Classifying covering types in homotopy type theory

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Abstract

Covering spaces are a fundamental tool in algebraic topology because of the close relationship they bear with the fundamental groups of spaces. They are namely in correspondence with the subgroups of the fundamental group: this is known as the *Galois correspondence*. In particular, the covering space corresponding to the trivial group is the universal covering, which is a "1-connected" variant of the original space, in the sense that it has the same homotopy groups, except for the first one which is trivial. In this article, we formalize this correspondence in homotopy type theory, a variant of Martin-Löf type theory in which types can be interpreted as spaces (up to homotopy). In passing, we develop a an n-dimensional generalization of covering spaces. Moreover, in order to witness for the applicability of our approach, we formally classify the covering of lens spaces and explain how to construct the Poincaré homology sphere.

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1 Introduction

The notion of covering space is a fundamental tool in algebraic topology. Namely, it provides a canonical way to remove the low-dimensional homotopy structure of a space (the universal covering has a trivial fundamental group) and it bears a close relationship with the fundamental group: coverings are classified by subgroups of the fundamental group of the original space, which is known as the *Galois correspondence*. The setting of homotopy type theory [13] allows one to perform geometric constructions in a synthetic way: all constructions on types correspond to manipulations of spaces, and are guaranteed to be invariant under homotopy of spaces by construction. It is thus natural to expect that the definition of covering space and associated properties can be developed in this framework, and we explain here that this is indeed the case. The notion of (universal) covering of a type was first introduced by Harper and Favonia in [8], and the Galois correspondence was recently independently shown by Wemmenhove, Manea, and Portegies [16]. Here, we develop further the theory of covering spaces, by explaining their relationship with the connected/truncated factorization, generalizing to n-coverings (we recover the usual case by setting $n = 0$), and computing their homotopy groups. Compared to [16], our proof of the Galois correspondence departs the tradition one in algebraic topology [7], providing arguments which are shorter, more conceptual, and should be amenable to generalizations (in particular, we leave the general classification of n-coverings for future works, handling only the case $n = 0$ here). Finally, we apply our constructions by classifying the covering spaces of lens spaces (which provide deloopings of cyclic groups) and constructing important spaces due to Poincaré (the hypercubical manifold and the homology sphere) as quotients of coherent actions of their fundamental group on their universal covering.

Plan of the paper. After recalling basic notations and concepts in homotopy type theory (Section 2), we define and study higher covering types (Section 3), and prove the classification of covering types (Section 4). Finally, as an application, we classify the covering types of lens spaces, construct the Poincaré homology sphere (Section 5) and conclude (Section 6).

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2 Homotopy type theory

We begin by briefly recalling the main notations and tools of homotopy type theory. Detailed presentations can be found in [13, 12].

Universe. We write $\mathcal{U}$ for the *universe* type, whose elements are small types, which is supposed to be closed under dependent sum and product types. Given a type $A : \mathcal{U}$ and a type family $B : A \to \mathcal{U}$, we write $\Sigma A.B$ or $\Sigma(x : A).B\,x$ for dependent sum types and $\Pi A.B$ or $\Pi(x : A).B\,x$ or $(x : A) \to B\,x$ for dependent product types. As customary, we respectively write $A \times B$ and $A \to B$ for product and arrow types, which correspond to non-dependent particular cases of the previous constructions. We respectively write 0 and 1 for the initial and terminal types.

Identities. The type theory features a notion of definitional equality and we write $t \equiv u$ when two terms t and u are definitionally equal. It also features a notion of propositional equality: given $x, y : A$, we write $x = y$ for the type of *identities* or *paths* between x and y in A. For any point $x : A$, there is a path $\mathrm{refl} : x = x$ witnessing for reflexivity. Given paths $p : x = y$ and $q : y = z$, we can construct paths $p \cdot q : x = z$ corresponding to concatenation or transitivity, and $p^{-1} : y = x$ corresponding to inverse or symmetry.

Pointed types. A *pointed type* is a type A together with a distinguished element, often written $\star_A$ or even $\star$ (equivalently, the distinguished element can be specified by providing a map $1 \to A$). A *pointed* map $f : A \to B$ between pointed types A and B is a map between the underlying types together with an identification $f(\star_A) = \star_B$. We write $A \to_\star B$ for the corresponding type of pointed maps.

Univalence. A map $f : A \to B$ is an *equivalence* when it admits both a left and a right inverse. We write $A \simeq B$ for the type of equivalences between A and B. Any identity between two types canonically induces an equivalence between them. The *univalence* axiom states that the corresponding map $(A = B) \to (A \simeq B)$ is itself an equivalence: in particular, any equivalence induces an identity.

Homotopy levels. A type A is *contractible* when it is equivalent to 1. A type A is a *proposition* (resp. a *set*, resp. a *groupoid*) when for any elements $x, y : A$ the type $x = y$ is contractible (resp. a proposition, resp. a set). The type of sets is denoted Set. More generally, we can define a notion of n-type for $n \in \{-2, -1\} \cup \mathbb{N}$ by stating that a (-2)-type is a contractible one, and an $(n{+}1)$-type is a type A in which $x = y$ is an n-type for every $x, y : A$. We write $\mathrm{isType}_n(A)$ for the predicate indicating that A is an n-type, which can be shown to be a proposition. We write $\mathcal{U}_n \equiv \Sigma\mathcal{U}.\,\mathrm{isType}_n$ for the type of n-types.

Truncation. The *n-truncation* $\|A\|_n$ is the universal way of turning a type A into an n-type: it comes equipped with a map $|-|_n : A \to \|A\|_n$ such that any map $f : A \to B$, whose target B is an n-type, induces a unique map $\tilde{f} : \|A\|_n \to B$ such that $\tilde{f}(|x|_n) = f(x)$ for $x : A$. A type A is *n-connected* when $\|A\|_n = 1$. In particular, a type is *connected* (resp. *simply connected*) when it is 0-connected (resp. 1-connected). The *connected component* of an element a of A is $\Sigma(x : A).\|a = x\|_{-1}$.

Loop space. The *circle* S^1 is the free pointed type containing a path $\mathrm{loop} : \star = \star$. Given a pointed type A, its *loop space* $\Omega\,A$ is $\star = \star$, which can be shown to coincide with the type $\mathrm{S}^1 \to_\star A$. Its *fundamental group* is the type $\pi_1(A) \equiv \|\,\Omega\,A\|_0$, which is canonically equipped with a group structure induced by path concatenation. When A is a pointed connected groupoid, its loop space coincides with its fundamental group, so that it is a group. A *delooping* of a group G is a type $\mathrm{B}\,G$ equipped with an isomorphism of groups $\Omega\,\mathrm{B}\,G \cong G$ (such a type always exists and is unique, thus the notation). It is easily shown that a map $\mathrm{B}\,G \to \mathrm{Set}$ corresponds to a set equipped with an *action* of G in the usual sense.

Fiber sequences. Given a map $f : A \to B$ and $y : B$, the *fiber* of f at y is the type $\mathrm{fib}_f\, y \equiv \Sigma(x : A).(f(x) = y)$. When B is pointed, the *kernel* of f is $\ker f \equiv \mathrm{fib}_f \star$. A composable pair of morphisms $F \longrightarrow A \xrightarrow{f} B$ is a *fiber sequence* when F is the kernel of f and the map $F \to A$ is the first projection. A map is n-connected (resp. n-truncated) when all its fibers are.

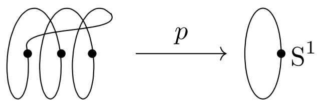
The Grothendieck duality. A fundamental property in homotopy type theory is that, given a type A, *fibrations* over A correspond both to types over A and to type families indexed by A: this is the *Grothendieck duality.* We write $\mathcal{U}/A \equiv \Sigma(B : \mathcal{U}).(B \to A)$ for the type of *types over* A and $A \to \mathcal{U}$ for the type of *type families* indexed by A. We have a function $\mathrm{fib} : \mathcal{U}/A \to (A \to \mathcal{U})$, which to $p : B \to A$ associates the fiber $\mathrm{fib}_p : A \to \mathcal{U}$, and a function $\int : (A \to \mathcal{U}) \to (\mathcal{U}/A)$, which to a family $F : A \to \mathcal{U}$ associates the first projection map $\pi : \Sigma A.F \to A$. The following is shown in [13, Section 4.8]:

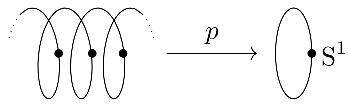
▶ **Theorem 1.** *Given a type A, the above functions induce an equivalence $\mathcal{U}/A \simeq (A \to \mathcal{U})$. Moreover, this correspondence is functorial in the sense that given $p : B \to A$ and $q : C \to A$, a morphism $f : B \to C$ with $q \circ f = p$ corresponds to a family of maps $(x : A) \to \mathrm{fib}_p\, x \to \mathrm{fib}_q\, x$ in a way which preserves identities and composition (and thus equivalences).*

3 Higher covering types

3.1 Covering spaces in topology

We briefly recall here the traditional notion of covering space in topology and refer to standard textbooks for details [7]. Given a topological space A, a *covering* is a space B together with a map $p : B \to A$ which is *locally trivial.* This means that every point $x : B$ admits an open neighborhood such that the preimage $p^{-1}(U)$ is homeomorphic to a space of the form $U \times F$, where F is a set equipped with the discrete topology and the restriction of p to $p^{-1}(U)$ is the first projection. When B is connected, which will be the case of interest here, the cardinal of F has to be the same for every point x and is called the number of *sheets* of the covering. Below we have figured a covering with 3 sheets (on the left) and with countably many sheets (on the right):





In order for the constructions of coverings to be well-behaved, we will implicitly assume in the following that A satisfies reasonable assumptions (namely being connected, locally path-connected, semilocally simply-connected). We will also assume that it comes equipped with a distinguished point $\star$. An important feature of coverings is that they have the *path lifting property*: given a path $\pi : x \rightsquigarrow y$ in A and an element $\tilde{x} \in \tilde{A}$ with $p(\tilde{x}) = x$ there is a unique path $\tilde{\pi} : \tilde{x} \to \tilde{y}$ with $p(\tilde{\pi}) = \pi$. This implies that we have an action of the fundamental group $\pi_1(A)$ on the fiber $p^{-1}(\star)$ and, in fact, this characterizes coverings:

▶ **Proposition 2** ([7, Section 1.3, p. 70])**.** *Coverings of A are in bijections with sets equipped with a action of $\pi_1(A)$.*

The *universal covering* is the only covering $p : \tilde{A} \to A$ with $\tilde{A}$ simply connected. For instance, the universal covering of S^1 is the "helix" pictured on the right above. This space can be shown to be unique up to isomorphism and can be constructed as follows:

▶ **Proposition 3** ([7, Section 1.3, p. 64])**.** *The universal cover $\tilde{A}$ of A can be constructed as the space whose points are pairs consisting of a point $x : A$ and a path $p : \star \rightsquigarrow x$ up to homotopy, equipped with a suitable topology, the map $p : \tilde{A} \to A$ being given by first projection.*

The *fundamental group* $\pi_1(A)$ of A is the group whose elements are homotopy classes of paths $\star \rightsquigarrow \star$ in A, with concatenation as multiplication and constant paths as neutral elements. By the construction of Proposition 3, given $x : A$, the fiber $p^{-1}(x)$ is $\pi_1(A)$, we have an action of $\pi_1(A)$ on this fiber given by left multiplication, which induces an action of $\pi_1(A)$ on $\tilde{A}$.

▶ **Proposition 4** ([7, Theorem 1.38])**.** *The quotient $\tilde{A}/\pi_1(A)$ of the universal cover under the above action is precisely A.*

The action can be shown to be free, so that the quotient above coincides with the homotopy quotient.

3.2 The universal fibration

We begin by describing a situation in homotopy type theory, which is close to coverings, and fundamental with respect to the characterization of identity types. This construction might seem a bit artificial at first, but we will see that the point of view nicely generalizes to coverings, in the sense that what we describe in this section is a kind of "universal ∞-covering".

Suppose given a pointed type A. A *pointed type over* A is a pointed morphism $p : B \to_\star A$. A *morphism* between two such pointed types $p : B \to_\star A$ and $q : C \to_\star A$ is a map $f : B \to C$ together with an equality $f \circ p = q$. Such morphisms compose in the expected way, which is compatible with the composition of underlying maps.

▶ **Definition 5.** *A pointed type $p : B \to_\star A$ is* universal *when for every pointed type $q : C \to A$, the type of morphisms from p to q is contractible.*

Since it satisfies a universal property, the universal pointed type is unique, and it exists thanks to the following characterization:

▶ **Proposition 6.** *A pointed type $p : B \to_\star A$ is universal if and only if B is contractible.*

Proof. By definition of morphisms and initiality of 1, the type 1 is initial among pointed types over A. An universal pointed type $p : B \to A$ is also initial among pointed types over A, by the universal property, and thus B is equivalent to 1, i.e. contractible. ◀

By immediate computations, we have:

▶ **Proposition 7.** *The fiber of the universal pointed type $p : 1 \to A$ is $\Omega\, A$.*

It is very illuminative to translate the previous definitions and properties under the Grothendieck duality (see Theorem 1). A pointed type over A corresponds to a fibration $P : A \to \mathcal{U}$ together with a distinguished element $\star_P$ of $P\star$, and, by Proposition 6, such a fibration is universal precisely when the total space $\Sigma A.P$ is contractible. The universal pointed type thus corresponds to the *universal fibration*, which is the map $\mathcal{F}_A : A \to \mathcal{U}$ defined by $\mathcal{F}_A\, x \equiv (\star = x)$, and pointed by $\mathrm{refl}_\star$. By Theorem 1, a morphism between fibrations $P, Q : A \to \mathcal{U}$ corresponds to a family of maps $f : (x : A) \to P\, x \to Q\, x$ together with an identification $f \star \star_P = \star_Q$. The initiality property of universal pointed types (Definition 5) then translates as the *fundamental theorem of identity types* [12, Theorem 11.2.2]:

▶ **Theorem 8.** *Suppose given a type family $P : A \to \mathcal{U}$ pointed by $\star_P : P\star$, together with a family of maps*

$$F : (x : A) \to (\star = x) \to P\, x$$

and an identification $F \star \mathrm{refl}_\star = \star_P$. Then F is a family of equivalences if and only if the total space $\Sigma A.P$ is contractible.

Proof. Under the Grothendieck duality of Theorem 1, the type family P corresponds to the pointed type $p : \Sigma A.P \to A$ over A, given by the first projection. The family of maps F then corresponds to a morphism between the universal type over A and p, i.e. to a pointed map $f : 1 \to \Sigma A.P$. Then F is a family of equivalences if and only if the induced map f is an equivalence, i.e. if and only if $\Sigma A.P$ is contractible. ◀

▶ Remark 9. In the situation of the above theorem, the map P can be thought of as being *representable*, in the sense that $Px = (\star = x)$ (and identities can be thought of as homs in types). This notion was actually used by Voevodsky [14] in order to first define identity types.

3.3 Higher covering types

We now introduce the notion of n-covering type, for any natural number n, which can be understood as an n-truncated variant of the pointed types of previous section. The traditional notion of covering is the particular case $n = 0$, as we indicate in remarks.

An *n-covering* of A is a map $p : B \to A$ whose fibers are n-types; such a map is also said to be *n-truncated.* We write

$$\mathrm{Covering}_n(A) \quad \equiv \quad \Sigma(B : \mathcal{U}).\Sigma(f : B \to A).(x : A).\,\mathrm{isType}_n(\mathrm{fib}_f\, x)$$

for the type of coverings of A. Under the Grothendieck duality, those can also be defined as families of n-types.

▶ **Lemma 10.** *We have an equivalence* $\mathrm{Covering}_n(A) \simeq (A \to \mathcal{U}_n)$.

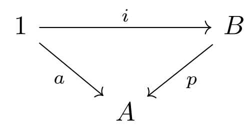
Proof. Follows immediately from Grothendieck duality (Theorem 1). ◀

▶ Remark 11. For $n = 0$, we recover the definition covering types of [8, Definition 1], as maps $A \to \mathrm{Set}$. Since Set is a groupoid [13, Theorem 7.1.11], the universal property of groupoid truncation provides us with an equivalence $(A \to \mathrm{Set}) \simeq (\|A\|_1 \to \mathrm{Set})$. A covering of A thus corresponds to a map $\|A\|_1 \to \mathrm{Set}$. Moreover, we have by Proposition 30 that $\|A\|_1$ is a $\mathrm{B}\,\pi_1 A$, from which deduce that a covering of A corresponds to a set equipped with an action of $\pi_1 A$. We thus recover the traditional Proposition 2, which was formalized in homotopy type theory in [8, Theorem 4].

A *morphism* f between n-coverings $p : B \to A$ and $q : C \to A$ is a map $f : B \to C$ together with an equality $p = q \circ f$.

A *pointed* n-covering is a pointed map $p : B \to_\star A$ between pointed types, whose underlying map is an n-covering. This corresponds exactly to the following notion:

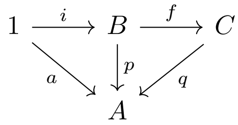
▶ **Definition 12.** *A* pointed n-covering *is a factorization*



of the pointing map $a : 1 \to A$ as $a = p \circ i$ where p is n-truncated.

In the following, we sometimes assimilate the covering to the map $p : B \to A$ leaving the data of the pointing $i : 1 \to B$ of B implicit. As expected, a morphism f between pointed

n-coverings p and q is a pointed map which is a morphism between the underlying n-coverings, i.e. such that the factorization of the target is $q \circ (f \circ i)$:



▶ **Definition 13.** *A pointed n-covering $p : B \to A$ is* universal *when for every n-covering $q : C \to A$ there exists a unique map $f : B \to C$ of pointed n-coverings.*

▶ Remark 14. For $n = 0$, we recover the characterization of pointed universal coverings as being initial in the category of pointed coverings [8, Lemma 12].

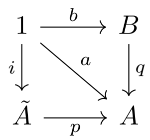
We sometimes write $\tilde{A}$ for the universal n-covering. This is justified by the fact that, being defined by a universal property, it is uniquely characterized:

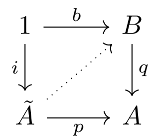
▶ **Lemma 15.** *Any two universal pointed n-coverings are uniquely isomorphic.*

We have the following characterization of universal n-coverings:

▶ **Theorem 16.** *Given a pointed type A with pointing map $a : 1 \to A$, any factorization $a = p \circ i$ as n-connected map i and followed by an n-truncated map p exhibits p as a universal n-covering. Moreover, such a factorization always exists and is unique.*

Proof. By [13, Theorem 7.6.6], the map $a : 1 \to A$ admits a unique factorization $a = p \circ i$ as required. Moreover, given a n-covering $q : B \to A$ pointed by $b : 1 \to B$, we have a commuting square as on the left





The commutation of the two triangles being given by the fact that we have two pointed n-coverings of A pointed by a. By [13, Theorem 7.6.7], because i is n-connected and p is n-truncated, there is a unique map $\tilde{A} \to B$ making the two triangles on the right commute, and p is thus universal in the sense of Definition 13. ◀

In practice, the universal n-covering can be constructed as follows. We recall from [13, Definition 7.6.3] that, given a map $f : B \to A$ and a natural number n, its *n-image* is

$$\mathrm{im}_n\, f \quad \equiv \quad \Sigma(x : A).\|\,\mathrm{fib}_f\, x\|_n$$

We write $i_n : B \to \mathrm{im}_n\, f$ for the canonical map such that $i_n(x) = (f(x), |\,\mathrm{refl}_x\,|_n)$ and $p_n : \mathrm{im}_n\, f \to A$ for the first projection map.

▶ **Proposition 17.** *The factorization $a = p_n \circ i_n$ of the pointing map a as above, exhibits $p_n : \mathrm{im}_n\, a \to A$ as the universal covering of A.*

Proof. By [13, Lemma 7.6.4], this factorization satisfies the conditions of Theorem 16. ◀

▶ Remark 18. For $n = 0$, we recover the usual definition of the universal covering space as the type of homotopy classes of paths from the distinguished point:

$$\tilde{A} \quad = \quad \Sigma(x : A).\|\star = x\|_0$$

For $n = -1$, the universal covering of A is its set of connected components. Finally, for $n = \infty$ (we adopt the convention that $\|A\|_\infty \equiv A$), we recover the universal pointed type of Section 3.2 by contractibility of singletons [13, Lemma 3.11.8].

▶ **Example 19.** The universal covering of $A \equiv S^1$ is $\Sigma(x : A).(\star = x)$ (we can remove the 0-truncation because S^1 is a groupoid) and thus contractible by [13, Lemma 3.11.8] (see Lemma 26 for a generalization of this argument).

▶ Remark 20. The factorization results used above hold more generally for any map $a : B \to A$ where B is not necessarily contractible. In this sense, given an arbitrary map $a : B \to A$, we can think of $\mathrm{im}_n\, a$ as the "universal n-cover of A relative to a".

The universal n-covering can also be characterized as the covering whose total space is $(n+1)$-connected. This can be shown using the following lemma proved in appendix.

▶ **Lemma 21.** *A pointed connected type A is $(n+1)$-connected (resp. $(n+1)$-truncated) if and only if the pointing map $1 \to A$ is n-connected (n-truncated).*

▶ **Theorem 22.** *Given a pointed connected type A, the universal pointed n-covering is the $(n+1)$-connected pointed n-covering of A.*

Proof. By Definition 12, a pointed n-covering $p : B \to A$ is a factorization of the pointing map $a : 1 \to A$ as $a = p \circ i$ with p n-truncated. By Theorem 16, it is universal if and only if $i : 1 \to B$ is n-connected which, by Lemma 21 is equivalent to the fact that B is $(n+1)$-connected. ◀

▶ Remark 23. For $n = 0$, we recover the fact that the universal covering is the only 1-connected covering of a type [8, Lemma 11].

We now formalize the intuition that the universal n-covering $\tilde{A}$ of A provides a way to "kill" all the homotopy in dimension $i \leq n+1$. This is based on what we call the *fundamental fibration* associated to the universal n-covering:

▶ **Theorem 24.** *Writing $\tilde{A}$ for the universal n-covering, we have a fiber sequence*

$$\tilde{A} \xrightarrow{\quad p \quad} A \xrightarrow{|-|_{n+1}} \|A\|_{n+1}$$

Proof. We have

$$\ker |-|_{n+1} \quad\equiv\quad \Sigma(x : A).(|\star|_{n+1} = |x|_{n+1}) \quad=\quad \Sigma(x : A).\|\star = x\|_n \quad=\quad \tilde{A}$$

where middle equality is [13, Theorem 7.3.12] and right one is Proposition 17. ◀

This was actually taken to be the definition of n-coverings in [2]. As a corollary, we have the following characterization of the homotopy groups of the universal n-covering:

▶ **Proposition 25.** *We have $\pi_i(\tilde{A}) = 1$ for $i \leq n+1$ and $\pi_i(\tilde{A}) = \pi_i(A)$ for $i > n+1$.*

In particular, this suggests that the universal covering should be contractible when A has no homotopy in dimension $i > n+1$:

▶ **Lemma 26.** *Given a $(n+1)$-truncated pointed type A, its universal n-covering is contractible.*

Proof. Since A is supposed to be $(n+1)$-truncated, the pointing map $a : 1 \to A$ is n-truncated by Lemma 21 and the factorization $a = a \circ \mathrm{id}_1$ of the pointing map as an n-connected map followed by an n-truncated map has to be the factorization of the universal covering (Theorem 16) by uniqueness. ◀

We would like to end this section with the following conjecture, which would allow constructing deloopings of higher groups based on the previous construction. Note that, below, the coverings are not supposed to be pointed.

▶ **Conjecture 27.** *Given a connected type A, the connected component of $\tilde{A}$ in $\mathrm{Covering}_n(A)$ is $\|A\|_{n+1}$.*

Provided that this conjecture holds, the connected component of $\tilde{A}$ in A would be a pointed $(n{+}1)$-connected groupoid, which can thus be thought of as a delooping of the fundamental n-group of A. The following proposition shows that we have right underlying type (it would remain to be shown that we have the right higher operations for the n-group):

▶ **Proposition 28.** *Given a pointed connected type A, the type of automorphisms of the universal n-covering $\tilde{A}$ is $\|\Omega A\|_n$.*

Proof. By universal property of $\tilde{A}$ (Definition 5), an automorphism $f : \tilde{A} \to \tilde{A}$ is uniquely determined by $f\star$ which is an element of $\mathrm{fib}_p \star$. The type of automorphisms of $\tilde{A}$ is thus $\mathrm{fib}_p \star$. Now, by Proposition 17, we can consider that p is the first projection $p : \Sigma(x : A).\|\mathrm{fib}_a x\|_n \to A$ whose fiber at $\star$ is $\|\mathrm{fib}_a \star\|_n$ by [13, Lemma 4.8.1], i.e. $\|\Omega A\|_n$. ◀

▶ Remark 29. Let us explain why the conjecture does hold in the case $n = 0$. Namely, writing

$$\mathrm{Comp}(p_0) \quad\equiv\quad \Sigma(p : \mathrm{Covering}(A)).\|p_0 = p\|_{-1}$$

for the connected component of the universal 0-covering $p_0 : \tilde{A} \to A$, we have

$$\Omega_{p_0} \mathrm{Comp}(p_0) = \Omega_{p_0} \mathrm{Covering}(A) = \|\Omega A\|_0 \equiv \pi_1(A)$$

where the first equality follows from the fact that the canonical projection map from $\mathrm{Comp}(p_0)$ to $\mathrm{Covering}(A)$ is an embedding [13, Lemma 7.6.4], and the second one is due to Proposition 28. Moreover, this identity is compatible with the group structures on both sets.

Given a group G, consider a delooping $A \equiv \mathrm{B}\,G$. By Grothendieck duality, the type of coverings of $\mathrm{B}\,G$ coincide with maps $\mathrm{B}\,G \to \mathrm{Set}$ (see Remark 11), i.e. with the type Set_G of sets equipped with an action of G [8, Theorem 4]. Moreover, under this identification, the universal covering corresponds to the principal G-set P_G, which is the set G equipped with the canonical action induced by right multiplication. Namely, $\mathrm{B}\,G$ being a 1-truncated pointed type, its universal covering is contractible by Lemma 26 and is thus the pointing map $p : 1 \to \mathrm{B}\,G$, and the corresponding map $\phi : \mathrm{B}\,G \to \mathrm{Set}$ is $\phi \equiv \mathrm{fib}_p \equiv (x \mapsto \star = x)$. In particular, we have $\phi(\star) = (\star = \star) = \Omega\,\mathrm{B}\,G = G$. Moreover, given an element a of G, seen as path $a : \star = \star$, we have $(\phi^{=} a)^{\to}(b) = ab$ by the formula for transport in identity types [13, Lemma 2.11.2]. We thus have that the connected component of the principal G-set

$$\mathrm{Comp}(P_G) \equiv \Sigma(X : \mathrm{Set}_G).\|P_G = X\|_{-1}$$

is a delooping of G. This type is known as the type of *G-torsors* [1, 3, 6, 15].

4 The Galois correspondence

4.1 The Galois fibration

From now on, we restrict ourselves to the case $n = 0$ of n-coverings. In order to define the Galois correspondence, we first need to define the action of the fundamental group $\pi_1(A)$ of a pointed connected type A on its universal cover $\tilde{A}$. This means that we want to define a map $F : \mathrm{B}\,\pi_1 A \to \mathcal{U}$ such that $F\star = \tilde{A}$. By the Grothendieck duality (Theorem 1), this amounts to define a map $f : X \to \mathrm{B}\,G$, for some type X, whose fiber is $\tilde{A}$. Moreover, the source X has to be the homotopy quotient of $\tilde{A}$ under this action, which is known to coincide with the strict quotient because the action is free, and should thus be A itself by Proposition 4, see Section 3.1. Another important observation, is that we have a very convenient model for $\mathrm{B}\,\pi_1 A$, namely:

▶ **Proposition 30.** *Given a pointed connected type A, we have that $\|A\|_1$ is a $\mathrm{B}\,\pi_1 A$.*

Proof. We take $|\star|_1$ to be the distinguished point of $\|A\|_1$. Connectedness is preserved by truncation: we have $\|\|A\|_1\|_0 = \|A\|_0 = 1$ (the first equality is [13, Lemma 7.3.15] and the second one is the fact that A is connected) and thus $\|A\|_1$ is connected. Finally, we have $\Omega\,\|A\|_1 = \|\,\Omega\,A\|_0 \equiv \pi_1 A$ by [13, Corollary 7.3.13]. ◀

The previous discussion suggests defining:

▶ **Definition 31.** *Given a pointed connected type A, the associated* Galois fibration *is the map $|-|_1 : A \to \|A\|_1$, which we write g_A in the following.*

Namely, its target is $\mathrm{B}\,\pi_1 A$ by Proposition 30 and we have the expected fiber as the case $n = 0$ of Theorem 24:

▶ **Proposition 32.** *We have $\ker g_A = \tilde{A}$, i.e. we have a fiber sequence $\tilde{A} \longrightarrow A \xrightarrow{g_A} \mathrm{B}\,\pi_1 A$.*

As an interesting immediate consequence of this result, we recover the fact that the fibers of the universal covering are the fundamental group:

▶ **Proposition 33.** *We have a fiber sequence $\pi_1 A \longrightarrow \tilde{A} \longrightarrow A$.*

Proof. By [13, Section 8.4], the fiber sequence of Proposition 32 can be extended on the left by $\Omega\,\mathrm{B}\,\pi_1 A$, which is $\pi_1 A$ by definition of the delooping. ◀

A careful reader could wonder why the action encoded by Proposition 32 is actually the "right" one in the sense that it corresponds to the traditional one in topology. Namely, we can expect that there are other such actions, i.e. maps $f : A \to \mathrm{B}\,\pi_1 A$ with $\ker f = \tilde{A}$ (what will be important here is that $\tilde{A}$ is 1-connected by Remark 23). In fact, it turns out is only one possible such action, up to an automorphism of $\mathrm{B}\,\pi_1 A$:

▶ **Proposition 34.** *Given a pointed connected type A and 1-connected map $f : A \to \mathrm{B}\,\pi_1 A$, there is an equivalence $e : \mathrm{B}\,\pi_1 A \to \mathrm{B}\,\pi_1 A$ for which there is a commuting triangle*

$$\begin{array}{ccc} & A & \\ {\scriptstyle g_A}\swarrow & & \searrow{\scriptstyle f} \\ \mathrm{B}\,\pi_1 A & \xrightarrow[e]{\sim} & \mathrm{B}\,\pi_1 A \end{array}$$

Proof. By naturality of truncation [13, equation (7.3.4)], we have a commutative square

$$\begin{array}{ccc} A & \xrightarrow{|-|_1} & \|A\|_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \|f\|_1} \\ \mathrm{B}\,\pi_1 A & \xrightarrow[|-|_1]{} & \|\,\mathrm{B}\,\pi_1 A\|_1 \end{array}$$

By definition, the upper map is g_A. The lower map is an equivalence because $\mathrm{B}\,\pi_1 A$ is a groupoid [13, Corollary 7.3.7]. The right vertical map is also an equivalence by [13, Lemma 7.5.14], because f is supposed to be 1-connected. Finally, f is an equivalence as a composite of equivalences. ◀

The situation above is a bit subtle. Namely, it states that a 1-connected map as f has to be the Galois fibration. However, this is up to an automorphism of $\mathrm{B}\,\pi_1 A$, which might itself bear some information. Namely, pointed automorphisms of $\mathrm{B}\,\pi_1 A$ correspond to group automorphisms of $\pi_1 A$ which might be non-trivial. In all the applications below, insights from the corresponding constructions in algebraic topology allow us to make sure that we indeed have the right action.

4.2 The Galois correspondence

In algebraic topology, we have seen in Proposition 4 that, given a nice pointed space A, there is an action of its fundamental group on its cover. This is in fact the basis of a classification of coverings: those correspond to the subgroups of the fundamental group, see for instance [7, Theorem 1.38]. In homotopy type theory, the action is encoded by the Galois fibration (Proposition 32) and our aim is now to develop a similar classification of coverings.

In this section, we write $\mathrm{Covering}(A)$ for the coverings of A which are pointed and *connected*, i.e. whose total space is connected. Given a group G, we also write $\mathrm{Subgroup}(G)$ for the type of subgroups of G, i.e. injective maps $i : H \to G$ for some group H.

▶ **Lemma 35.** *Given a subgroup $i : H \to G$, the fiber of $\mathrm{B}\,i$ merely is the set G/H.*

▶ **Theorem 36.** *There is an equivalence between subgroups of $\pi_1(A)$ and pointed connected coverings of A:*

$$\mathrm{Subgroup}(\pi_1(A)) \simeq \mathrm{Covering}(A)$$

Proof. Given a subgroup $i : G \hookrightarrow \pi_1(A)$, the corresponding covering X is obtained as pullback of the Galois fibration along the delooping of the inclusion of groups:

$$\begin{array}{ccc} C_G & \longrightarrow & \mathrm{B}\,G \\ {\scriptstyle p_G}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{B}\,i} \\ A & \xrightarrow[g_A]{} & \mathrm{B}\,\pi_1(A) \end{array}$$

We need to show that X is a connected covering, i.e. it is connected and p_G has 0-truncated fibers. The fiber of the covering is given by glueing pullbacks

$$\begin{array}{ccccc} F_G & \longrightarrow & C_G & \longrightarrow & \mathrm{B}\,G \\ \big\downarrow & \lrcorner & {\scriptstyle p_G}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{B}\,i} \\ 1 & \xrightarrow[\star]{} & A & \xrightarrow[g_A]{} & \mathrm{B}\,\pi_1(A) \end{array}$$

and thus we have

$$F_G \equiv \ker p_G = \ker \mathrm{B}\,i = \pi_1(A)/G$$

by Lemma 35 (to be precise, we only have the existence of such an equality, which is sufficient for our purposes exposed in next sentence). As a consequence, the fibers of p_G are sets. Also, by vertical preservation of the fiber,

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow \\ C_G & \longrightarrow & \mathrm{B}\,G \\ {\scriptstyle p_G}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{B}\,i} \\ A & \xrightarrow[g_A]{} & \mathrm{B}\,\pi_1(A) \end{array}$$

we have the upper square which is a pullback, i.e.

$$\tilde{A} \longrightarrow C_G \longrightarrow \mathrm{B}\,G$$

which says, by action-fibration duality, that

$$C_G = \tilde{A} /\!\!/ G$$

Therefore, C_G is connected as a homotopy quotient of a connected space.

Conversely, given a connected covering $f : X \to A$, we have

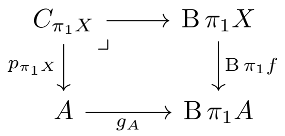
$$\pi_1(f) : \pi_1(X) \to \pi_1(A)$$

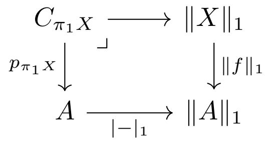
This is a mono, because we have the long exact sequence associated to the fibration:

$$\pi_1(F) \longrightarrow \pi_1(X) \xrightarrow{\pi_1(f)} \pi_1(A)$$

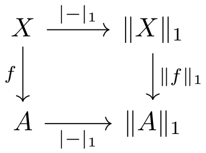
where $F \equiv \ker f$ is a set (because f is covering) and thus $\pi_1(F) = 1$.

We now have to show that these two operations are mutually inverse. Given a connected covering $f : X \to A$, the associated connected covering (by performing the two operations) is obtained as the pullback on the left, which can be rewritten as on the right

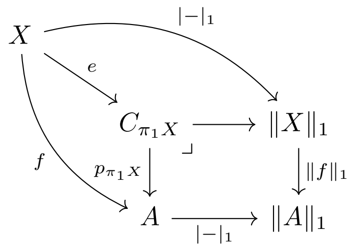




Moreover, we have a commuting square



and thus a universal map $e : X \to C_{\pi_1 X}$ such that



More explicitly,

$$C_{\pi_1 X} \quad \equiv \quad \Sigma(a : A).\Sigma(y : \|X\|_1).(|a|_1 = \|f\|_1(y))$$

and $e(x) = (f(x), |x|_1, \mathrm{refl}_{|f(x)|_1})$. Because the lower-left triangle commutes, e amounts to a family of maps

$$\begin{aligned} e_a : \mathrm{fib}_f\, a &\to \mathrm{fib}_{\pi_1 X}\, a \\ (x, p) &\mapsto (f(x), |x|_1, \mathrm{refl}, p) \end{aligned}$$

indexed by $a : A$, with $p : a = f(x)$. Let us consider the case $a \equiv \star$. We have

$$\mathrm{fib}_{\pi_1 X}\, \star \quad \equiv \quad \Sigma(a : A).\Sigma(y : \|X\|_1).\Sigma(q : |a|_1 = \|f\|_1(y)).(\star = a)$$

We can construct an inverse map

$$\begin{aligned} e'_\star : \mathrm{fib}_{\pi_1 X}\, \star &\to \mathrm{fib}_f\, \star \\ (a, y, q, p) &\mapsto (x, p \cdot \tilde{q}) \end{aligned}$$

where we assume that $y = |x|_1$ because we are eliminating to $\mathrm{fib}_f\, \star$ which is a set (and thus a groupoid). Above, we have $p : |a|_1 = \|f\|_1(x)$, which is equivalent to $\|a = f(x)\|_0$ and we can thus suppose that $q = |\tilde{q}|_0$ with $\tilde{q} : a = f(x)$ because we are eliminating to a set. We have

$$e' \circ e(x, p) = (x, p \cdot \mathrm{refl}) = (x, p)$$

Conversely,

$$e \circ e'(a, |x|_1, |q|_0, p) = e(x, p \cdot q) = (f(x), |x|_1, \mathrm{refl}, p \cdot q)$$

(we can suppose that the second and third arguments are truncations as above, because we are eliminating to a set). We thus have to show

$$(f(x), |x|_1, \mathrm{refl}, p \cdot q) = (a, |x|_1, |q|_0, p)$$

with $p : \star = a$ and $q : a = f(x)$. Abstracting over a, we can suppose that q is refl by J, and we conclude immediately.

Conversely, given a subgroup $i : G \hookrightarrow \pi_1 A$, the associated subgroup (by performing the two transformations) is

$$\pi_1(p_G) : \pi_1 C_G \to \pi_1 A$$

we thus have to show that $\pi_1 C_G = G$ and the map $\pi_1 p_G = i$ (note to be precise, we should identify the sources and the targets of the maps up to equality). We have a pullback square

$$\begin{array}{ccc} C_G & \longrightarrow & \mathrm{B}\,G \\ {\scriptstyle p_G}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{B}\,i} \\ A & \xrightarrow[g_A]{} & \mathrm{B}\,\pi_1 A \end{array}$$

which can be extended as (see above)

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow \\ C_G & \longrightarrow & \mathrm{B}\,G \\ {\scriptstyle p_G}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{B}\,i} \\ A & \xrightarrow[g_A]{} & \mathrm{B}\,\pi_1(A) \end{array}$$

The map $C_G \to \mathrm{B}\,G$ is 1-connected because the fiber is $\tilde{A}$ which is 1-connected by Theorem 22. By [13, Lemma 7.5.14], it thus induces an equivalence $\|C_G\|_1 \simeq \|\mathrm{B}\,G\|_1$, and thus $\pi_1(C_G) = \pi_1(\mathrm{B}\,G) = G$. We should have the fact that $\pi_1 p_G = i$ similarly, by applying π_1 to the above square (which is not anymore a pullback but remains commutative). ◀

▶ **Example 37.** Consider the case $A \equiv \mathrm{S}^1$. The associated Galois fibration is

$$\mathrm{S}^1 \xrightarrow{|-|_1} \|\,\mathrm{S}^1\,\|_1 \xrightarrow{\sim} \mathrm{B}\,\pi_1\,\mathrm{S}^1 \simeq \mathrm{B}\,\mathbb{Z}$$

and is thus an equivalence. The subgroups of $\mathbb{Z}$ are $i_n : \mathbb{Z} \to \mathbb{Z}$ with $i_n(k) = n \times k$ with $n > 0$ or $i_0 : 0 \to \mathbb{Z}$. And thus covering of the circle are of the form

$$\begin{array}{ccc} C_n & \longrightarrow & \mathrm{B}\,\mathbb{Z} \\ {\scriptstyle p_n}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{B}\,i_n} \\ \mathrm{S}^1 & \xrightarrow[g_{\mathrm{S}^1}]{} & \mathrm{B}\,\mathbb{Z} \end{array}$$

Since the lower arrow is an equivalence, the pullback is $\mathrm{B}\,\mathbb{Z}$, i.e. S^1. And $p_n : \mathrm{S}^1 \to \mathrm{S}^1$ is the pointed map sending the loop to loop^n.

▶ **Lemma 38.** *For any 1-connected map $f : X \to \mathrm{B}\,G$ its 1-truncation $\|f\|_1 : \|X\|_1 \to \mathrm{B}\,G$ is an equivalence.*

5 Applications

5.1 Coverings of lens spaces

Lens spaces were defined in homotopy type theory by the authors of this paper in [9]. We briefly recall here their definition. Given natural numbers l and n with l prime with n, we write

$$\phi_n^l : \mathrm{S}^1 \to \mathrm{B}\,\mathbb{Z}_n$$

for the pointed map sending loop to the loop in $\mathrm{B}\,\mathbb{Z}_m$ corresponding to $l : \mathbb{Z}_m$. We have a fiber sequence

$$\mathrm{S}^1 \longrightarrow \mathrm{S}^1 \xrightarrow{\phi_n^l} \mathrm{B}\,\mathbb{Z}_n$$

which is obtained by delooping the exact sequence $\mathbb{Z} \xrightarrow{-\times n} \mathbb{Z} \xrightarrow{-\times l} \mathrm{B}\,\mathbb{Z}_n$, see [9, Section 6.2].

▶ **Definition 39.** *Given a sequence $l_1, \ldots, l_k$ of natural numbers all prime with n, the associated* lens space $L_n^{l_1,\ldots,l_k}$ *is the source of the map*

$$\phi_n^{l_1} * \ldots * \phi_n^{l_k} : L_n^{l_1,\ldots,l_k} \to \mathrm{B}\,\mathbb{Z}_n$$

which we simply write as $\phi_n^{l_1,\ldots,l_k}$ in the following.

It can be shown that the fiber of this map is S^{2k-1} [9, Section 6.2], which is thus $2k-2$ connected. In particular, we have that $\pi_1 L_n^{l_1,\ldots,l_k} = \pi_1 \mathrm{B}\,\mathbb{Z}_n = \mathbb{Z}_n$.

We now classify covering spaces of lens spaces. We have seen in Theorem 36 that they correspond to subgroups of $\mathbb{Z}_n$, which are the $\mathbb{Z}_m$ with $m \mid n$. Given such a $\mathbb{Z}_m$, the corresponding covering C_m is obtained by taking the inclusion $i_m : \mathbb{Z}_m \to \mathbb{Z}_n$, delooping it, and pulling back along the Galois fibration:

$$\begin{array}{ccc} C_m & \longrightarrow & \mathrm{B}\,\mathbb{Z}_m \\ {\scriptstyle p_m}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{B}\,i_m} \\ L_n^{l_1,\ldots,l_k} & \xrightarrow[g_{L_n^{l_1,\ldots,l_k}}]{} & \mathrm{B}\,\mathbb{Z}_n \end{array}$$

In order to perform computations, we first note that, by Proposition 34, we can replace the map $g_{L_n^{l_1,\ldots,l_k}}$ at the bottom by $\phi_n^{l_1,\ldots,l_k}$ (up to an automorphism of the target).

▶ **Proposition 40.** *For natural numbers l, m, n, p with l prime with n, and $n = mp$, we have a pullback square*

$$\begin{array}{ccc} \mathrm{S}^1 & \xrightarrow{\phi_m^l} & \mathrm{B}\,\mathbb{Z}_m \\ {\scriptstyle s_p}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{B}\,i_m} \\ \mathrm{S}^1 & \xrightarrow[\phi_n^l]{} & \mathrm{B}\,\mathbb{Z}_n \end{array}$$

where the vertical map s_p sends loop *to* loop^p.

Proof. We write X for pullback of ϕ_n^l and $\mathrm{B}\,i_m$ as shown on the left below:

$$\begin{array}{ccc} X & \xrightarrow{(\mathrm{B}\,i_m)^*\phi_n^l} & \mathrm{B}\,\mathbb{Z}_m \\ {\scriptstyle (\phi_n^l)^*\mathrm{B}\,i_m}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{B}\,i_m} \\ \mathrm{S}^1 & \xrightarrow[\phi_n^l]{} & \mathrm{B}\,\mathbb{Z}_n \end{array} \qquad\qquad \begin{array}{ccc} \Omega\,X & \longrightarrow & \mathbb{Z}_m \\ \downarrow & \lrcorner & \downarrow{\scriptstyle i_m} \\ \mathbb{Z} & \xrightarrow[-\times l]{} & \mathbb{Z}_n \end{array}$$

The type X is pointed (because both maps ϕ_n^l and $\mathrm{B}\,i_m$ are) and connected (this can be shown as in [9, Lemma 26]). Finally, since loop spaces commute to pullbacks because they are right adjoints, we have a pullback square of groups as on the right above. From this, we can deduce that $\Omega\,X \cong \mathbb{Z}$ as follows. The preceding pullbacks means that we have

$$\Omega\,X \quad = \quad \Sigma((a,b) : \mathbb{Z} \times \mathbb{Z}_m).(al =_{\mathbb{Z}_n} bp)$$

We write $f : \mathbb{Z} \to \Omega\,X$ for the map sending 1 to (p, l), and we claim that this is an isomorphism.

First, f is injective because (p, l) is free. Indeed, suppose that $f(x) \equiv (xp, xl) = 0$. In particular, $xp =_{\mathbb{Z}} 0$ and since $p \neq 0$ we have $x = 0$. Second, f is surjective. Namely, fix $(a, b) : \Omega X$. There is $y \in \mathbb{Z}$ such that $al = bp + yn = p(b + ym)$. Therefore $p \mid al$, but l is prime with n and $p \mid n$ so that l is prime with p and thus $p \mid a$. There thus exists x such that $a = px$. Thus $pxl - bp =_{\mathbb{Z}_n} 0$, i.e. $p(xl - b) =_{\mathbb{Z}_n} 0$, that is $mp \equiv n \mid p(xl - b)$ and thus $m \mid xl - b$, i.e. $b =_{\mathbb{Z}_m} xl$. Finally, $(a, b) = (xp, xl) \equiv f(x)$ and f is surjective. The maps of the pullback are the projections from ΩX and are thus the expected ones. ◀

▶ **Theorem 41.** *For $l_1, \ldots, l_k$ natural numbers prime with n, and $k > 1$, the coverings of $L_n^{l_1,\ldots,l_k}$ are precisely the $L_m^{l_1,\ldots,l_k}$ with $m \mid n$, and the projections maps are given by $s_p *_{\mathrm{B}\,\mathbb{Z}_n} \cdots *_{\mathrm{B}\,\mathbb{Z}_n} s_p$.*

Proof. By Theorem 36, the coverings of $L_n^{l_1,\ldots,l_k}$ correspond to subgroups of $\pi_1 L_n^{l_1,\ldots,l_k}$, i.e. the subgroups of $\mathbb{Z}_n$, and those are of the form $\mathbb{Z}_m$ for $m \mid n$. More precisely, by Theorem 36, given such a subgroup $i_m : \mathbb{Z}_m \to \mathbb{Z}_n$, the corresponding covering space is given by the pullback of the delooping this map along the Galois fibration:

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{B}\,\mathbb{Z}_m \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{B}\, i_m} \\ L_n^{l_1,\ldots,l_k} & \xrightarrow[g_{L_n^{l_1,\ldots,l_k}}]{} & \mathrm{B}\,\mathbb{Z}_n \end{array}$$

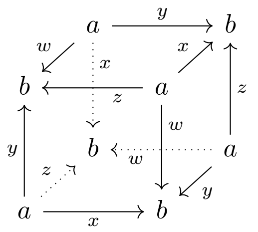
As remarked in Proposition 34, we can replace the bottom map by $\phi_n^{l_1,\ldots,l_k}$. By [9, Theorem 25], pullback commute with joins of maps, so the map $X \to \mathrm{B}\,\mathbb{Z}_n$ can be computed as the iterated join of the maps obtained by pulling back $\phi_n^{l_i}$ along $\mathrm{B}\, i_m$, which, by Proposition 40, are the $\phi_m^{l_i} : \mathrm{S}^1 \to \mathrm{B}\,\mathbb{Z}_m$. The pullback of $\phi_n^{l_1,\ldots,l_k}$ along $\mathrm{B}\, i_m$ is thus $\phi_m^{l_1,\ldots,l_k}$. Similarly, by [9, Theorem 25] and Proposition 40, the vertical map $X \to L_n^{l_1,\ldots,l_k}$ is obtained as the join of n instances of s_p. Finally, we obtain the pullback square

$$\begin{array}{ccc} L_m^{l_1,\ldots,l_k} & \xrightarrow{\phi_m^{l_1,\ldots,l_k}} & \mathrm{B}\,\mathbb{Z}_m \\ {\scriptstyle s_p^{*^k_{\mathrm{B}\,\mathbb{Z}_n}}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{B}\, i_m} \\ L_n^{l_1,\ldots,l_k} & \xrightarrow[g_{L_n^{l_1,\ldots,l_k}}]{} & \mathrm{B}\,\mathbb{Z}_n \end{array}$$

◀

5.2 Constructing the hypercubical manifold and the homology sphere

We would like to illustrate here another kind of situation where the Galois fibration occurs when constructing types corresponding to well-known spaces, albeit in a somewhat hidden way. The general idea is the following. Suppose that we have a group G and we want to define a coherent action of G on a type A, which is not supposed to be truncated (in particular, it might not be a set). This amounts to define a map $\psi : \mathrm{B}\, G \to \mathcal{U}$ such that $\psi \star = A$, which requires eliminating to a type which is not a groupoid, and thus difficult to perform directly. But we can use the action-fibration duality, which brings a fresh point of view on the problem. By the Grothendieck duality, provided we can construct the homotopy quotient $\overline{A}$ of A under the action of G, constructing the action amounts to defining a map $\phi : \overline{A} \to \mathrm{B}\, G$. However, it is not clear which map this should be. When the type A is simply connected, we know that ϕ has to be the Galois fibration by Proposition 34.



The hypercubical manifold. We have defined and studied in [10] a type corresponding to the *hypercubical manifold*. Topologically, this manifold K was defined by Poincaré as a space obtained by identifying the opposite faces of a cube after a quarter-turn rotation, and can be pictured as on the right. The fundamental group of this space is the quaternion group Q and its universal covering is the 3-sphere S^3. Thus, K can also be obtained as a quotient of S^3 under Q.

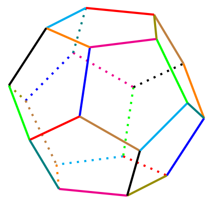
In homotopy type theory, a type corresponding to K is easily defined as a higher inductive type. However, in order to work with it and validate its construction, we need to show that it can be obtained as a quotient of S^3 under the action of Q, i.e. from a map $\mathrm{B}\,Q \to \mathcal{U}$ such that the image of $\star$ is S^3. However, such a map is difficult to construct directly using the elimination principle of $\mathrm{B}\,Q$ because S^3 is not n-truncated for any n. As explained above, we can instead adopt a fibrational point of view and construct a map $\phi : K \to \mathrm{B}\,Q$ whose fiber is S^3, i.e. a fiber sequence

$$\mathrm{S}^3 \longrightarrow K \xrightarrow{\phi} \mathrm{B}\,Q$$

thus showing that K is a quotient of S^3 under an action of Q. Details can be found in [10].

The homology sphere. When investigating the notion of homology, it was not clear at first whether homology would be fine enough in order to characterize homotopy types. It turns out that this is not the case, which was first shown by Poincaré by introducing a space, the *homology sphere* (also known as the *Poincaré dodecahedral space*), which has the same homology type as the sphere S^3, but not the same homotopy type [11].

Following [5], we define the homology sphere D as a higher inductive type corresponding to a dodecaedron where each face is identified with the opposite one after a rotation of a fifth-turn. The classes of 0-, 1- and 2-cells respectively have 4, 3 and 2 elements, and edges are identified as indicated by the colors on the right. The resulting space has one 3-cell, 6 2-cells, 10 1-cells and 5 0-cells. We have a fibration



$$\mathrm{S}^3 \longrightarrow D \xrightarrow{\phi} \mathrm{B}\,\pi_1(D)$$

The fundamental group $\pi_1(D)$ is the group of order 120, known as the *binary icosahedral group*, which can be presented as $\langle r, s, t \mid r^2 = s^3 = t^5 = rst \rangle$. As for the hypercubical manifold, we can compute the fundamental group of D in two steps: we first construct the fundamental groupoid (whose 0-, 1- and 2-cells are generated by the 0-, 1- and 2-cells involved in the description of D as a cellular complex), and then we contract 1-cells in order to obtain a model with only one 0-cell, which is thus a delooping of a group, i.e. $\mathrm{B}\,\pi_1(D)$. The map ϕ is then the canonical one induced by this process. Finally, since D is constructed by attaching cells, i.e. has a canonical description as a colimit, we can use the flattening lemma in order to compute its fiber, which we claim to be S^3. This will be detailed in subsequent works.

6 Future works

We have defined and studied n-covering types, and formalized their classification for $n = 0$. In the future, we would like to provide an explicit construction of covering types of many interesting and natural types (such as the hypercubical manifold and the homology sphere presented above). In passing, we would like to mention here that we show in [3, 4] that Cayley complexes are universal coverings (and Cayley graphs are universal (-1)-coverings). A natural question is also whether the Galois correspondence can be extended to higher coverings. Its exploration is left for future works.

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A Additional proofs

Proof of Lemma 21. First, consider a property $P : (x, y : A) \to (p : x = y) \to \mathcal{U}$ such that $P\,x\,y\,p$ is a proposition for any $x, y : A$ and $p : x = y$. Since A is pointed connected, the following are equivalent

(i) $P\,x\,y\,p$ for any x, y and p,

(ii) $P \star \star p$ holds for any $p : \star = \star$.

Namely, the second is a particular case of the first. Conversely, suppose (ii) holds, and that we are given x, y, and p. Since we want to show $P\,x\,y\,p$ which is a proposition, we can suppose given paths $p_x : \star = x$ and $p_y : \star = y$ because A is connected, we then deduce that $P\,x\,y\,p$ holds from (ii) by transport.

Now, we have that A is $(n{+}1)$-connected if and only if $x = y$ is n-connected for any $x, y : A$ (see Section 2) if and only if $\Omega\,A$ is n-connected (by the preceding observation, based on the fact that being n-connected is a proposition [13, Theorem 7.1.10]) if and only if the pointing map $a = 1 \to A$ is n-connected (because we have $\Omega\,A = \mathrm{fib}_a \star$). The reasoning is similar for the truncated version. ◀

Proof of Proposition 25. Since $\tilde{A}$ is $(n{+}1)$-connected by Theorem 22, we have $\pi_i(\tilde{A}) = 1$ for $i \leq n + 1$ [13, Lemma 8.3.2]. For $i > n + 1$, this is a consequence of the long exact sequence induced by the fiber sequence of Theorem 24, see [13, Theorem 8.4.6]:

$$\cdots \longrightarrow \pi_{i+1}\|A\|_{n+1} \longrightarrow \pi_i\tilde{A} \longrightarrow \pi_i A \longrightarrow \pi_i\|A\|_{n+1} \longrightarrow \pi_{i+1}\tilde{A} \longrightarrow \cdots$$

We namely have $\pi_i\|A\|_{n+1} = 1$ and the short exact sequence

$$1 \longrightarrow \pi_i\tilde{A} \longrightarrow \pi_i A \longrightarrow 1$$

shows that $\pi_i\tilde{A} = \pi_i A$. ◀

The following lemma is used in the subsequent proof of Lemma 35:

▶ **Lemma 42.** *Suppose given a group morphism $i : H \to G$ and a function $f : G \to X$, where X is a set, which is invariant under the action of H, and such that $\mathrm{fib}_f\,x$ is the group H equipped with its canonical action on itself for every $x : X$. Then $X = G/H$.*

Proof. Recall that the quotient is the set truncation of the homotopy coequalizer

$$G \times H \overset{\pi}{\underset{\alpha}{\rightrightarrows}} G \dashrightarrow G/\!\!/H$$

with π the first projection and α the right action of H on G (we have $\alpha(a, b) \equiv a \times i(b)$), i.e. $G/H = \|G/\!\!/H\|_0$. Otherwise said, G/H satisfies the same universal property as the above equalizer but restricted to sets. Our aim is now to show that $f : G \to X$ satisfies this universal property. The hypothesis that f is invariant under the action of H precisely means that it coequalizes the two maps. Suppose given another coequalizing map $g : G \to Y$ with Y a set. We need to show that there exists a unique map $h : X \to Y$ such that $h \circ f = g$:

$$\begin{array}{ccccc} G \times H & \overset{\pi}{\underset{\alpha}{\rightrightarrows}} & G & \xrightarrow{f} & X \\ & & & \searrow^{g} & \downarrow{\scriptstyle h} \\ & & & & Y \end{array}$$

The existence of such an h is implied by the fact that we have, for every $x : X$, an element $a : \mathrm{fib}_f\,x$ such that for every element $b : \mathrm{fib}_f\,x$, we have $g\,a = g\,b$. In turn, under the identification of $\mathrm{fib}_f\,x$ with H, we can take A to be the neutral element of H and the facts that the action of H on itself is transitive and that g preserves this action ensures that any other image will be equal to this one. ◀

Proof of Lemma 35. Recall that we have

$$\ker \mathrm{B}\, i \quad \equiv \quad \Sigma(x : \mathrm{B}\, H).(\star = \mathrm{B}\, i\, x)$$

We define a map

$$f : G \to \ker \mathrm{B}\, i$$

by $f(a) \equiv (\star, p_a)$ for $a : G$ corresponding to a path $p_a : \star = \star$ in $\mathrm{B}\, G$. The fibers of this map are

$$\begin{aligned} \mathrm{fib}_f(x, p) &\equiv \Sigma(a : G).((x, p) = (\star, p_a)) \\ &= \Sigma(a : G).\Sigma(q : x = \star).(p \cdot (\mathrm{B}\, i)^{=}(q) = p_a) \end{aligned}$$

In particular, for $x \equiv \star$, we have $p \equiv p_b$ for some $b : G$ and the fiber is

$$\begin{aligned} \mathrm{fib}_f(\star, p_b) &= \Sigma(a : G).\Sigma(c : H).(b \times i(c) = a) \\ &= \Sigma(c : H).\Sigma(a : G).(b \times i(c) = a) \\ &= H \end{aligned}$$

Since $\mathrm{B}\, H$ is connected, we can deduce that $\mathrm{fib}_f(x, p)$ merely is H for any $(x, p) : \ker \mathrm{B}\, i$. By Lemma 42, we deduce that $\ker \mathrm{B}\, i$ merely is H. ◀