subgroups of* $\pi_1(X, x)$,
- *coverings of* (X, x).

$$\mathrm{Aut}(p) \quad \cong \quad (\mathbf{Cov}(X))^{\mathrm{op}}$$

Proof.

To a subgroup of $G \subseteq \mathrm{Aut}(p)$, we associate the covering

$$p/G \quad : \quad \tilde{X}/G \to X$$

where $p : \tilde{X} \to X$ is the universal covering.

To a covering $q : Y \to X$, we associate $q_*(\pi_1(Y, y))$ for some $y \in p^{-1}(x)$. □

Degree and index

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Proposition

The degree of a covering is the index of the corresponding subgroup in $\pi_1(X, x)$.

Normal covering

A covering $p : (\tilde{X}, \tilde{x}) \to (X, x)$ is **normal** if its action on the fiber $p^{-1}(x)$ is **transitive**:

$$\forall y, z \in p^{-1}(x), \exists f \in \mathrm{Aut}(p), \quad y \cdot f = z$$

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The intuition of a normal covering: a given loop gets unfolded a given number of time, uniformly.

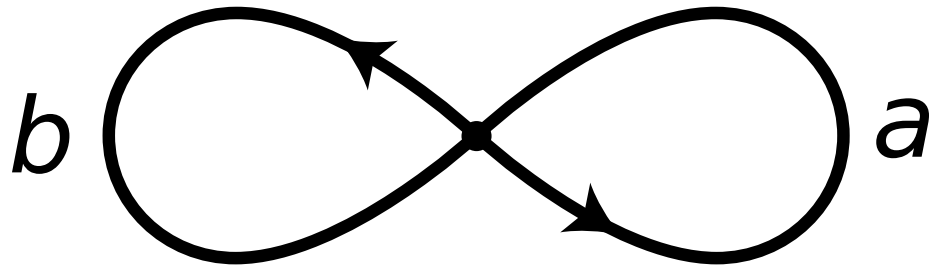
Separated covering

A covering is **separated** when the action is free...

...which is always the case (as we have seen).

An example

Consider the space $X = S_1 \vee S_1$:

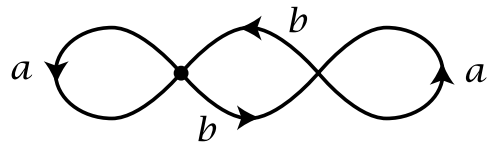


Its fundamental group is

$$\pi_1(X) \quad = \quad \langle a, b \mid \rangle$$

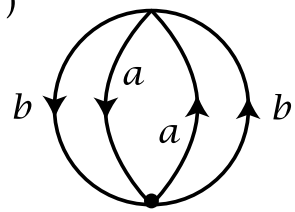
Some fundamental groups

(1)



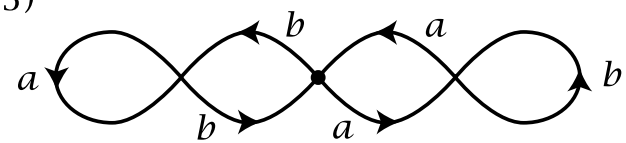
$\langle a, b^2, bab^{-1} \rangle$

(2)



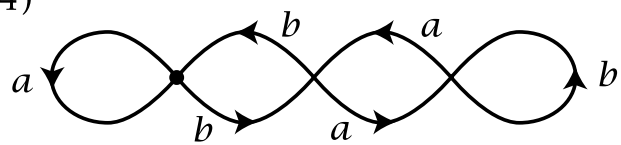
$\langle a^2, b^2, ab \rangle$

(3)



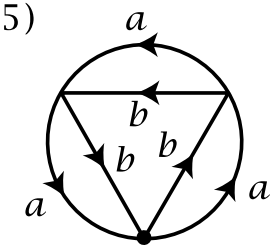
$\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$

(4)



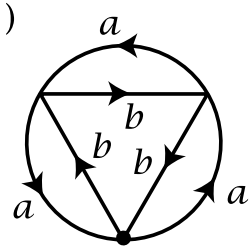
$\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$

(5)



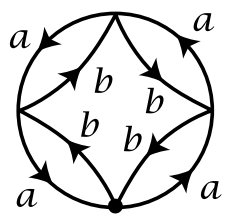
$\langle a^3, b^3, ab^{-1}, b^{-1}a \rangle$

(6)



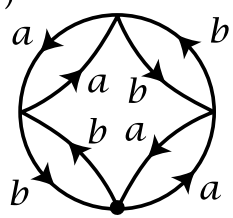
$\langle a^3, b^3, ab, ba \rangle$

(7)



$\langle a^4, b^4, ab, ba, a^2b^2 \rangle$

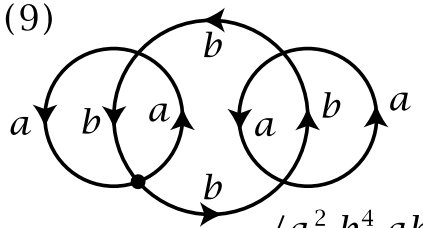
(8)



$\langle a^2, b^2, (ab)^2, (ba)^2, ab^2a \rangle$

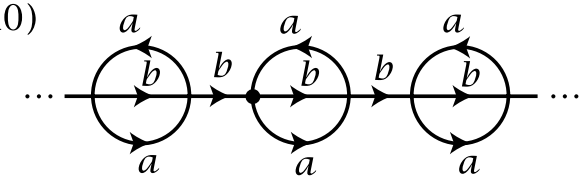
Some fundamental groups

(9)



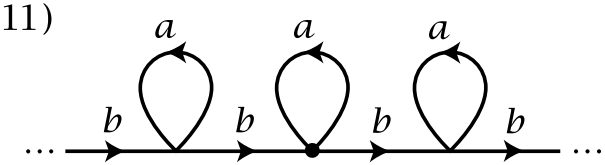
$\langle a^2, b^4, ab, ba^2b^{-1}, bab^{-2} \rangle$

(10)



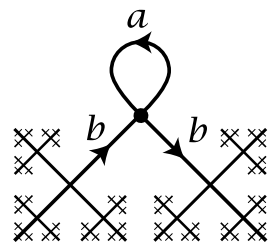
$\langle\, b^{2n}ab^{-2n-1}, b^{2n+1}ab^{-2n} \mid n \in \mathbb{Z} \,\rangle$

(11)



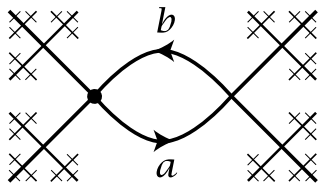
$\langle\, b^n ab^{-n} \mid n \in \mathbb{Z} \,\rangle$

(12)



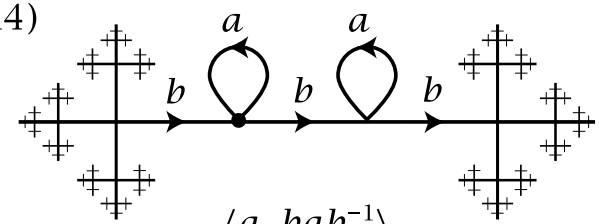
$\langle a \rangle$

(13)



$\langle ab \rangle$

(14)



$\langle a, bab^{-1} \rangle$

Let's try to drop
connectedness assumptions
(on E and on B)

Subgroups vs transitive actions

We have seen that

$$\begin{array}{c}\text{connected covering spaces}\\ \text{of } B\end{array} \quad \cong \quad \begin{array}{c}\text{subgroups}\\ \text{of } \pi_1(B)\end{array}$$

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This can be reformulated as

$$\begin{array}{c}\text{connected covering spaces}\\ \text{of } B \text{ with fiber } F\end{array} \quad \cong \quad \begin{array}{c}\text{transitive actions}\\ \text{of } \pi_1(B) \text{ on } F\end{array}$$

Subgroups vs transitive actions

The two points of view are the same on connected coverings:

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- given an action $\pi_1(B) \times F \to F$, we define

$$H \quad = \quad \mathrm{Stab}(x) \quad = \quad \{y \in \pi_1(B) \mid y \cdot x = x\}$$

  for some $x \in F$.

Non-connected covering spaces

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If we consider non-connected covers (but B still is), we get

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If we consider non-connected E and B, we get

$$\begin{array}{c}\text{covering spaces}\\ \text{of } B \text{ with fiber } F\end{array} \quad \cong \quad \begin{array}{c}\text{functors}\\ \Pi_1(B) \to \mathbf{Set}\end{array}$$

More categorically

In fact, this has mostly nothing to do with topology: everything can be done at the level of the fundamental groupoid $\pi_1(X)$.

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A functor $F : \mathcal{E} \to \mathcal{B}$ between groupoids is **covering**, or a **discrete opfibration**, when for every $f : x \to y$ in $\mathcal{B}$ and $\tilde{x} \in \mathcal{E}$ with $p(\tilde{x}) = x$, there exists a unique $\tilde{f} : \tilde{x} \to \tilde{y}$ such that $F(\tilde{f}) = f$.

$$\begin{array}{ccc} \tilde{x} & \overset{\tilde{f}}{\dashrightarrow} & \tilde{y} \\ & & \\ x & \xrightarrow{f} & y \end{array}$$

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$$\tilde{x} \overset{\tilde{f}}{\dashrightarrow} \tilde{y}$$

$$x \xrightarrow{f} y$$

Typical example: when $p : E \to B$ is a covering map,

$$p_* : \pi_1(E) \to \pi_1(B)$$

is a covering functor.

Covering functors

All previous theorem can be shown in this setting.

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Theorem

There is an equivalence between the categories of

- *discrete opfibrations over* $\mathcal{B}$
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This means that a covering functor

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Bonus: this works even when $\mathcal{B}$ is a category (not a groupoid)!

RELATING THOSE

Duality between geometry and algebra

Fix a field K.

To any space X, one can associate the commutative algebra

$$\mathcal{O}(X) \quad = \quad X \Rightarrow K$$

For instance:

$$(f+g)(x) \quad = \quad f(x)+g(x)$$

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