Division by two, in homotopy type theory

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Natural numbers as sets

The **natural numbers** $\mathbb{N}$ can be defined as the equivalence classes of finite sets under isomorphism (= cardinals).

For instance,

$$3 = \{a \quad b \quad c\}$$

$$= \{x \quad y \quad z\}$$

Operations on sets

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- **addition** is the quotient of disjoint union:

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- **product** is the quotient of cartesian product:

$$3 \times 2 \quad = \quad a \;\; b \;\; c \;\times\; \begin{matrix} x \\ y \end{matrix} \quad = \quad \begin{matrix} (a,x) & (b,x) & (c,x) \\ (a,y) & (b,y) & (c,y) \end{matrix} \quad = \quad 6$$

Operations on sets

When we have an operation on natural number we can therefore ask:

is the quotient of some operation on sets?

This is satisfactory when it is the case because

- this is more “constructive”: we replace equality by isomorphism,
- we have an extension of the operations to infinite sets,
- we can study which axioms of set theory we need to perform this.

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We see that this approach feels more constructive!

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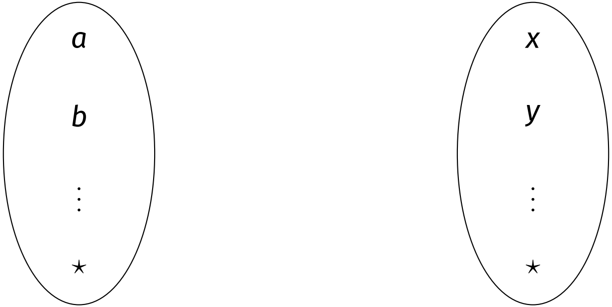
$$A \sqcup \{\star\} \xrightarrow{\quad f \quad} B \sqcup \{\star\}$$



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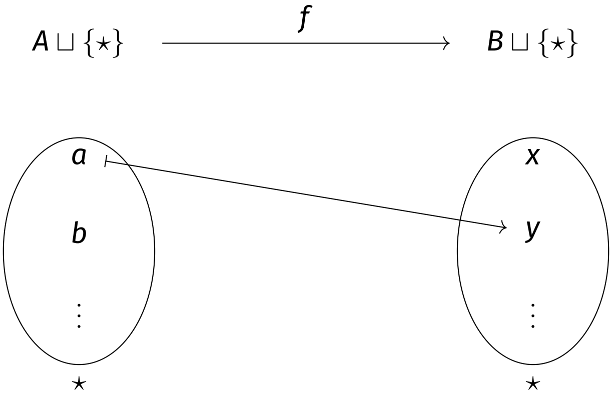

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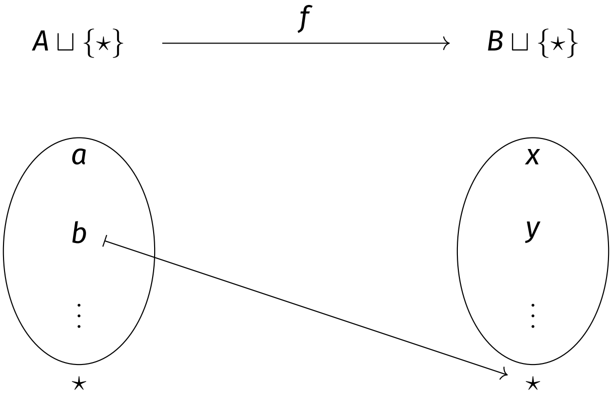

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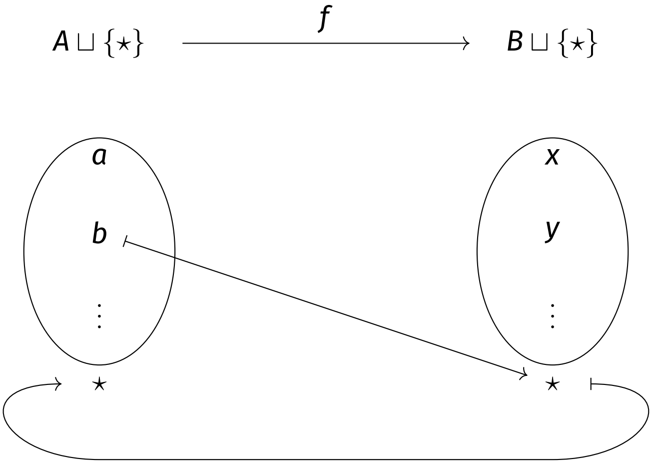

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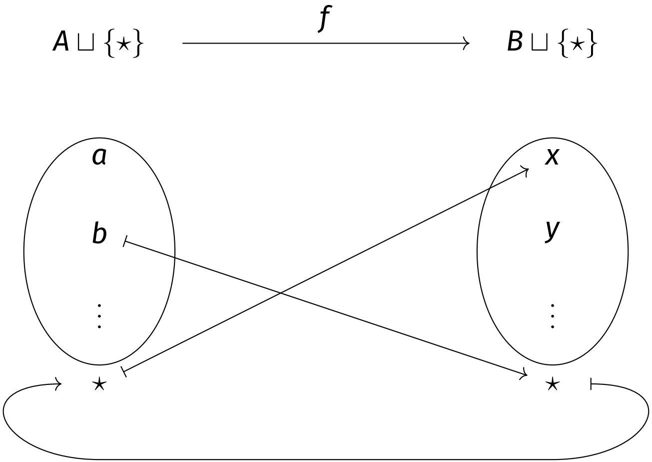

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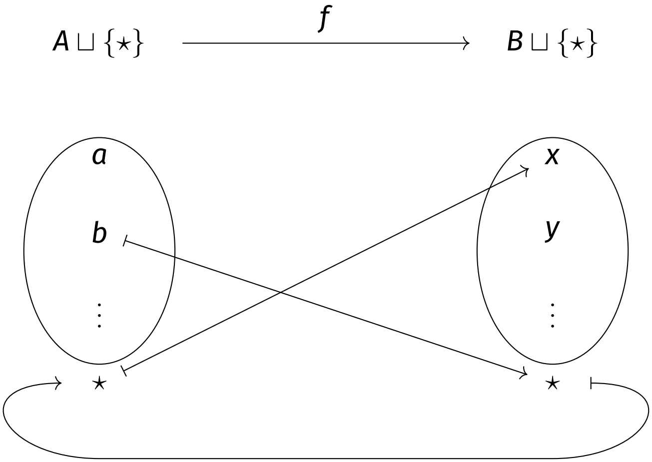

(trace!)

Division by 2

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And this is indeed the case:

- if the two sets are finite, we are essentially working with natural numbers,
- otherwise we have $A \simeq A \sqcup A \simeq B \sqcup B \simeq B$.

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It turns out excluded-middle seems unavoidable so that we focus on AC.

History of division

- 1901: Bernstein gives a construction of **division by 2** in ZF
- 1922: Serpiński simplifies the construction
- 1926: Lindenbaum and Tarski construct **division by *n***
- 1943: Tarski forgets about the construction finds a new one
- 1994: Conway and Doyle manage to reinvent the 1926 solution
- 2015: Doyle, Qiu and Schartz further simplify the construction
- 2018: Swan shows that **excluded middle** is unavoidable
  by exhibiting a non-boolean topos in which $\times \mathbf{2}$ is not regular

Still an active research topic :)

In this work

We started from Conway and Doyle's 1994 paper *Division by three*:

- we focus on division by 2,
- we formalize the results in Agda,
- we generalize from sets to *spaces*.

The Conway-Doyle-Serpiński construction of division by 2

Suppose given a bijection

$$A \times \mathbf{2} \underset{g}{\overset{f}{\rightleftarrows}} B \times \mathbf{2}$$

with $\mathbf{2} = \{-, +\}$. We want to construct a bijection

$$A \rightleftarrows B$$

without using the axiom of choice.

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- the elements of A and B are edges: for $a \in A$,

$$(a, -) \xrightarrow{a} (a, +)$$

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- the elements of A and B are edges: for $a \in A$,

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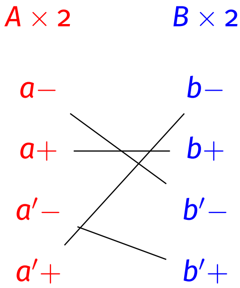
  with $\mathbf{2} = \{-, +\}$
- we identify any two vertices related by the bijection.

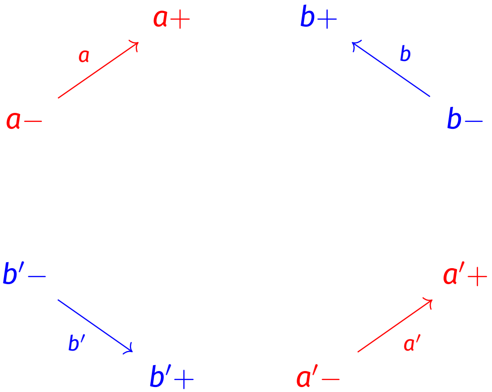
The bijection as a graph

For instance, suppose

$$A = \{a, a'\} \qquad B = \{b, b'\}$$

and consider the bijection





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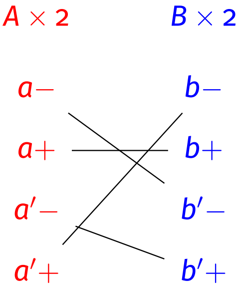
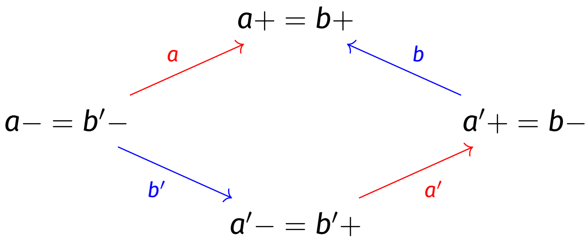
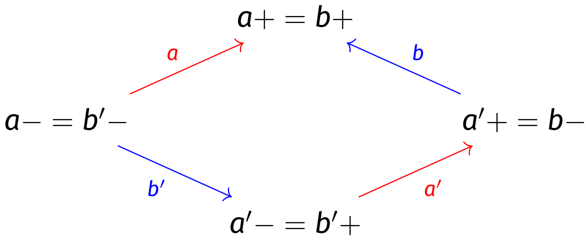




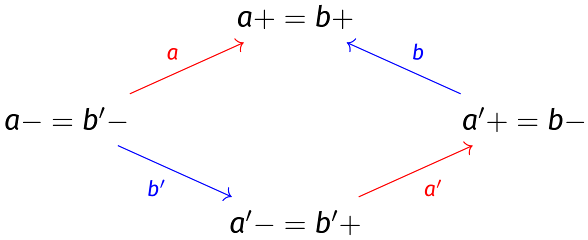
Properties of the graph



Note that:

- every vertex is connected to exactly two edges
- in a path, edges alternate between elements of A and B

Chains



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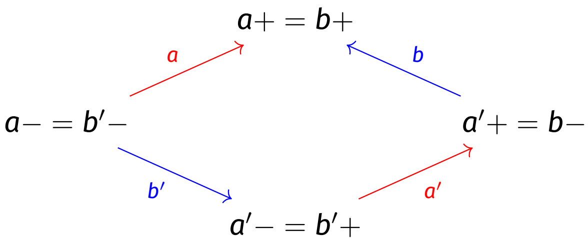

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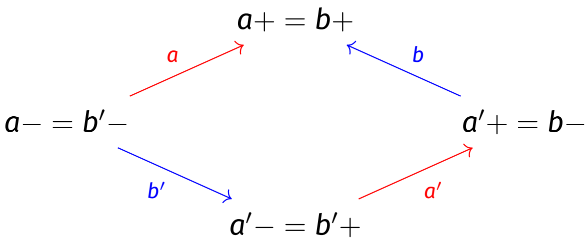

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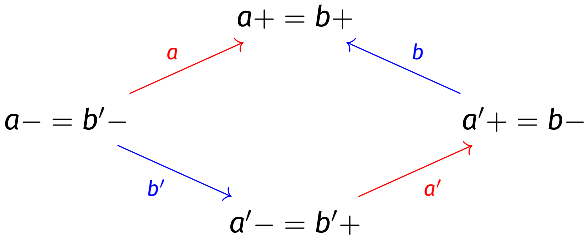

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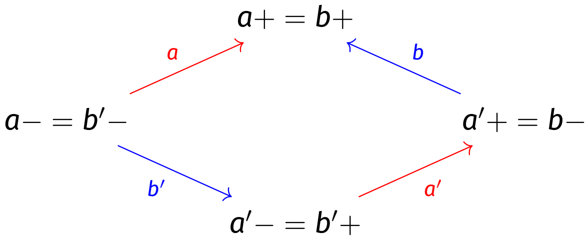

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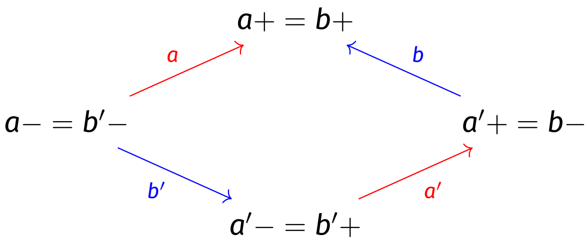

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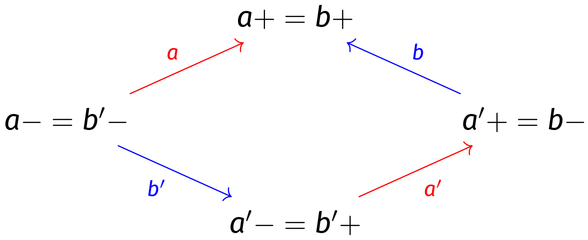

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We thus only need to pick an **orientation** in every chain …
which is not obvious without choice!

Bracketing

Consider a chain

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We can interpret arrows as brackets, which does not require an orientation:

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    we have a canonical choice of an arrow for orientation!

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In each case we can pick an orientation without choice.

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From sets to spaces

We have formalized the original result:

Theorem
For any two types `A` *and* `B` *which are sets,*

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Note: we should use **equivalences** instead of isomorphisms for types.

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- `Arrows = A ⊎ B`
- `Ends = Arrows ×` $\mathbb{2}$ `= dArrows`

The idea:

$$(\texttt{a} \;,\; \texttt{src}) \cdot \xrightarrow{\;\texttt{a}\;} \cdot (\texttt{a} \;,\; \texttt{tgt})$$

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The idea:

$$\mathtt{(a\ ,\ src)} \cdot \xrightarrow{\mathtt{a}} \cdot \mathtt{(a\ ,\ tgt)}$$

We also have functions

$$\begin{aligned} \mathtt{arr} : \mathtt{dArrows} &\to \mathtt{Arrows} \\ \mathtt{(a,src)} &\mapsto \mathtt{a} \\ \mathtt{(a,tgt)} &\mapsto \mathtt{a} \end{aligned} \qquad \begin{aligned} \mathtt{fw} : \mathtt{Arrows} &\to \mathtt{dArrows} \\ \mathtt{a} &\mapsto \mathtt{(a,src)} \end{aligned}$$

Reachability

$$\cdots \longrightarrow \cdot \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdot \longleftarrow \cdots$$

We can then define a function:

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iterate : ℤ → dArrows → dArrows
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reachable e e' = Σ[ n ∈ ℤ ] (iterate n e ≡ e')
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as well as

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is-reachable e e' = ∥ reachable e e' ∥
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Revealing rechability

Recall,

$$\texttt{reachable e e' = } \Sigma\texttt{[n} \in \mathbb{Z} \texttt{] (iterate n e} \equiv \texttt{e')}$$

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Proof.

Since `A` and `B` are <u>sets</u>, so is `dArrows = (A ⊎ B) × 𝟚`.

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Conversely, `is-reachable e e' → reachable e e'`

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Since `A` and `B` are <u>sets</u>, so is `dArrows = (A ⊎ B) × 𝟚`.

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Revealing rechability

Recall,

$$\texttt{reachable e e' = }\Sigma\texttt{[n} \in \mathbb{Z}\texttt{] (iterate n e} \equiv \texttt{e')}$$

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Thus `reachable e e'` is a proposition,
which is moreover decidable because we are <u>classical</u>.
Supposing `reachable e e'`, since we have a way to enumerate $\mathbb{Z}$,
we can therefore find an `n :` $\mathbb{Z}$ such that `iterate n e` $\equiv$ `e'`. □

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A satisfactory definition of chains

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and similarly, we define chains as

```
Chains = Arrows / is-reachable-arr
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Building the bijection chainwise

Given a chain `c`, we write `chainA c` (resp. `chainB c`) for the type of its elements in `A` (resp. `B`).

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Lemma
If, for every chain `c`*, we have* `chainA c ≃ chainB c`*, then* `A ≃ B`.

Proof.
Given a relation `R` on a type `A`, the type is the union of its equivalence classes:

```
A ≃ Σ[ c ∈ A / R ] (fiber [_] c)
```

The result can be deduced from this and standard equivalences. □

Types of chain

Recall that a chain c can be

- well-bracketed:

$$\cdots \ {\color{red}\xrightarrow{(}} \cdot {\color{blue}\xrightarrow{(}} \cdot {\color{red}\xleftarrow{)}} \cdot {\color{blue}\xrightarrow{(}} \cdot {\color{red}\xleftarrow{)}} \cdot {\color{blue}\xleftarrow{)}} \ \cdots$$

- a switching chain:

$$\cdots \ {\color{red}\longleftarrow} \cdot {\color{blue}\longleftarrow} \cdot {\color{red}\longleftarrow} \cdot {\color{blue}\longrightarrow} \cdot {\color{red}\longrightarrow} \cdot {\color{blue}\longrightarrow} \ \cdots$$

- a slope:

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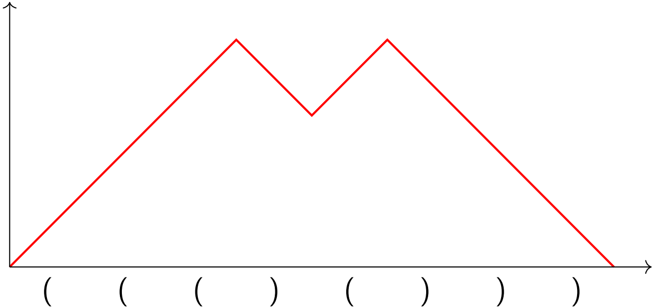
It only remains to show `chainA c` $\simeq$ `chainB c` in each case (we will only present well-bracketing).

Well-bracketing

A word over $\{(,)\}$ may be interpreted as a *Dyck path*:



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The **height** of the following path is 4:

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An arrow `a` is **matched** when it satisfies

```
Σ[ n ∈ ℕ ] (
  height (suc n) (fw a) ≡ 0 ∧
  ((k : ℕ) → k < suc n → ¬ (height k (fw x) ≡ 0)))
```

Well-bracketing

The chain of an arrow $\circ$ is **well-bracketed** when every arrow reachable from $\circ$ is matched.

Proposition
Being well-bracketed for a reachable arrow is a proposition, which is independent of the choice of $\circ$.

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A chain is **well-bracketed** when each of its arrow is well-bracketed in the above sense.

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Since

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in order for this definition to make sense:

- we need to eliminate to a set (by definition of chains as *quotients*): here, we eliminate to `HProp`, which is a set, of which being well-bracketed is an element!
- we need to show that this is independent of the choice of the representative for the origin `o`.

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- we need to show that this is independent of the choice of the representative for the origin `o`.

Proposition
Given a well-bracketed chain `c`, *we have an equivalence* `chainA c` $\simeq$ `chainB c`.

The two other cases

- switching chains
- slopes

are handled similarly.

Division by 2

Theorem
For any two types `A` *and* `B` *which are sets,*

$$\mathtt{A} \times \mathbb{2} \simeq \mathtt{B} \times \mathbb{2} \quad \to \quad \mathtt{A} \simeq \mathtt{B}.$$

Our aim is now to generalize the theorem to the situation where $\mathtt{A}$ and $\mathtt{B}$ are arbitrary types (as opposed to sets).

We suppose fixed an equivalence $\mathtt{A} \times \mathbb{2} \simeq \mathtt{B} \times \mathbb{2}$.

The set truncation

Given a type `A`, we write $\| \texttt{A} \|_0$ for its **set truncation**:

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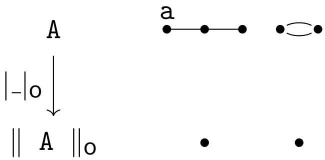
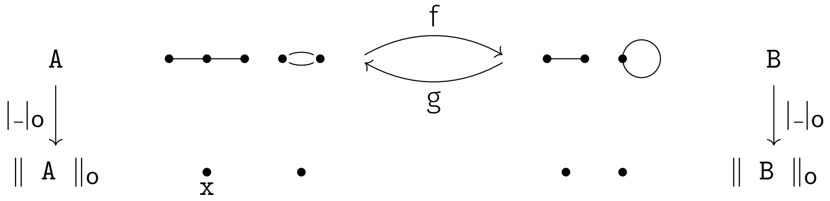
The picture we should have in mind is



Given `a : A`,

- $|$ `a` $|_0$ is its connected component,
- `fiber` $|\_|_0$ $|$ `a` $|_0$ are the elements of this connected component.

Equivalences and set truncation



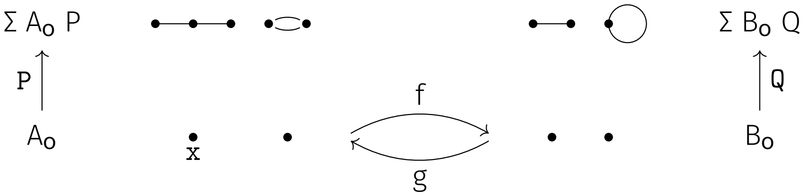
Proposition

Suppose given an equivalence $\mathtt{A} \simeq \mathtt{B}$ *(with* $\mathtt{f : A \to B}$*).*

- *There is an induced equivalence* $\| \mathtt{A} \|_0 \simeq \| \mathtt{B} \|_0$.
- *Given* $\mathtt{x} : \| \mathtt{A} \|_0$*, we have an equivalence*

$$\mathtt{fiber}\ |\_|_0\ \mathtt{x} \simeq \mathtt{fiber}\ |\_|_0\ (\|\|_0\mathtt{-map\ f\ x})$$

Equivalences and set truncation



Proposition

Given an equivalence $\mathtt{A_0} \simeq \mathtt{B_0}$ *(with* $\mathtt{f} : \mathtt{A_0} \to \mathtt{B_0}$*), and type families* $\mathtt{P} : \mathtt{A_0} \to \mathtt{Type}$ *and* $\mathtt{Q} : \mathtt{B_0} \to \mathtt{Type}$*, such that for* $\mathtt{x} : \mathtt{A}$*, we have*

$$\mathtt{P\ x} \simeq \mathtt{Q\ (f\ x)}$$

Then

$$\Sigma\ \mathtt{A_0}\ \mathtt{P} \simeq \Sigma\ \mathtt{B_0}\ \mathtt{Q}$$

Reachability and equivalence

Proposition
Given directed arrows $\texttt{a}$ *and* $\texttt{b}$ *in* $\| \ \texttt{dArrows} \ \|_{\texttt{O}}$ *reachable from the other, we have*

$$\texttt{fiber} \ |\_|_{\texttt{O}} \ \texttt{a} \simeq \texttt{fiber} \ |\_|_{\texttt{O}} \ \texttt{b}$$

Proof.
We can define functions

$$\texttt{next} : \texttt{dArrows} \to \texttt{dArrows} \qquad \texttt{prev} : \texttt{dArrows} \to \texttt{dArrows}$$

sending a directed arrow to the next one (in the direction), which form an equivalence, thus

$$\texttt{fiber} \ |\_|_{\texttt{O}} \ \texttt{a} \simeq \texttt{fiber} \ |\_|_{\texttt{O}} \ (\| \ \texttt{next} \ \|_{\texttt{O}} \ \texttt{a})$$

by previous proposition and we conclude by induction. □

Dividing homotopy types by 2

Theorem
Given types $\mathtt{A}$ *and* $\mathtt{B}$*, we have*

$$\mathtt{A} \times \mathbb{2} \simeq \mathtt{B} \times \mathbb{2} \quad \to \quad \mathtt{A} \simeq \mathtt{B}$$

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Since this bijection sends a directed arrow a to a reachable one b,

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Proof.

$$\begin{gathered} \mathtt{A} \times \mathbb{2} \simeq \mathtt{B} \times \mathbb{2} \\ \| \mathtt{A} \times \mathbb{2} \|_0 \simeq \| \mathtt{B} \times \mathbb{2} \|_0 \\ \| \mathtt{A} \|_0 \times \mathbb{2} \simeq \| \mathtt{B} \|_0 \times \mathbb{2} \\ \| \mathtt{A} \|_0 \simeq \| \mathtt{B} \|_0 \end{gathered}$$

Since this bijection sends a directed arrow `a` to a reachable one `b`,

$$\mathtt{fiber}\ |\_|_0\ \mathtt{a} \simeq \mathtt{fiber}\ |\_|_0\ \mathtt{b}$$

thus $\mathtt{A} \simeq \Sigma[\ \mathtt{a} \in \mathtt{A}\ ]\ (\mathtt{fiber}\ |\_|_0\ \mathtt{a}) \simeq \Sigma[\ \mathtt{b} \in \mathtt{B}\ ]\ (\mathtt{fiber}\ |\_|_0\ \mathtt{b}) \simeq \mathtt{B}$

About the LPO

We required to work in classical logic, but it might be the case that a weaker principle (implied by excluded middle, but not provable in intuitionistic logic) could be sufficient.

Moreover, we could not show that having division by 2 implies LEM.

A good candidate is the **limited principle of omniscience** (LPO):

> *Given a sequence* `f : ℕ → Bool`,
>
> - *either* `∀ (n : ℕ) ¬ (P n)`,
> - *or* ` (n : ℕ) (P n)`.

Questions?