Guaranteed Topological Methods for Dynamical Systems

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CEA, LIST

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People in the business

Errors are mine, ideas are not.

This is not my work:

• Rigorous geometrical methods for chaos: Marian Mrozek, Piotr Zgliczynski, …
• Taylor method for rigorous integration of flows: Martin Berz, …
• …
We want to study **dynamical systems**:

- physics provide us with lots of **differential equations** to describe gaz, heat, etc.
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- these systems can be sensitive (and even chaotic): we have to use \textit{guaranteed methods} to handle floating-point errors
The big picture

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- the topology of those systems can be very helpful!
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**LMeASI** should be able to help!
Part I

Dynamical systems
Dynamical systems

Suppose given

• a topological space \( X \)
• a time domain \( T \in \{\mathbb{R}, \mathbb{R}^+, \mathbb{Z}, \mathbb{N}\} \).

Definition

A **dynamical system** (or **flow**) is a continuous

\[
\varphi : X \times T \to X
\]

such that

• \( \varphi(x, 0) = x \)
• \( \varphi(\varphi(x, t_1), t_2) = \varphi(x, t_1 + t_2) \)
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The ds is **continuous** when $T \in \{\mathbb{R}, \mathbb{R}^+\}$ or **discrete** when $T \in \{\mathbb{Z}, \mathbb{N}\}$. 
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**Remark**

A discrete dynamical system is characterized by $f = x \mapsto \varphi(x, 1)$. 
Flows vs. vector fields
Examples of dynamical systems

- Free fall

\[ m \dot{v} = mg \]
Examples of dynamical systems

• Free fall

\[ \dot{v} = g \]
Examples of dynamical systems

- Free fall

\[ \varphi(v, t) = gt + v_0 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \]
Examples of dynamical systems

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- Pendulum

[insert your favorite physical system here]
Examples of dynamical systems

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- Pendulum
- [insert your favorite physical system here]
Limit behaviors

Theorem (Poincaré-Bendixon)

Given a differentiable real dynamical system defined on an open subset of the plane, then every non-empty compact ω-limit set of an orbit, which contains only finitely many fixed points, is either

- a fixed point,
- a periodic orbit,
- or a connected set composed of a finite number of fixed points together with homoclinic and heteroclinic orbits connecting these.

Moreover, there is at most one orbit connecting different fixed points in the same direction. However, there could be countably many homoclinic orbits connecting one fixed point.
Some dynamical systems exhibit much more complex limit behaviors...
Chaotic systems

Some dynamical systems exhibit much more complex limit behaviors. . .

Definition
A dynamical system is chaotic if

1. it is sensitive to initial conditions
2. it exhibits topological mixing
3. it has a dense set of periodic orbits
Examples of dynamical systems – Chaotic

- The logistic map

\[ x_{n+1} = r x_n (1 - x_n) \]
Examples of dynamical systems – Chaotic

- The logistic map
- Lorenz equations (used to model weather)

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y \\
\dot{z} &= xy - \beta z
\end{align*}
\]
Examples of dynamical systems – Chaotic

- The logistic map
- Lorenz equations (used to model weather)
- Hénon map (simplification of Lorenz)

\[
\begin{align*}
x_{n+1} & = y_n + 1 - ax_n^2 \\
y_{n+1} & = bx_n
\end{align*}
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Examples of dynamical systems – Chaotic

- The logistic map
- Lorenz equations (used to model weather)
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- The tinkerbell map

\[
\begin{align*}
    x_{n+1} &= x_n^2 - y_n^2 + ax_n + by_n \\
    y_{n+1} &= 2x_ny_n + cx_n + dy_n
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• Smale’s horseshoe map
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- Smale’s horseshoe map
- Wikipedia’s
Part II

Computing invariants of dynamical systems
Invariant sets

The **trajectory** of a point \( x \in X \) is

\[
\varphi(x) = \{ \varphi(x, t) \mid t \in T \}
\]

A point \( x \) can be

• periodic: \( \exists t \in T, \varphi(x, t) = x \)

• stationary: \( \forall t \in T, \varphi(x, t) = x \) (i.e. \( \varphi(x) = \{ x \} \))
Invariant sets

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**Definition**

Given $N \subseteq X$, its **invariant part** is

$$\text{Inv}(N, \varphi) = \{x \in N \mid \varphi(x) \subseteq N\}$$

and $N$ is **invariant** when $\text{Inv}(N, \varphi) = N$. 
We want to study the structure of these invariant sets!
The exit set $N^-$ of $N \subseteq X$ is

$$N^- = \{ x \in N \mid \exists \varepsilon > 0, \forall 0 < t < \varepsilon, \varphi(x, t) \not\in N \}$$
The kind of thing we will use

The exit set $N^-$ of $N \subseteq X$ is

$$N^- = \{ x \in N \mid \exists \varepsilon > 0, \forall 0 < t < \varepsilon, \varphi(x, t) \notin N \}$$

"Theorem" (not the exact hypothesis but you get the idea)
If $N$ is connected and $N^-$ is either empty or not connected then $N$ admits a fixpoint.
Stationary points: a simple example

Suppose that $X = \mathbb{R}$, $T = \mathbb{R}$ and consider a dynamical system

$$\varphi : X \times T \rightarrow X$$

defined as the solution of

$$\dot{x} = f(x)$$

which should be thought as a tangent vector field on $X$. 

Proposition
If we can find an interval $[a, b]$ such that $f(a) < 0$ and $f(b) > 0$ then by the intermediate value theorem there exists a point $s \in [a, b]$ which is stationary:

$$\dot{x} = f(x) = 0$$
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Remark

This methodology can be extended to guaranteed methods
In higher dimensions?

When $X = \mathbb{R}^2$ the previous method cannot be used anymore…
In higher dimensions?

When $X = \mathbb{R}^2$ the previous method cannot be used anymore... ...we have to use some topological tools.
Theorem

Every continuous map $h : D^2 \rightarrow D^2$ admits a fixpoint.
Brouwer fixpoint theorem

Theorem

Every continuous map \( h : D^2 \rightarrow D^2 \) admits a fixpoint.

Proof.
By reduction ad absurdum.

- Write \( r \) for the map \( r : D^2 \rightarrow S^1 \) sending \( x \) to the intersection of the half-line from \( h(x) \), going through \( x \), with \( \partial D^2 \).

- Show that \( r \) is a deformation retract.

- Impossible because \( \pi_1(S^1) = \mathbb{Z} \neq 1 = \pi_1(D^2) \).
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*Every continuous map* $h : D^2 \rightarrow D^2$ *admits a fixpoint.*

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So, every discrete dynamical system $\varphi : X \times \mathbb{T} \rightarrow X$ with $X = D^2$ and $\mathbb{T} = \mathbb{Z}$ admits a stationary fixpoint.
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So, every discrete dynamical system \( \varphi : X \times \mathbb{T} \to X \) with \( X = D^2 \) and \( \mathbb{T} = \mathbb{Z} \) admits a stationary fixpoint.

This is the same kind of theorem!
Algebraic topology in a hurry

- The standard interval \( I = [0, 1] \).
Algebraic topology in a hurry

- The **standard interval** $I = [0, 1]$.
- A **path** on a topological space $X$ is a cont. map $p : I \rightarrow X$. 

\[ \text{Diagram of a path} \]
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- More generally two maps \( f, g : X \to Y \) are **homotopic** when there exists \( h : I \to Y^X \) such that \( h(0) = f \) and \( h(1) = g \).
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- More generally two maps \( f, g : X \to Y \) are **homotopic** when there exists \( h : I \to Y \) such that \( h(0) = f \) and \( h(1) = g \).
- \( A \subseteq X \) is a **deformation retract** of \( X \) when \( \text{id}_X : X \to X \) is homotopic to a retraction \( r : X \to A \) of \( X \) onto \( A \) (i.e., \( r(X) = A \) and \( r|_A = \text{id}_A \)).
The fundamental group

- The **concatenation** of paths $p$ and $q$ such that $p(1) = q(0)$ is defined by

\[
(p \cdot q)(t) = \begin{cases} 
p(2t) & \text{if } 0 \leq t \leq 1/2 \\
q(2t - 1) & \text{if } 1/2 \leq t \leq 1.
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- The **fundamental group** $\pi_1(X, x_0)$ is the group of homotopy classes of paths $p$ such that $p(0) = p(1) = x_0$. 

Example

- $\pi_1(D_n) = 0$
- $\pi_1(S^1) = \mathbb{Z}$
- $\pi_1(S^1 \vee S^1) = \mathbb{Z} \ast \mathbb{Z}$
- $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$

Lemma

If $A \subseteq X$ is a deformation retract of $X$ then $\pi_1(A, x_0) \sim = \pi_1(X, x_0)$. 
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**Lemma**

*If $A \subseteq X$ is a deformation retract of $X$ then $\pi_1(A, x_0) \cong \pi_1(X, x_0)$.***
In practice, homotopy groups are very hard to compute so we compute their **homology groups**.

- $H(X)$ is a sequence $H_n(X)$ of groups.
Homology

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- $H(X)$ is a sequence $H_n(X)$ of groups.
- Every finitely generated group $G$ can be decomposed as

$$G = \mathbb{Z}^n \oplus \mathbb{Z}_{q_1} \oplus \ldots \oplus \mathbb{Z}_{q_k}$$

$n = \text{rank}(G)$ is called the **rank** of $G$ (≈ dimension of a vector space).
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- The **relative homology** $H(X, Y)$ of $X$ wrt $Y \subseteq X$ is the homology of $X$ where $Y$ has been contracted to a point.
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- The **relative homology** $H(X, Y)$ of $X$ wrt $Y \subseteq X$ is the homology of $X$ where $Y$ has been contracted to a point.
- **Cohomology** $H^*(X)$ is defined in a “similar” (dual) way.
• The exit set $N^-$ of $N \subseteq X$ is

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N^- = \{ x \in N \mid \exists \varepsilon > 0, \forall 0 < t < \varepsilon, \varphi(x, t) \notin N \}
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Ważewski theorem

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Ważewski theorem

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Theorem

If $N$ is an isolating block and $N^-$ is not a deformation retract of $N$ then there exists $x \in N$ such that $\varphi(x) \subseteq N$. 

Remark

This can be verified by ensuring that $H^*(N, N^-) \neq 0$. This gives invariant points in $N$ for continuous dynamic systems.
Ważewski theorem

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**Theorem**

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**Remark**

This can be verified by ensuring that $H_*(N, N^-) \not\cong 0$.

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The Conley index

- generalizes this construction
- can be generalized to discrete dynamic systems
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Let’s see the horseshoe map first!
Part III

Chaos in the horseshoe map
The horseshoe map

Definition

The **horseshoe map** is the discrete dynamical system defined on a square as follows:
The horseshoe map

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It can be extended to a dds on the whole plane $\mathbb{R}^2$ and we are interested in $\text{Inv}(N, \varphi)$. 
Invariant points and binary strings

Write $f = x \mapsto \varphi(x, 1)$ and $N_0 \uplus N_1 = f^{-1}(N)$:

- Obviously, $\text{Inv}(N, \varphi) \subseteq N_0 \uplus N_1$.  

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\end{figure}
Invariant points and binary strings

Write $f = x \mapsto \varphi(x, 1)$ and $N_0 \cup N_1 = f^{-1}(N)$:

- Obviously, $\text{Inv}(N, \varphi) \subseteq N_0 \cup N_1$.
- This defines a map $\rho : \text{Inv}(N, \varphi) \to \Sigma_2$ with $\Sigma_2 = \{0, 1\}^\mathbb{Z}$. 
Invariant points and binary strings

Write $f = x \mapsto \varphi(x, 1)$ and $\mathcal{N}_0 \sqcup \mathcal{N}_1 = f^{-1}(\mathcal{N})$:

- Obviously, $\text{Inv}(\mathcal{N}, \varphi) \subseteq \mathcal{N}_0 \sqcup \mathcal{N}_1$.
- This defines a map $\rho : \text{Inv}(\mathcal{N}, \varphi) \to \Sigma_2$ with $\Sigma_2 = \{0, 1\}^\mathbb{Z}$.
- This map satisfies $\rho \circ f = \sigma \circ \rho$, where $\sigma$ is the shift map

\[
\sigma : \Sigma_2 \rightarrow \Sigma_2 \\
(n \mapsto s_n) \mapsto (n \mapsto s_{n+1})
\]
Symbolic Dynamics – A chaotic map

The set $\Sigma_2$ admits a metric defined by

$$d(s, t) = \sum_{n=-\infty}^{\infty} \frac{|s_n - t_n|}{2|n|}$$
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**Theorem**

*The shift map $\sigma : \Sigma_2 \rightarrow \Sigma_2$ is chaotic.*
Symbolic Dynamics – A chaotic map

The set $\Sigma_2$ admits a metric defined by

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Theorem

The shift map $\sigma : \Sigma_2 \rightarrow \Sigma_2$ is chaotic.

Theorem

There exists an homeomorphism $\rho : \text{Inv}(N, \varphi) \rightarrow \Sigma_2$ (the important part is that $\rho$ is a continuous surjection) such that $\rho \circ f = \sigma \circ \rho$ (it’s called a topological conjugacy).
Part IV

The Conley index
The Conley index

- An **isolating neighborhood** \( N \) is a compact set such that \( x \in \text{bd}(N) \) implies \( \varphi(x) \not\subseteq N \), i.e.

\[
\text{Inv}(N, \varphi) \subseteq \text{int}(N)
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The Conley index

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- An **isolated invariant set** $S$ is a compact set such that $S = \text{Inv}(N, \varphi)$ for some isolating neighborhood $N$. 


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• An **isolated invariant set** $S$ is a compact set such that $S = \text{Inv}(N, \varphi)$ for some isolating neighborhood $N$.

**Theorem**

*For every isolating neighborhood $N$ of $S$ there exists an isolating block $S \subseteq M \subseteq N$ and $H^*(M, M^-)$ only depends on $S$ (or $N$), where $H^*$ denotes the Alexander-Spanier cohomology (with coefficients in $\mathbb{Q}$).*
Generalizing to discrete systems

Suppose given a dds $\varphi$, and write $f = x \mapsto \varphi(x, 1)$.

Definition

An **index pair** $(P_1, P_2)$ of an isolated invariant set $S$ is a pair of compact sets such that

$$f(P_2) \cap P_1 \subseteq P_2$$
$$P_1 \cap \text{cl}(f(P_1) \setminus P_1) \subseteq P_2$$
$$S = \text{Inv}(\text{cl}(P_1 \setminus P_2), f) \subseteq \text{int}(P_1 \setminus P_2)$$

(intuition: $P_2$ is an exit set for $P_1$).
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\end{align*}
\]

(intuition: $P_2$ is an exit set for $P_1$).

**Problem:** $H^*(P_1, P_2)$ is *not* an invariant...
Index pairs

- Given a fd vector space $V$, the **generalized kernel** of $\alpha : V \to V$ is

$$\text{gker } \alpha = \bigcup_{n \in \mathbb{N}} \ker \alpha^n$$
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**Definition**

An **index quadruple** $(P_1, P_2, \overline{P_1}, \overline{P_2})$ consists of

• an index pair $(P_1, P_2)$

• we have

$$P_1 \cup f(P_1) \subseteq \overline{P_1} \quad \quad \quad \quad P_2 \cup f(P_2) \subseteq \overline{P_2}$$

• the inclusion $\iota : (P_1, P_2) \hookrightarrow (\overline{P_1}, \overline{P_2})$ is an excision:

$$\iota^* : H^*(P_1, P_2) \xrightarrow{\sim} H^*(\overline{P_1}, \overline{P_2})$$
The discrete Conley index

**Theorem**

For every isolating neighborhood $N$ of $f$ there exists an index quadruple such that $\text{Inv}(N, f) \subseteq P_1 \subseteq \overline{P_1} \subseteq N$ and the Conley index of $f$ in $N$ is

$$\text{Con}(N, f) = L(H^*(P_1, P_2), I_P)$$

with $I_P = f^* \circ (i^*)^{-1} : H^*(P_1, P_2) \rightarrow H^*(P_1, P_2)$ where

$$(P_1, P_2) \xrightarrow{f} (\overline{P_1}, \overline{P_2}) \xleftarrow{i} (P_1, P_2)$$

and this does not depend on the choice of the index quadruple.
Example: simple Conley indexes

- \( \text{Con}(N, f) = 0 \)
Example: simple Conley indexes

- $\text{Con}(N, f) = 0$

- $\text{Con}(N, f) = \mathbb{Q}$
Example: the horseshoe map

We set $P_1 = N$, $P_2 = N \setminus (N_1 \uplus N_2)$ and $P_i = P_i \cup f(P_i)$:
Example: the horseshoe map

We set $P_1 = N$, $P_2 = N \setminus (N_1 \cup N_2)$ and $\overline{P_i} = P_i \cup f(P_i)$:

$H_1(P_1, P_2)$ has two generators $\alpha$ and $\beta$. The index map is

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$
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$$H_1(P_1, P_2)$$

has two generators $\alpha$ and $\beta$. The index map is

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

We have $A^2 = 0$ and therefore $\ker A = \mathbb{Q}$ and $\text{Con}(N, f) = 0$. 
Chaos with Conley

Theorem

If \( N = N_0 \cup N_1 \) is an isolating neighborhood with \( N_0 \cap N_1 = \emptyset \). If for \( i \in \{0, 1\} \),

\[
\text{Con}(N_i, f)_n = \begin{cases} 
(Q, \text{id}) & \text{if } n = 1 \\
0 & \text{otherwise}
\end{cases}
\]

and the map parts of 

\[
\text{Con}(N_{00,01,11}, f) \quad \text{and} \quad \text{Con}(N_{00,10,11}, f)
\]

are different from the identity then there exists a continuous surjection \( \rho : \text{Inv}(N, f) \to \{0, 1\}^\mathbb{Z} \) such that

\[
\rho \circ f^d = \sigma \circ \rho
\]

for some \( d \in \mathbb{N} \).
Part V

Guaranteed methods
Suppose given a Galois connection

\[ \mathcal{P}(\mathbb{R}^n) \xleftarrow{\perp} D \xrightarrow{\perp} \mathcal{P}(\mathbb{K}_{\mathbb{R}^n}) \]

Typical example: the elements of \( D = \mathcal{P}(\mathbb{K}_{\mathbb{R}^n}) \) are sets of cubes.
Abstract interpretation

Suppose given a Galois connection

\[ \mathcal{P}(\mathbb{R}^n) \xleftrightarrow{\alpha} D \]

Typical example: the elements of \( D = \mathcal{P}(\mathcal{K}_{\mathbb{R}^n}) \) are sets of cubes.

Every map

\[ f : X \to Y \]

can be approximated as a map

\[ F : X \to \mathcal{P}(\mathcal{K}_Y) \]

such that

\[ \forall x \in X, \quad \alpha \circ f(x) \leq F(x) \]

and previous computations can be done on approximated maps.
Guaranteed computations on dds

Given a dds $f : X \rightarrow X$, we “replace” $f$ by an approximation $F : X \rightarrow \mathcal{P}(\mathcal{K}_X)$ in the computations (previous definitions are adapted to the approximated case).

**Theorem**

If $N$ is an isolating neighborhood of $F$ and $(P_1, P_2)$ is an index pair for $F$ in $N$, then for every function $f$ approximated by $F$, $\gamma(N)$ is an isolating neighborhood for $f$ and $(\gamma(P_1), \gamma(P_2))$ is an index pair for $f$.

**Theorem**

...similarly for index quadruples...
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Guaranteed homology of a map

We also have to compute the homology of a map to compute the Conley index!
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- $F : X \to \mathcal{P}(\mathcal{K}_X)$ is **acyclic-valued** if for every $x \in X$, $\gamma(F(x))$ is acyclic.
- $F : X \to \mathcal{P}(\mathcal{K}_X)$ is **lower-continuous** if

\[
\forall x \in X, \quad F(x) = \bigcap \{F(Q) \mid x \in Q \in \mathcal{K}_X\}
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$$\forall x \in X, \quad F(x) = \bigcap \{F(Q) \mid x \in Q \in \mathcal{K}_X\}$$

**Theorem**

*If $F$ is lower-continuous and acyclic-valued, then for every chain map $f$ approximated by $F$ we have*

$$H_*(f) = H_*(F)$$
Guaranteed homology of a map

A function $F : X \rightarrow Y$ can be represented by its graph

$$G(F) = \{(x, y) \in X \times Y \mid y \in F(x)\}$$
Guaranteed homology of a map

A function $F : X \to Y$ can be represented by its graph

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which makes the diagram commute.

Theorem $H^\ast(f) = H^\ast(q) - H^\ast(p)$. 

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Guaranteed homology of a map

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which makes the diagram commute.

**Theorem**

$$H_\ast(f) = H_\ast(q)H_\ast(p)^{-1}.$$
Images are approximated by finite sets of cubes, one can devise very fast methods for computing the (cubical) homology...
Part VI

Theorems
A few more definitions

Given a continuous dynamical system \( \varphi \), we define the following.

- The **return time** \( t_{\varphi, A} : A \to \mathbb{R}^+ \) of \( \varphi \) in \( A \subseteq X \) is
  \[
  t_{\varphi, A}(x) = \inf\{ t > 0 \mid \varphi(x, t) \in A \}
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- The **Poincaré map** $P_{\varphi, A} : \{x \in A \mid 0 < t_{\varphi, A}(x) < \infty\} \rightarrow A$

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- $A \subseteq X$ is a **Poincaré section** when $P_{\varphi,A}$ is continuous and not empty.

- Given a boolean matrix $A$ of size $n \times n$, we define
  \[ \Sigma(A) = \{s \in \{0, \ldots, n-1\}^\mathbb{Z} \mid \forall i \in \mathbb{Z}, A(s_i, s_{i+1}) = 1\} \]
  i.e. the paths in the graph defined by $A$. 
The kind of theorems we get

Theorem

Consider the **Lorenz equations** and the plane $P = \{(x, y, z) \mid z = 27\}$. For all parameter values in a sufficiently small neighborhood of $(\sigma, \rho, \beta) = (28, 10, 8/3)$ there exists a Poincaré section $N \subset P$ such that the associated Poincaré map $g$ is Lipschitz and well defined. Furthermore, for

$$A = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}$$

there is a continuous surjection $\rho : \text{Inv}(N, g) \to \Sigma(A)$ such that $\rho \circ g = \sigma \circ \rho$. In particular $h(\text{Inv}(N, g)) \geq 0.48$. Moreover, for every $\alpha \in \Sigma(A)$ which is periodic there exists an $x \in \text{Inv}(N, g)$ on a periodic trajectory such that $\rho(x) = \alpha$. 
The kind of theorems we get

Theorem
Consider the Hénon map $h : \mathbb{R}^2 \to \mathbb{R}^2$ given by the formula $h(x, y) = (1 + y/5 - ax^2, 5bx)$ at the classical parameter values $a = 1.4$ and $b = 0.2$. The discrete dynamical system induced by the Hénon map admits an invariant set $S$ semiconjugate with a subshift of finite type on 8 symbols and topological entropy $h = 0.28$. Moreover, if

$$A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{pmatrix}$$

then for each periodic sequence $\theta \in \Sigma(A)$ with period $p$ the set $\rho^{-1}(\theta)$ contains a periodic orbit with period $p$. In particular $h(S) \geq 0.28$. 
Part VII

Improving abstract interpretation
Alternatives to cubical sets

- Cubical sets ($\approx$ intervals) are easy to implement
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- In particular, they can be represented by bitmaps for which efficient algorithms to compute homology can be devised
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- However, they don’t keep dependencies: very small grids need to be used
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Other domains can be used:
- zonotopes

\[ \hat{x} = c_0 + \sum_i c_i \varepsilon_i \]
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Other domains can be used:
- zonotopes
  \[ \hat{x} = c_0 + \sum_i c_i \varepsilon_i \]
- Taylor models
  \[
  f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x - x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(\zeta)}{(n + 1)!} (x - x_0)^{n+1}
  \]
  \[
  = P_n^f (x - x_0) + I_n^f
  \]
Thanks!

Questions?