# Discrete Morse Theory

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- $f$ is a **Morse function** if it has no degenerate critical point.
- The **index** of a n-d critical point $p$ is the dimension of the largest subspace of $T_p M$ such that $H_f(p)$ is negative definite (the number of negative eigenvalues).

# MORSE THEORY

Lemma (Morse Lemma)

*Given an non-degenerate critical point* $p$*, there exists a chart* $(x_1, \ldots, x_n)$ *on a neighborhood* $U$ *of* $p$ *such that* $x_i(p) = 0$ *for every* $i$*, and*

$$f(x) \quad = \quad f(p) - x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2$$

*on* $U$*, and* $k$ *is the index of* $f$*.*

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Corollary

*Non-degenerate critical points are isolated.*

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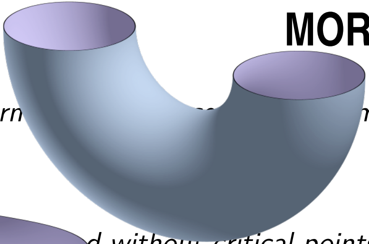

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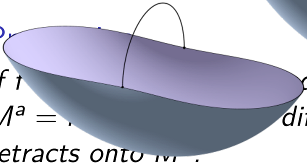

### Pr…

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### Proposition

*Suppose that p is a critical point of index $\gamma$, $f(p) = q$, $f^{-1}([q-\varepsilon, q+\varepsilon])$ is compact and contains no other critical point than p. Then $M^{q+\varepsilon}$ is homotopy equivalent to $M^{q-\varepsilon}$ with a $\gamma$-cell attached.*

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### Corollary

*Any differentiable manifold is a CW-complex with an n-cell for each critical point of index n.*

# MORSE THEORY

Proposition (Morse inequalities)

*We write $c_i$ for the number of cp of index $i$ and $b_i$ for the $i$-th Betti number. Then*

$$c_i - c_{i-1} + c_{i-2} - \ldots + (-1)^i c_0 \quad \geq \quad b_i - b_{i-1} + b_{i-2} - \ldots + (-1)^i b_0$$

# MORSE HOMOLOGY

We suppose given

- a smooth manifold $M$,
- a smooth Morse function $f : M \to \mathbb{R}$ and
- a smooth Riemannian metric on $M$

$$g \quad : \quad \prod_{p \in M} (T_p M \times T_p M \to \mathbb{R})$$

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We write $\psi_s : M \to M$ (with $s \in \mathbb{R}$) for the flow associated to $-\nabla_g f$.

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Given two critical points $p, q \in M$, we write

$$\mathcal{M}(p,q) \quad = $$
$$\left\{ u : \mathbb{R} \to M \mid \frac{du}{ds} = -\nabla_g f(u), \lim_{s \to -\infty} u(s) = p, \lim_{s \to +\infty} u(s) = q \right\}$$

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# MORSE HOMOLOGY

Given a *generic* (Morse-Smale) pair $(f, g)$, we define

- $C_k$: the $\mathbb{Z}$-module generated by critical points of index $k$,
- $\partial : C_k \to C_{k-1}$ by $\partial(p) = \sum_{q \in CP(k-1)} |\mathcal{M}(p, q)| \cdot q$

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Lemma

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Proposition

*The homology is equal to the singular homology of $M$ with coefficients in $\mathbb{Z}$. In particular, it does not depend on $(f, g)$.*

# MORSE HOMOLOGY

Example

- Morse complex of a 2-sphere:

$$\ldots \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z}$$

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- Morse complex of a torus:

$$\ldots \to 0 \to \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} \mathbb{Z}$$

with $\partial_1(c_0^2) = 2(c_0^1 - c_1^1)$ and $\partial_0(c_i^1) = 2c_0$ (???)

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Definition

A **discrete Morse function** $f : K \to \mathbb{R}$ should satisfy for every $\sigma \in K_p$:

1. there is at most one $\tau \in K_{p+1}$ such that $\tau > \sigma$ and $f(\tau) \leq f(\sigma)$,
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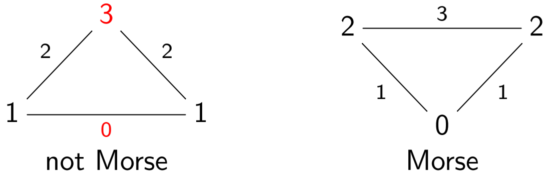

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Definition

A cell $\sigma \in K_p$ is **critical** (of **index** $p$) if

1. there is no $\tau \in K_{p+1}$ such that $\tau > \sigma$ and $f(\tau) \leq f(\sigma)$,
2. there is no $\upsilon \in K_{p-1}$ such that $\upsilon < \sigma$ and $f(\upsilon) \geq f(\tau)$.

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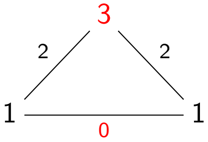
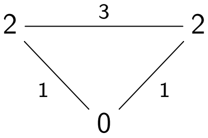



not Morse




Morse

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Proposition

*Suppose that $[a, b]$ is an interval which does not contain any critical value of $f$, then $M^a$ is a deformation retract of $M^b$. Moreover, $M^b$ simplicially collapses onto $M^a$.*

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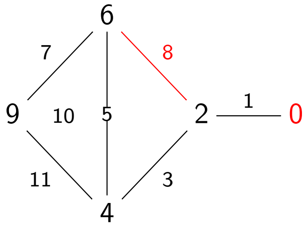
**Proposition**

*A simplicial complex with a discrete Morse function is homotopy equivalent to a CW-complex with one cell of dimension $p$ for each critical simplex of dimension $p$.*

# AN EXAMPLE

Consider the simplicial complex



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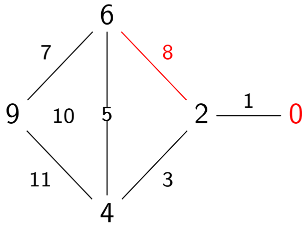


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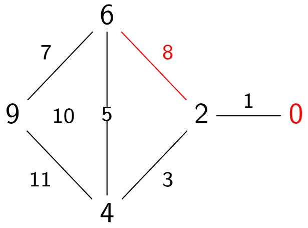
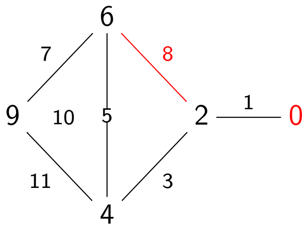



- Critical cells are in red.
- The complex is therefore homotopy equivalent to the 1-sphere


obtained by "collapsing" all the connected black parts.

# THE DISCRETE GRADIENT VF

From the simplicial complex

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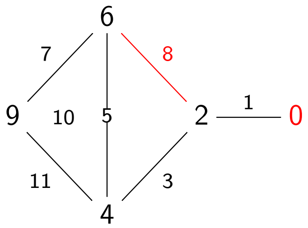


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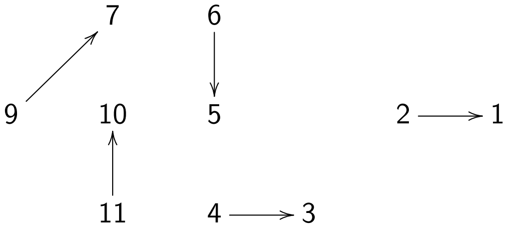


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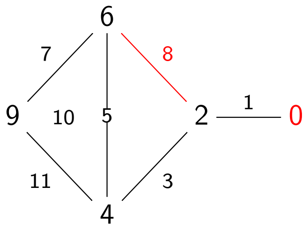


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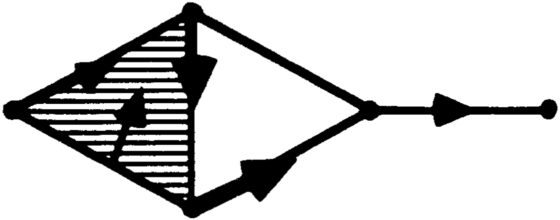

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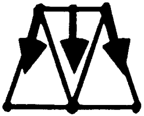
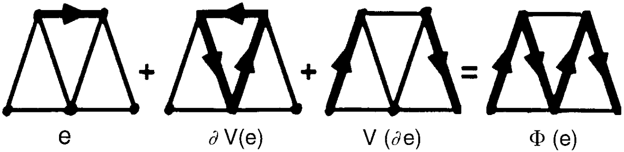
The associated **flow** is $\phi : C_p \to C_p$ defined by

$$\phi = 1 + \partial V + V\partial$$

Example

Consider $V$ defined by . The associated flow $\phi(e)$ is



# THE MORSE COMPLEX

We write $C_p^\phi \subseteq C_p$ for the $p$-chains $c$ such that $\phi(c) = c$.

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Remark

The complexes $C_p^\phi$ can also be defined as spanned by critical $p$-cells.

Since all the information we need about the Morse function is encoded in the discrete gradient vector field, this is what we are going to start with in the following.

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Define a weighted DAG $G(C_\bullet)$ with vertices $X = \cup_{i \geq 0} X_i$ and edges

$$X_i \ni c \xrightarrow{[c:c']} c' \in X_{i-1}$$

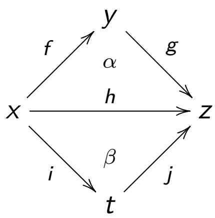
whenever $[c : c'] \neq 0$.

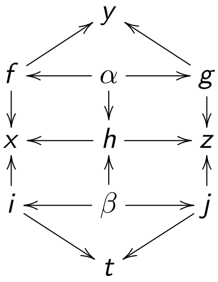
# ACYCLIC MATCHINGS

A set $\mathcal{M} \subseteq E$ of $G(C_\bullet) = (X, E)$ is an **acyclic matching** when

1. For each $c \xrightarrow{[c:c']} c'$ in $\mathcal{M}$, $[c : c']$ in the center, invertible
2. Each vertex lies in a most one edge of $\mathcal{M}$
3. The graph $G_{\mathcal{M}} = (X, E_{\mathcal{M}})$ has no directed cycle with

$$E_{\mathcal{M}} \quad = \quad (E \setminus \mathcal{M}) \cup \left\{ c' \xrightarrow{-1/[c:c']} c \mid c \to c' \in \mathcal{M} \right\}$$



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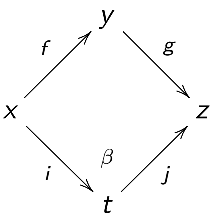


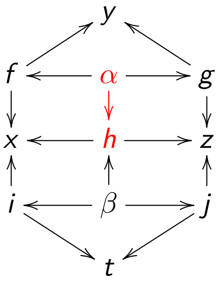

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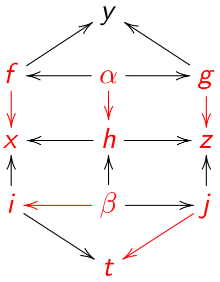

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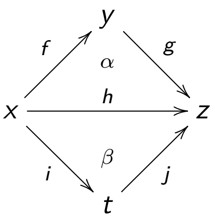


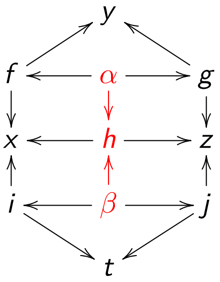

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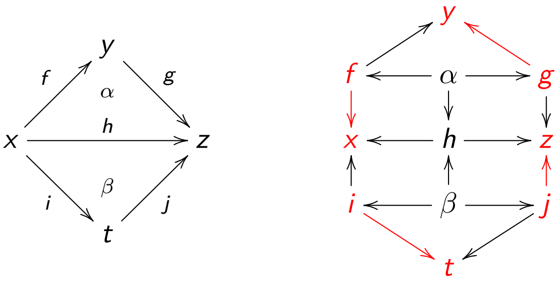
1. For each $c \xrightarrow{[c:c']} c'$ in $\mathcal{M}$, $[c : c']$ in the center, invertible
2. Each vertex lies in a most one edge of $\mathcal{M}$
3. The graph $G_{\mathcal{M}} = (X, E_{\mathcal{M}})$ has no directed cycle with

$$E_{\mathcal{M}} = (E \setminus \mathcal{M}) \cup \left\{ c' \xrightarrow{-1/[c:c']} c \mid c \to c' \in \mathcal{M} \right\}$$



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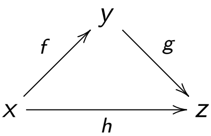


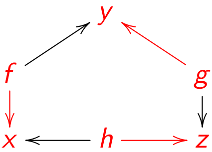

# ACYCLIC MATCHINGS

Consider $G(C_\bullet)$ together with an acyclic matching $\mathcal{M}$.

- When $e \to f \in \mathcal{M}$, $e$ is **collapsible** and $f$ is **redundant**.
- A vertex $c \in X$ is **critical** when it lies in no edge of $\mathcal{M}$.
- We write $X_i^{\mathcal{M}} \subseteq X_i$ for the critical vertices.
- The **weight** of a path is

$$w(c_1 \to c_2 \to \ldots \to c_r) \quad = \quad \prod_{i=1}^{r-1} w(c_i \to c_{i+1})$$

  with $w(c \xrightarrow{\ell} c') = \ell$.
- We write

$$\Gamma(c, c') \quad = \quad \sum_{p \in \mathrm{path}(c,c')} w(p)$$

# THE MORSE COMPLEX

The **Morse complex** $C_\bullet^{\mathcal{M}} = (C_i^{\mathcal{M}}, \partial_i^{\mathcal{M}})$ is defined by $C_i^{\mathcal{M}} = RX_i^{\mathcal{M}}$ and $\partial_i^{\mathcal{M}} : C_i^{\mathcal{M}} \to C_{i-1}^{\mathcal{M}}$ by

$$\partial_i^{\mathcal{M}}(c) \quad = \quad \sum_{c' \in X_{i-1}^{\mathcal{M}}} \Gamma(c, c')c'$$

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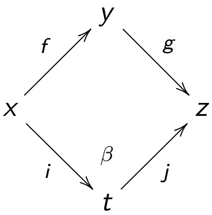


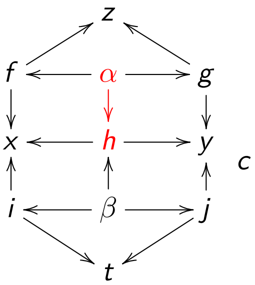


$$\partial_2^{\mathcal{M}}(\beta) = i + j - f - g$$

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Theorem

*The complex $C_\bullet^{\mathcal{M}}$ of free R-modules is homotopy equivalent to $C_\bullet$. The maps $f : C_\bullet \to C_\bullet^{\mathcal{M}}$ and $g : C_\bullet^{\mathcal{M}} \to C_\bullet$ give a chain homotopy (and thus a quasi-iso) between $C_\bullet$ and $C_\bullet^{\mathcal{M}}$:*

$$f_i(c) = \sum_{c' \in X_i^{\mathcal{M}}} \Gamma(c, c')c' \qquad g_i(c) = \sum_{c' \in X_i} \Gamma(c, c')c'$$

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Proposition

*If $\mathcal{M}$ is a set of edges with different source and targets, then $C_\bullet^{\mathcal{M}} \cong C_\bullet$ iff $\mathcal{M}$ is an acyclic matching.*

# GAUß ELIMINATION

- Fix a free chain complex

$$0 \to RX_k \xrightarrow{\partial} RX_{k-1} \to 0 \tag{1}$$

with $X_k = \{x_1, \ldots, x_m\}$ and $X_{k-1} = \{y_1, \ldots, y_n\}$.

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- We define a matrix $A \in R^{n \times m}$ with

$$a_{j,i} \quad = \quad [\partial x_i : y_j]$$

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- Then (1) has the same homology as

$$0 \to RX'_k \xrightarrow{A'} RX'_{k-1} \to 0$$

with $X'_k = X_k \setminus \{x_i\}$ and $X'_{k-1} = X_{k-1} \setminus \{y_j\}$.

# GAUß ELIMINATION

For instance

$$0 \to \mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^2 \to 0$$

with

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \end{pmatrix}$$

Taking $a_{2,2}$ as pivoting element,

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The homology is the same as

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$$M \quad = \quad \left( x_i \mid x_1 - \frac{a_{j,1}}{a_{j,i}} x_i \mid \ldots \mid \hat{0} \mid \ldots \mid x_m - \frac{a_{j,m}}{a_{j,i}} x_i \right)$$

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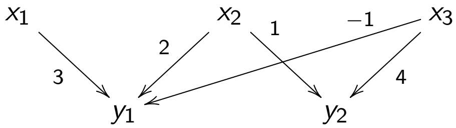

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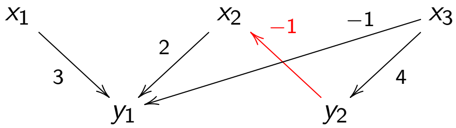

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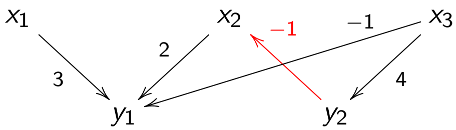


We have

$$A \quad \approx \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -9 \end{pmatrix}$$

and the flow from $x_3$ to $y_1$ is $-1 + 4 \times (-1) \times 2 = -9$, etc.

# TOWARDS THE CATEGORY OF COMPONENTS

So, if we start with a cell-complex, we can always hope to reduce it using an acyclic Matching.

Say we start from a cubic complex. The associated category of components is described by a subcomplex.

- Can this subcomplex be obtained by Morse reduction?
- Is there (in good situations) a notion of **minimal** Morse-equivalent complex?
- etc.