# Notes about Filters

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## 1 Filters and ultrafilters

**Definition 1.** *A* ***filter*** *$F$ on a poset $(L, \leq)$ is a subset of $L$ which is upward-closed and downward-directed (= is a filter-base):*

1. *for every $A \in F$ and $B \in L$ such that $A \leq B$, we have $B \in F$*
2. *$F$ is not empty*
3. *for every $A, B \in F$ there exists $C \in F$ such that $C \leq A$ and $C \leq B$*

*A filter $F$ is* ***proper*** *when there exists $A \in L$ such that $A \notin F$.*

*Remark* 2. When $L$ is a lattice, typically when $L = \mathcal{P}(U)$, $F$ is a filter iff

1. for every $A \in F$ and $B \in L$, $A \vee B \in F$
2. $\top \in F$
3. for every $A, B \in F$, $A \wedge B \in F$

**Definition 3.** *A filter is* ***prime*** *if its complement is an ideal. An* ***ultrafilter*** *is a filter is a maximal proper filter.*

*Remark* 4. In $L = \mathcal{P}(U)$, a filter $F$ is an ultrafilter iff for every $A \subseteq U$ either $A \in F$ or $U \setminus A \in F$. Alternatively, a set ultrafilter is $F \subseteq L$ such that

$$\forall A \in L, \qquad A \in F \quad \Leftrightarrow \quad \forall B_1, \ldots, B_n \in F, A \cap B_1 \cap \ldots \cap B_n \neq \emptyset$$

Equivalently, for every partition $X = X_1 \uplus \ldots \uplus X_n$ of $X$, there exists exactly one $i$ such that $X_i \in F$.[1]

*Remark* 5. In a distributive lattice, every ultrafilter is prime. In a boolean algebra, every prime filter is an ultrafilter.

In the following, we will be mostly interested in set ultrafilters.

**Definition 6.** *Given a set $X$ and $x \in X$, the* ***principal*** *ultrafilter on $X$ generated by $x$ is the set $P_x = \{F \subseteq X \mid x \in F\}$.*

[1] For other characterizations along this line, see this n-café post.

**Lemma 7.** *An ultrafilter containing a finite set is principal.*

**Proposition 8** (Ultrafilter lemma)**.** *Suppose that $A = (A_i)_{i \in I}$ is a collection of subsets of $X$. We say that $A$ has the* finite intersection property *if any finite subcollection $J \subseteq I$ has a non-empty intersection $\bigcap_{j \in J} A_j$. In this case, there exists an ultrafilter on $X$ such that $A \subseteq X$.*

## 2 Ultrafilters and topological spaces

**Definition 9.** *A **topological space** consists of a set $X$ together with a set $\tau$ of subsets of $X$, the* open sets *of $X$, such that*

1. *$\emptyset \in \tau$ and $X \in \tau$*
2. *$\tau$ is closed under finite intersections*
3. *$\tau$ is closed under arbitrary unions*

**Definition 10.** *A **neighborhood** $V \subseteq X$ of a point $x$ is a set such that exists an open $V$ containing $x$ for which $V \subseteq U$.*

*Remark* 11. Notice that the set of neighborhoods of a point $x \in X$ is always a filter, called the **neighborhood filter** of $x$.

**Definition 12.** *A filter $F$ **converges** to $x$ when for every neighborhood $U$ of $x$ belongs to $F$ (i.e. the neighborhood filter of $x$ is contained in $F$).*

**Definition 13.** *A **directed set** $X$ is a poset such that for every $x, y \in X$, there exists $z \in X$ such that $x \leq z$ and $y \leq z$.*

**Definition 14.** *A **net** in a set $X$ is a function $\nu : I \to X$ where $I$ is a directed set, that we often write $(x_i)_{i \in I}$. Given $Y \subseteq X$, a net $(x_i)$ is **eventually** in $Y$ if there exists $i \in I$ such that for every $j \geq i$, $x_j \in Y$. A point $x \in X$ is a **limit** of $(x_i)$ if for every neighborhood $U$ of $x$, $(x_i)$ is eventually in $U$.*

*Remark* 15. Every sequence of points $(x_i)_{i \in \mathbb{N}}$ is a net.

**Proposition 16.** *Every net $(x_i)_{i \in I}$ in $X$ defines an **eventuality filter** $F$ consisting of the subsets of $X$ such that $(x_i)$ is eventually in. Conversely, every filter $F$ defines a net such that $I$ is the set whose elements are $(U, x)$ with $U \in F$ and $x \in U$ (it is the disjoint union of $F$) such that $(U, x) \geq (V, y)$ whenever $U \subseteq V$ and $\nu : I \to X$ is the second projection.*

**Question 17.** *Can we make this a categorical equivalence? In particular, two nets have the same filter when they are both subsets of each other. Check the converse direction.*

**Proposition 18.** *TODO: convergence wrt nets is the same as convergence wrt filters.*

**Proposition 19.** *For every function $f : X \to Y$ between topological spaces, $f$ is* continuous *iff given any point $x \in X$ and net $(x_i)_{i \in I}$ converging towards $x$, the net $(f(x_i))_{i \in I}$ converges towards $f(x)$.*

*Remark* 20. The restriction of the above proposition to sequences of points is only valid in first-countable spaces. We recall that a topological space is **first-countable** when every point $x$ has a countable neighborhood basis, i.e. there exists a sequence $(U_i)_{i \in \mathbb{N}}$ of open neighborhoods of $x$ such that for every open neighborhood $V$ of $x$ there exists $i$ such that $U_i \subseteq V$. Notice that every metric space is first-countable.

**Lemma 21.** *An uncountable set (such as $\mathbb{R}$) together with the cofinite topology is not first-countable.*

*Proof.* Consider a point $x \in \mathbb{R}$. TODO... □

**Definition 22.** *A topological space is **compact** when*

*(i) each open cover contains a finite subcover*

*(ii) every convergent net has a convergent subnet*

*(iii) every filter on $X$ has a convergent refinement*

*(iv) every ultrafilter converges to at least one point*

**Question 23.** *Can we characterize the filters coming from sequences? Which generalization of non-standard analysis do we get?*

# 3 The ultrafilter monad

**Definition 24.** *The **ultrafilter monad** $\beta : \mathbf{Set} \to \mathbf{Set}$ sends every set $X$ to the set $\beta X$ of its ultrafilters. Given a morphism $f : X \to Y$, and $F \in \beta X$ we define $\beta f(F) = \{B \subseteq Y \mid \exists A \subseteq X, f(A) \subseteq B\}$ (i.e. the upward closure of the set of images, or equivalently $B \in \beta f(F)$ iff $f^{-1}(B) \in F$). The unit $\eta_X : X \to \beta X$ of the monad sends $x \in X$ to the principal ultrafilter $P_x$ generated by $x$. Given a set $A \subseteq X$, we write $[A]_X$ for the set of ultrafilters on $X$ which contain $A$. The multiplication $\mu_X : \beta\beta X \to \beta X$ sends $F$ (an ultrafilter on $\beta X$) to $\mu_X(F) = \{A \mid [A]_X \in F\}$. In other words,*

- *$A \in \eta_X(x)$ iff $x \in A$*
- *$A \in \mu_X U$ iff $\{F \in \beta X \mid A \in F\} \in U$*

**Proposition 25.** *This monad is generated from the adjunction*[2]

$$\mathbf{Set} \underset{Q}{\overset{P^{\mathrm{op}}}{\underset{\bot}{\rightleftarrows}}} \mathbf{Bool}^{\mathrm{op}}$$

[2] Some more details in here.

*where* **Bool** *is the category of boolean algebras, where* $P = \mathbf{Set}(-, 2) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Bool}$ *is the powerset functor (with the boolean algebra structure inherited pointwise from* $2 = \{0, 1\}$*) and* $Q = \mathbf{Bool}(-, 2) : \mathbf{Bool}^{\mathrm{op}} \to \mathbf{Set}$*.*

*Remark* 26. In the above proposition, one should remark that a boolean algebra morphism $\phi : B \to 2$ is uniquely determined by

- the maximal ideal $\phi^{-1}(0)$
- the ultrafilter $\phi^{-1}(1)$

so $Q(B)$ can be identified with the set of ultrafilters in $B$.

The monad structure on $\beta$ can be constructed abstractly, thanks to the following theorem by Börger [Bör87, HP91]:

**Theorem 27.** *The endofunctor* $\beta$ *is terminal among endofunctors* $\mathbf{Set} \to \mathbf{Set}$ *that preserve finite coproducts.*

**Corollary 28.** *There exists exactly one monad structure on* $\beta$ *and* $\beta$ *is terminal among monads* $\mathbf{Set} \to \mathbf{Set}$ *that preserve finite coproducts.*

**Definition 29.** *A topological space* $X$ *is* ***Hausdorff*** *(or* separated *or* $T_2$*) if it satisfies one of the following equivalent conditions*

*(i) for every points* $x, y \in X$ *there exists a neighborhood* $U$ *of* $x$ *and a neighborhood* $V$ *of* $y$ *such that* $U \cap V = \emptyset$

*(ii) every net* $\nu$ *admits at most one limit*

*(iii) every filter* $F$ *admits at most one limit*

*(iii) every ultrafilter* $F$ *admits at most one limit*

*Remark* 30. In a Hausdorff space $X$, given a set $U$ and a point $x \in X$, either $U$ is a neighborhood of $x$ or $X \setminus U$ is, so the neighborhood filter is an ultrafilter.

We thus get the following beautiful theorem by Manes [Man69]:

**Proposition 31.** *The Eilenberg-More category of* $\beta$ *is the category of compact Hausdorff spaces (a* $\beta$*-algebra is sometimes called a* ***compactum****).*

*Proof.* If $X$ is a compact Hausdorff space, then the corresponding algebra $\ell : \beta X \to X$ sends an ultrafilter to its limit. Conversely, given an algebra $\ell : \beta X \to X$, we define a topology on $X$ by $U \subseteq X$ is open when for every $x \in X$ and every $F \in \beta X$ such that $U \in F$, if $\ell(F) = x$ then $x \in U$ (i.e. $U$ is a neighborhood of every point it contains). □

*Remark* 32. The algebra part $\beta X \to X$ sends every filter to its limit!

*Remark* 33. Given a topological space $X$, $\beta X$ is the free compact Hausdorff space on $X$, called its **Stone-Čech compactification**.

To sum up: Hausdorff means every ultrafilter converges to at most one point and compact means every ultrafilter converges to at least one point (this also works if we say proper filter instead of ultrafilter). We can thus have more generality when using relations to relate a filter with its possible limits, and Proposition 31 was generalized by Barr [Bar70] to show that[3]

**Proposition 34.** *The category* **Top** *is isomorphic to the category of lax algebras of $\beta$ in* **Rel** *(which are sometimes called* relational $\beta$-modules*). Notice that in* **Rel** *the multiplication of $\beta$ is strict but the unit is oplax in the sense that*

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \beta X \\ {\scriptstyle r}\big\downarrow & \subseteq & \big\downarrow{\scriptstyle \beta r} \\ Y & \xrightarrow[\eta_Y]{} & \beta Y \end{array}$$

*By a lax algebra $a : \beta X \to X$, we mean here*

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \beta X \\ & {\scriptstyle \mathrm{id}}\searrow \;\subseteq & \big\downarrow{\scriptstyle a} \\ & & X \end{array} \qquad \begin{array}{ccc} \beta\beta X & \xrightarrow{\beta a} & \beta X \\ {\scriptstyle \mu_X}\big\downarrow & \supseteq & \big\downarrow{\scriptstyle a} \\ \beta X & \xrightarrow[a]{} & X \end{array}$$

*and a morphism $f : (X, a) \to (Y, b)$ between such algebras is a map $f : X \to Y$ such that*

$$\begin{array}{ccc} \beta X & \xrightarrow{\beta f} & \beta Y \\ {\scriptstyle a}\big\downarrow & \subseteq & \big\downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

# 4 The filter monad

Many properties of the filter monad are studied in [EF99].

**Definition 35.** *The filter monad $T$ :* **Set** $\to$ **Set** *is defined as in Definition 24 replacing ultrafilter by filter.*

**Question 36.** *Can we also define an ultrapower monad similarly? It doesn't seem so.*

**Definition 37.** *The* ***Scott topology*** *of a poset $L$ makes functions preserving directed suprema continuous: open sets $U$ are those which are*

– *upward closed*

[3] See `http://www2.mat.ua.pt/pessoais/dirk/artigos/kleislitop_sec.pdf`.

– *inaccessible by directed joins (all directed sets with supremum in $U$ have a non-empty intersection with $U$).*

*Remark* 38. Closed sets are the downward closed and closed under suprema of directed subsets.

**Definition 39.** *A poset $(E, \leq)$ is **continuous** if for every $x \in E$,*

1. *the set $\mathrm{wb}(x) = \{y \in E \mid y \ll x\}$ is directed*
2. *$\bigvee \mathrm{wb}(x)$ exists*
3. *$x = \bigvee \mathrm{wb}(x)$*

*Above, $x$ is **way-below** $y$, what we write $x \ll y$, when for every directed set $D \subseteq E$ with supremum such that $y \leq \bigvee D$, there exists $d \in D$ such that $x \leq d$. A **continuous lattice** is a complete lattice whose underlying poset is continuous. Equivalently, a complete lattice is continuous when for every $d \in D$*

$$d \quad = \quad \bigvee\{\bigwedge U \mid d \in U \text{ with } U \text{ Scott-open}\}$$

**Definition 40.** *A topological space is **Kolmogorov** (or $T_0$) if for every pair of distinct points of $X$, at least one of them has an open neighborhood not containing the other.*

It was shown by Day [Day75] that

**Proposition 41.** *The algebras of the filter monad over the category of $T_0$ topological spaces are the continuous lattices with the Scott topology. Moreover, the structure map $m_D : TD \to D$ is given by $m_D(\phi) = \bigvee\{\bigwedge U \mid U \in \phi\}$.*

The monadicity over sets is a bit less enlightening:

**Proposition 42.** *A D-lattice is a complete lattice $L$ such that for any family $(D_i)_{i \in I}$ of directed subsets*

$$\bigwedge\{\bigvee D_i \mid i \in I\} \quad = \quad \bigwedge\{\bigvee (d_i)_{i \in I} \mid d \in \prod_{i \in I} D_i\}$$

*and a morphism between those is a function preserving directed joins and arbitrary meets. The category of D-lattices is monadic over* **Set** *by $T$.*

# 5 Ultrapowers

**Definition 43.** *Given an ultrafilter $F$ on a set $X$, the **ultrapower** of a set $Y$ is the set $Y^X/F$, whose elements are equivalence classes $\langle f \rangle$ of functions $f \in \mathrm{Hom}(X, Y)$ under the relation $\sim$ such that $f \sim g$ whenever $\{x \in X \mid f(x) = g(x)\} \in F$.*

*Remark* 44. If $F$ is principal then $Y^X/F \cong Y$, so we are mainly interested in non-principal ultrafilters.

*Remark* 45. An ultrafilter $F$ on $X$ can be seen as a point in the Stone-Čech compactification $\beta X$ of the discrete space $X_d$ and ultrapowers can be interpreted as taking stalks[4]. Namely, the inclusion $\eta_X : X \to \beta X$ is a continuous functions

[4]Found this here.

and hence induces a geometric morphism ${\eta_X}_*$ between the sheaf toposes over the spaces. Then, under the composite

$$\mathbf{Set} \overset{\Delta}{\to} \mathbf{Set}/X \cong \mathrm{Sh}(X_d) \overset{{\eta_X}_*}{\to} \mathrm{Sh}(\beta X) \overset{\mathrm{stalk}_F}{\to} \mathbf{Set}$$

the set $Y$ is taken to $Y^X/F$.

**Theorem 46** (Łoś theorem)**.** *Given an ultrafilter $F$ on $I$, we have*

$$(M_i)/F \vDash \phi \qquad \textit{iff} \qquad \{i \in I \mid M_i \vDash \phi\} \in F$$

**Theorem 47** (Compactness theorem)**.** *A set $\Sigma$ of formulas has a model iff every finite subset of $\Sigma$ has a model.*

*Proof.* The direct implication is trivial. We consider the other direction. We write $U$ for the cardinal of $\Sigma$. Given a finite set $I \subseteq U$, we write $S_I = \{J \text{ finite} \mid J \supseteq I\}$. Then the set $\{S_I \mid I \text{ finite}\}$ has the finite intersection property. Therefore by the ultrafilter lemma, there exists an ultrafilter $F$ containing all the $S_I$ and we can form the ultraproduct $(M_I)/F$ where $M_I$ is the model associated to the finite set $I$. By Łoś theorem it is then easy to show that $(M_I)/F$ is a model of $\Sigma$. □

# 6 Stone duality

The duality for boolean algebras is handled in [BS81].

**Definition 48.** *A **boolean algebra** is ...*

**Definition 49.** *The **Stone space** $S(B)$ associated to a boolean algebra $B$ is the topological space of ultrafilters on $B$ (or equivalently the homomorphisms $\mathbf{Bool}(B, 2)$), with the topology generated by the basis $P_x = \{F \in S(B) \mid x \in F\}$.*

**Lemma 50.** *The spaces $S(B)$ are compact totally disconnected Hausdorff (such spaces are called **Stone spaces**).* Totally disconnected *means that the only connected subspaces are empty or reduced to a point.*

**Lemma 51.** *Given a topological space $X$, its collection of clopen (both closed and open) sets is a boolean algebra.*

**Theorem 52** (Stone's representation theorem)**.** *Any boolean algebra $B$ is isomorphic to the algebra of the clopen sets of the Stone space $S(B)$, and this extends contravariantly to morphisms:*

$$\mathbf{Stone} \quad \cong \quad \mathbf{Bool}^{\mathrm{op}}$$

There are many variants and extensions of this result:

**Definition 53.** *A topological space is **irreducible** when it cannot be written as the union of two proper closed sets. A topological space is **sober** when every irreducible closed subset is the closure of exactly one point.*

**Lemma 54.** *Hausdorff ($T_2$) implies sober implies Kolmogorov ($T_0$).*

**Definition 55.** *A **frame** $F$ is a Heyting algebra which is complete as a lattice. Equivalently, it is a complete lattice such that*

1. *it is a Heyting algebra, i.e. $x \wedge -$ has a right adjoint*
2. *for every $x \in F$ and $S \subseteq F$, $x \wedge \bigvee S = \bigvee\{x \wedge s \mid s \in S\}$*
3. *$F$ is a distributive lattice: $x \wedge (y \vee z) = (x \vee y) \wedge (x \vee z)$*

*A morphism of frames is a (necessarily monotone) function which preserves finite meets and arbitrary joins. The category of frames is denoted by* **Frm**. *The category of **locales** is the dual category.*

**Theorem 56** (Stone's representation theorem)**.** *The category of sober spaces and continuous functions is dual to the category of spacial frames.*

$$\mathbf{Sob} \quad \cong \quad \mathbf{SpFrm}^{\mathrm{op}}$$

*Proof.* We can define a functor $\Omega : \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$ sending a topological space $X$ to its frame of open sets (its is contravariant because we know that $f^{-1}(O)$ is an open set when $O$ is an open set and $f$ is continuous). Now, we'd like to define an adjoint for this functor. Given a space $X$, its points $x$ are morphisms $p_x : 1 \to X$, which are mapped to $\Omega(p_x) : \Omega(X) \to \Omega(1) \cong 2$. This suggests that we define a functor $\mathrm{pt} : \mathbf{Frm}^{\mathrm{op}} \to \mathbf{Top}$ as follows. Given a frame $F$, the points of $\mathrm{pt}(F)$ are the morphisms $p : F \to 2$ such a morphism is equivalently characterized by

– the principal meet-prime ideal $p^{-1}(0)$,

– the completely prime filter $p^{-1}(1)$.

The open sets of $\mathrm{pt}(F)$ are the sets of the form $\phi(x) = \{p \in \mathrm{pt}(F) \mid p(x) = 1\}$. It can be shown that $\Omega$ is left adjoint to pt and the images of the two functors can be characterized as indicated in the theorem. □

**Definition 57.** *The **spectrum** $\mathrm{Spec}(R)$ of a commutative ring $R$ is the set of proper prime ideals of $R$ with the Zariski topology, whose closed sets are of the form*

$$V(I) \quad = \quad \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$$

*for some ideal $I$. A **coherent space** (or **spectral space**) is a topological space which is isomorphic to the spectrum of a commutative ring. Equivalently, a coherent space $X$ is spectral when*

*(i) it is a projective limit of $T_0$ spaces*

*(ii) if we write $K^O(X)$ for the set of quasi-compact open subsets of $X$,*

   1. *$X$ is quasi-compact and $T_0$*

2. $K^O(X)$ *is a basis of open subsets of* $X$
3. $K^O(X)$ *is closed under finite intersections*
4. $X$ *is sober*

*(quasi-compact means that from every open cover we can extract a finite open cover)*

**Theorem 58.** *The category of coherent sober spaces (and coherent maps) is equivalent to the category of coherent locales which is in turn dual to the category of distributive lattices.*

$$\mathbf{Spec} \quad = \quad \mathbf{CohSp} \quad \cong \quad \mathbf{CohLoc} \quad \cong \quad \mathbf{DLat}^{\mathrm{op}}$$

Let us investigate the above theorem in details.

**Definition 59.** *A* ***Priestley space*** $(X, \leq)$ *consists of a topological space* $X$ *together with a partial order on the underlying set* $X$ *such that*

– $X$ *is compact*

– *if* $x \not\leq y$ *then there exists a clopen up-set* $U$ *of* $X$ *such that* $x \in U$ *and* $y \notin U$

**Definition 60.** *A space is* ***zero-dimensional*** *if it has a basis of clopen sets.*

**Lemma 61.** *A zero-dimensional Hausdorff space is necessarily totally disconnected.*

**Lemma 62.** *Every Priestley space is Hausdorff, zero-dimensional, and thus a Stone space.*

**Proposition 63.**

$$\mathbf{Pries} \quad \cong \quad \mathbf{DLat}^{\mathrm{op}}$$

**Proposition 64.** *The category of Priestley spaces is isomorphic to the category of spectral spaces*

$$\mathbf{Pries} \quad \cong \quad \mathbf{Spec}$$

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