Asynchronous Games
Innocence without Alternation

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CONCUR 2007
September 7, 2007

Denotational semantics

Giving properties of programs which are invariant during the execution.

Game semantics

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Game semantics

1. **formulas** A are interpreted by **games**
2. **proofs** $\pi : A \to B$ are interpreted by **strategies**

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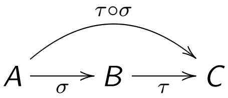
We also want composition (and other structures) to be preserved by the interpretation.



Concurrency in game semantics

Game semantics is a *trace semantics*.

The program P emits and receives moves

$$P \xrightarrow{m_0} P_1 \xrightarrow{m_1} P_2 \xrightarrow{m_2} \cdots$$

played in a game.

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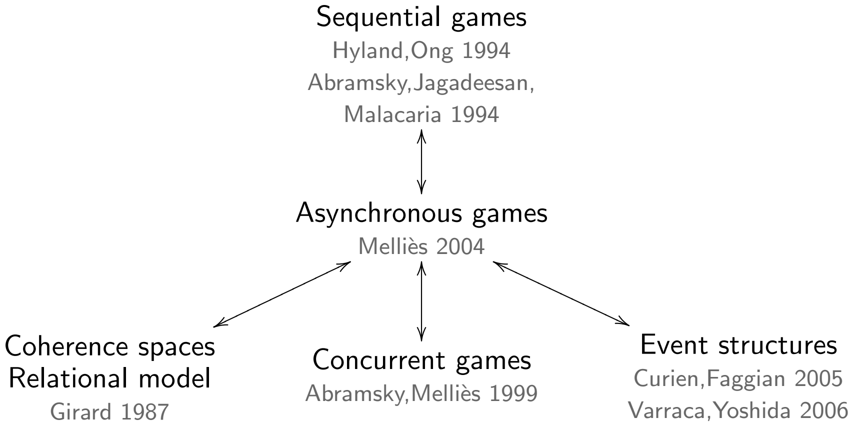
Here, we will refine it as

a Mazurkiewicz trace semantics for proofs

based on event structures.

Unifying semantics of linear logic



Part I

Asynchronous games

Asynchronous games

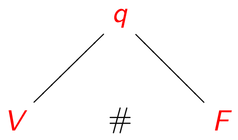
A *2-player* **event structure**

$$(M, \leq, \#, \lambda)$$

consisting of

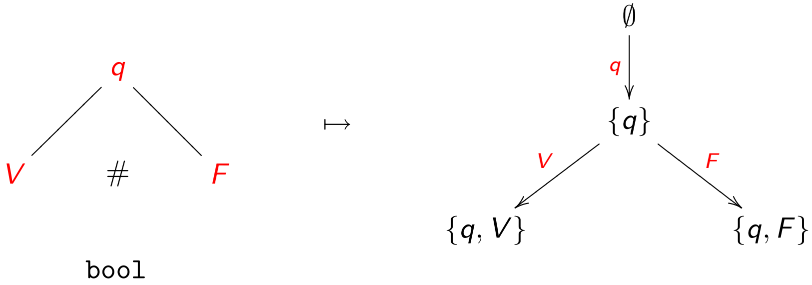
- a set of **moves** M
- a partial order $\leq$ expressing **causal dependencies**
- a symmetric relation $\#$ expressing **incompatibilities**
- a **polarization** of moves $\lambda : M \to \{O, P\}$

`bool` $=$



Playing in games

- **positions** are downward-closed sets of compatible moves
- **plays** are paths between positions, starting from $\emptyset$



Playing in games

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q

o

$\mapsto$

$$\emptyset \xrightarrow{q} \{q\}$$

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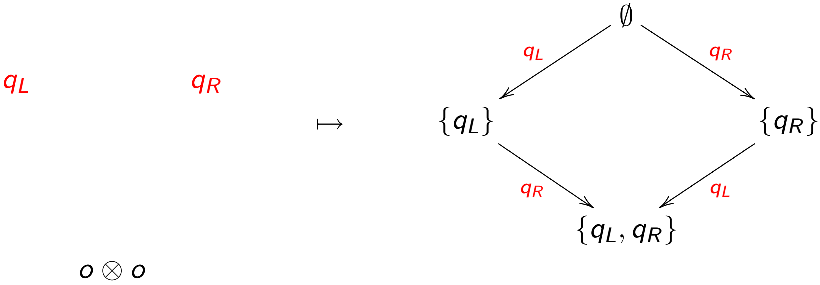

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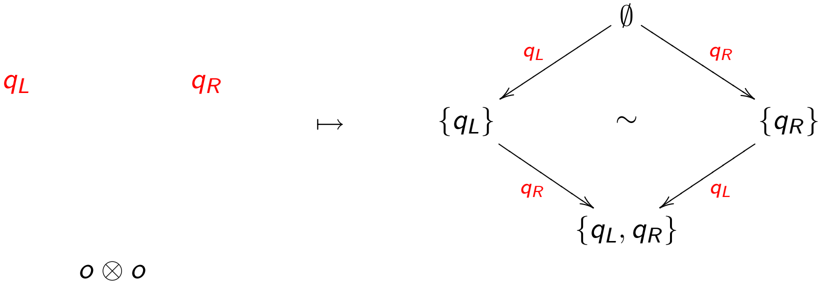
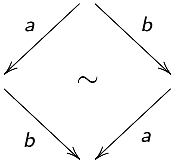
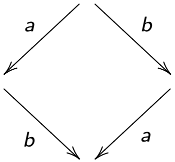


An approach to interferences

The Mazurkiewicz approach to *true concurrency*.

$a \| b$ vs. $a \cdot b + b \cdot a$



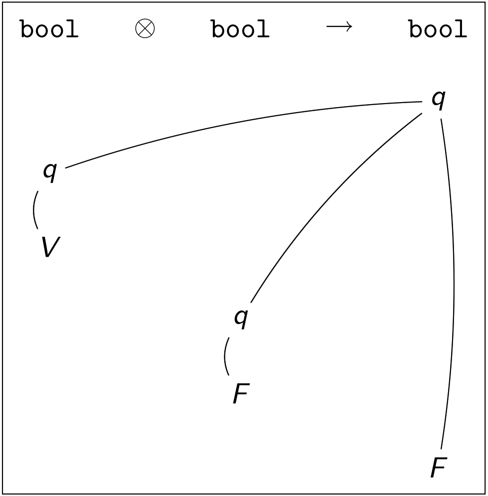


$x := 4 \| y := 5$ $x := 4 \| x := 5$

multiplicatives additives

Implementations of the conjuction

Left and



Implementations of the conjuction

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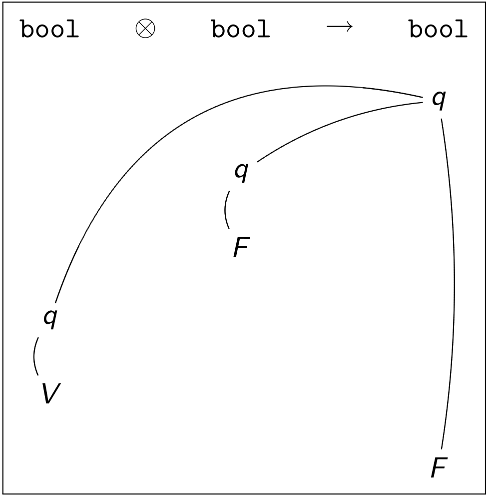

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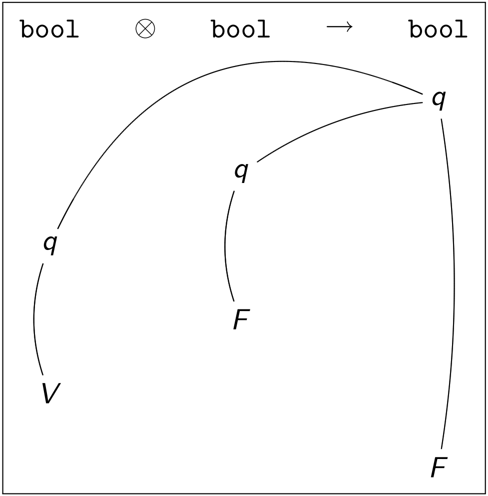

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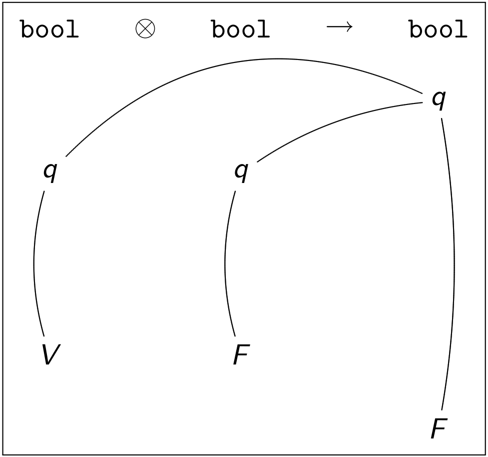

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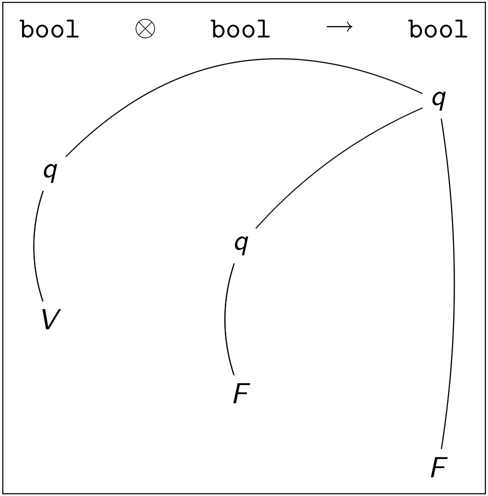

Asynchronous games

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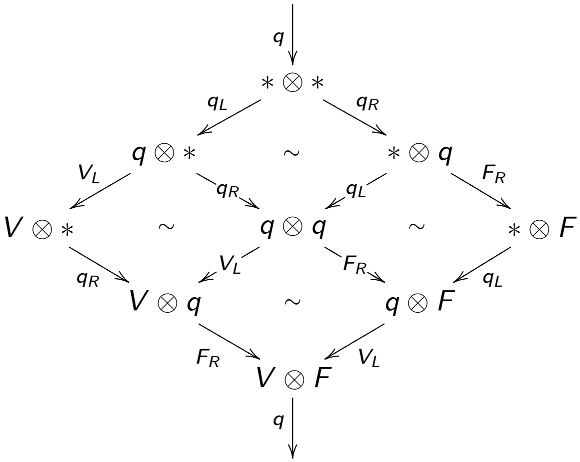

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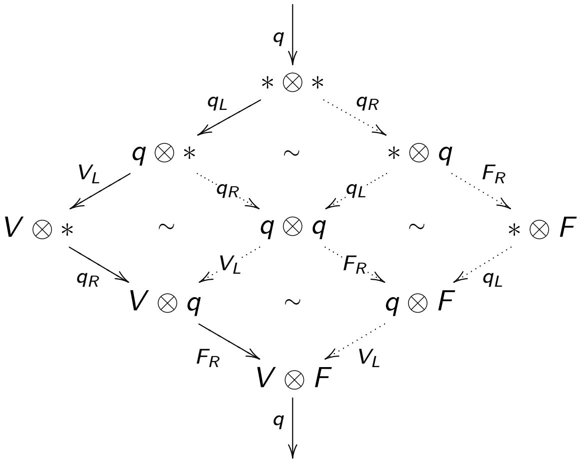
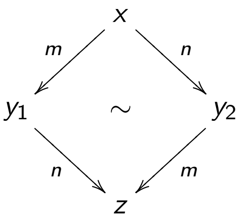


A **game** induces an *asynchronous graph* $\mathcal{G}$:

- vertices are **positions** (+ initial position $*$),
- edges are **moves**,
- 2-dimensional tiles



generate **homotopy** between paths.

A logic for game semantics

- we only consider formulas of MALL:

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}(\parr) \qquad \frac{\vdash \Gamma_1, A \qquad \vdash \Gamma_2, B}{\vdash \Gamma_1, \Gamma_2, A \otimes B}(\otimes)$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}(\&) \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}(\oplus)$$

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$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}(\&) \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}(\oplus)$$

- with explicit moves:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, \uparrow A}(\uparrow) \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, \downarrow A}(\downarrow)$$

From formulas to games

In linear logic, the formula corresponding to booleans is

$$\texttt{bool} \quad = \quad \uparrow(\downarrow 1 \oplus \downarrow 1)$$

which is like of $1 \oplus 1$ with explicit changes of polarities.

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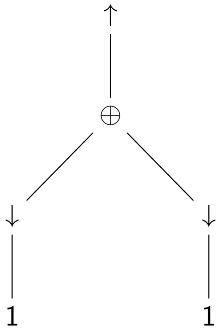
It can be drawn as



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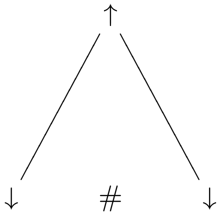

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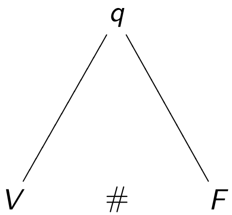

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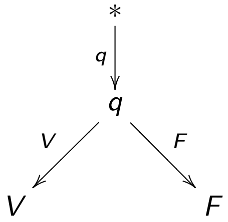

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The game associated to $\uparrow A$ is of the form

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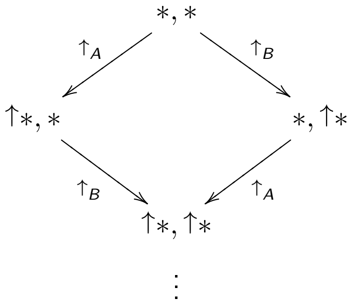
The corresponding asynchronous graph contains



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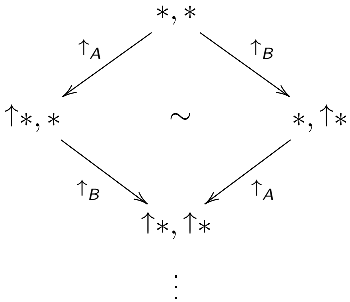

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Three proofs of $\uparrow A \parr \uparrow B$:

$$*, *$$

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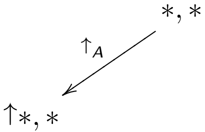
$$\dfrac{\overline{\vdash A, \uparrow B}}{\vdash \uparrow A, \uparrow B}(\uparrow)$$



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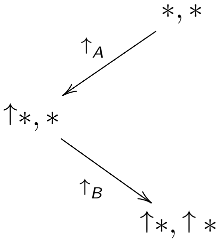

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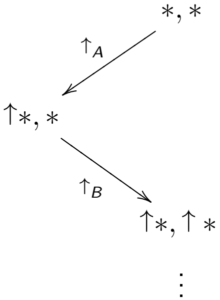

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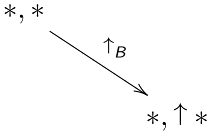

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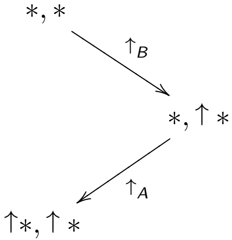

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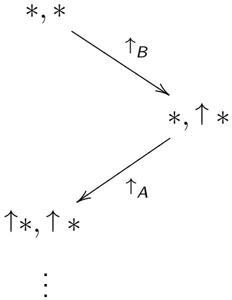

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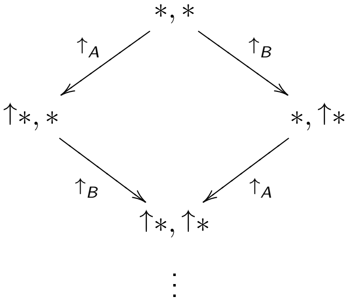

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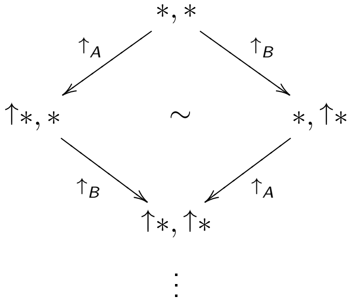

Proofs explore formulas

play	$=$	exploration of the formula
proof	$=$	strategy of exploration

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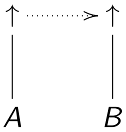

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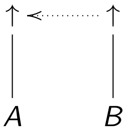

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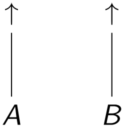


Towards innocence

Can we characterize the *definable* strategies?

We have to restrict the space of strategies.

innocent strategy = strategy behaving like a proof

Part II

Traces vs. event structures

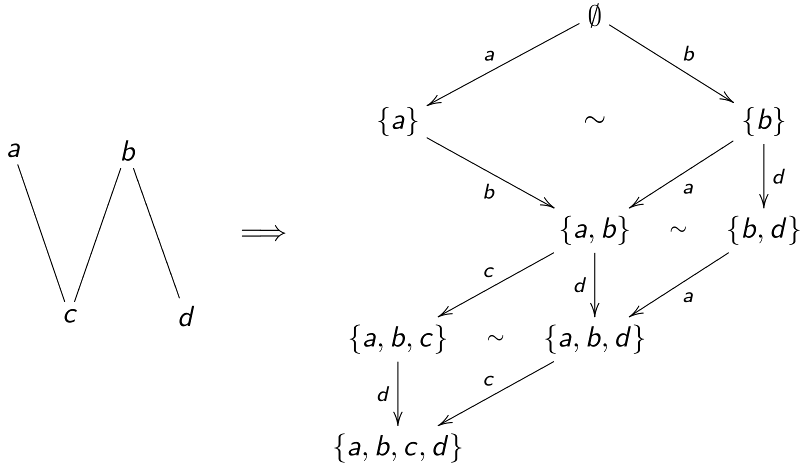
Traces vs. partial orders

formula = event structure on the moves

proof = refinement of the underlying partial order

From causal to sequential

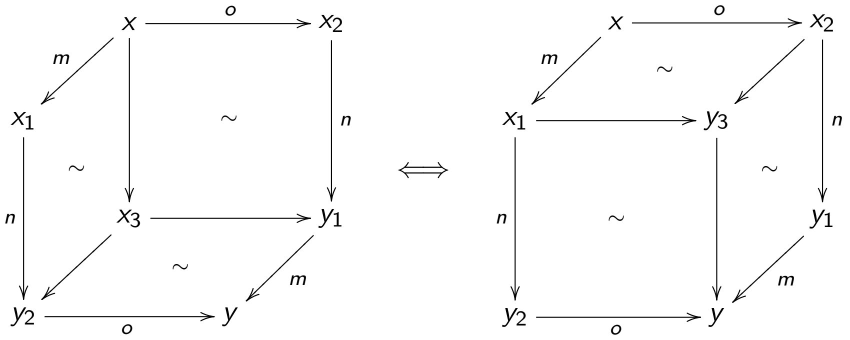
Every event structure defines an asynchronous graph.



From sequential to causal

Here, one needs the Cube Property.

The Cube Property



Theorem

Paths modulo homotopy are given by a partial order on their moves.

Asynchronous games

By definition, an **asynchronous game** is a rooted asynchronous graph satisfying the Cube Property.

Positional strategies

Definition
A **strategy** is a set of plays, closed under prefix.

Definition
A strategy is **positional** when its paths form a subgraph of the game.

Causal strategies

From now on, we consider *causal strategies* which

1. are positional
2. satisfy properties implying the Cube Property

Composition

Unfortunately, causal strategies do not compose...

Part III

A category of asynchronous games

Categories of games and strategies

$$A \multimap B \quad = \quad A^* \parr B \quad = \quad A^* \otimes B$$

The strategy **not**:

$$\begin{array}{ccc}
\texttt{bool} & \xrightarrow{\texttt{not}} & \texttt{bool} \\
 & & q \\
q & & \\
V & & \\
 & & F
\end{array}$$

Categories of games and strategies

$$A \multimap B \quad = \quad A^* \parr B \quad = \quad A^* \otimes B$$

The strategy **not**:

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\texttt{bool} & \xrightarrow{\texttt{not}} & \texttt{bool} \\
 & & q \\
q & & \\
F & & \\
 & & V
\end{array}$$

Composition

Traces compose by *parallel composition*

$\texttt{bool} \longrightarrow \texttt{bool}$ $\qquad$ $\texttt{bool} \longrightarrow \texttt{bool}$

q

q

F

V

Composition

Traces compose by *parallel composition*

`bool` $\longrightarrow$ `bool`   `bool` $\longrightarrow$ `bool`

q

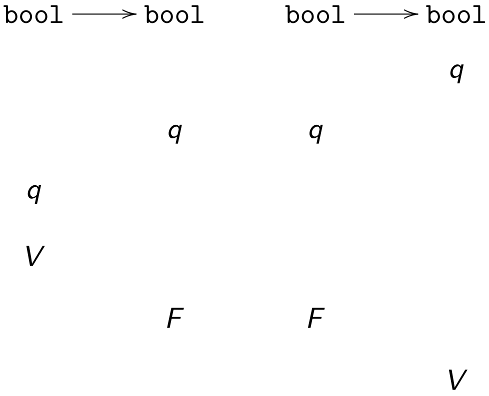
q

V

F

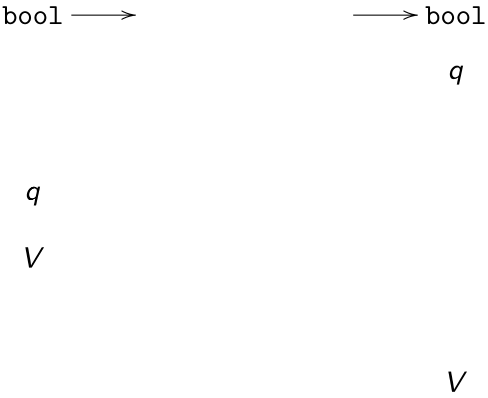
Composition

Traces compose by *parallel composition*



Composition

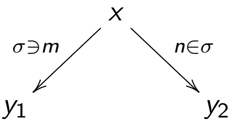
Traces compose by *parallel composition + hiding*.



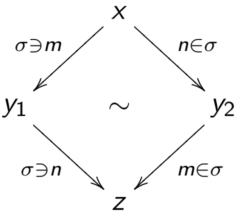
Determinism

Definition

A strategy $\sigma : A$ is **deterministic** when



implies



where m is a Proponent move.

Deterministic strategies do compose!

They form a monoidal category of asynchronous games.

Part IV

Concurrent strategies

Halting positions

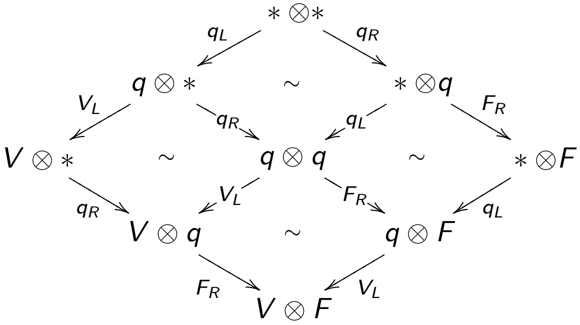
Definition

A position of a strategy σ is **halting** when there is no Proponent move $m : x \longrightarrow y$ in σ.

We write σ° for the set of halting positions of σ.

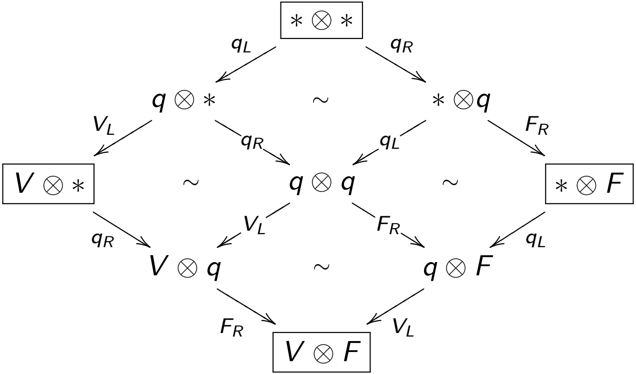
Halting positions

The game true ⊗ false.



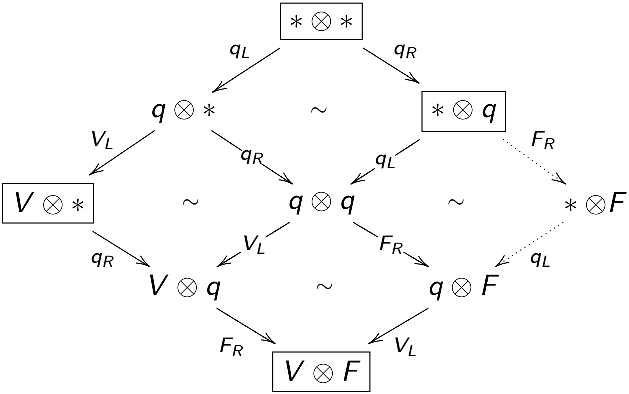
Halting positions

The *parallel* implementation of `true` and `false`.



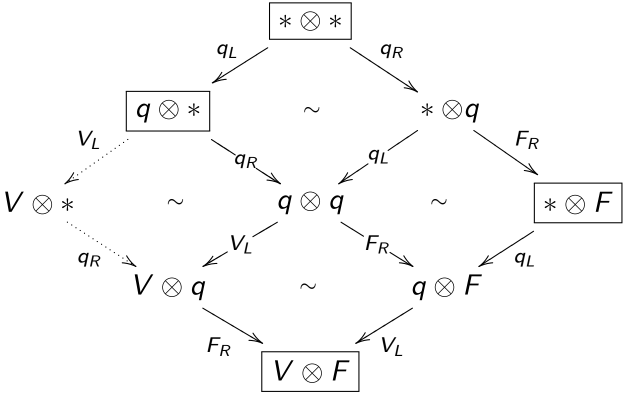
Halting positions

The *left* implementation of `true` and `false`.



Halting positions

The *right* implementation of `true` and `false`.



Ingenuous strategies

In the spirit of concurrent games Abramsky,Melliès 1999

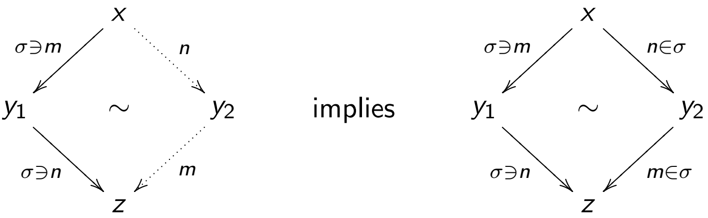
we would like strategies to be characterized by their *halting positions*.

Ingenuous strategies

Definition

A strategy σ is **ingenuous** when it is

1. causal,
2. deterministic,
3. *courteous*:



where m is a Proponent move.

Ingenuous strategies as relations

Theorem

Every ingenuous strategy σ is characterized by its set σ° of halting positions.

This set σ° describes a closure operator.

ingenuous strategies $\iff$ concurrent strategies

Part V

Innocence

Preserving composition

Unfortunately, we don't have

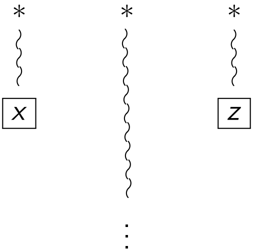
$$(\sigma;\tau)^\circ \quad = \quad \sigma^\circ;\tau^\circ$$

Preserving composition

The *livelock*:

$$(\sigma;\tau)^\circ \subseteq \sigma^\circ;\tau^\circ$$

$$A \xrightarrow{\sigma} B \xrightarrow{\tau} C$$

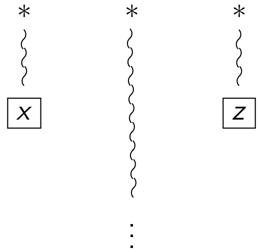


Preserving composition

The *livelock*:

$$(\sigma;\tau)^\circ \subseteq \sigma^\circ;\tau^\circ$$

$$A \xrightarrow{\sigma} B \xrightarrow{\tau} C$$



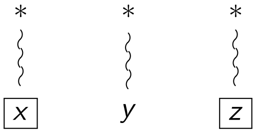
Solution: handle infinite positions

Preserving composition

The *deadlock*:

$$(\sigma;\tau)^\circ \supseteq \sigma^\circ;\tau^\circ$$

$$A \xrightarrow{\sigma} B \xrightarrow{\tau} C$$



Preserving composition

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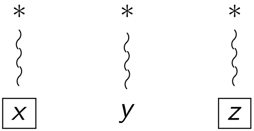
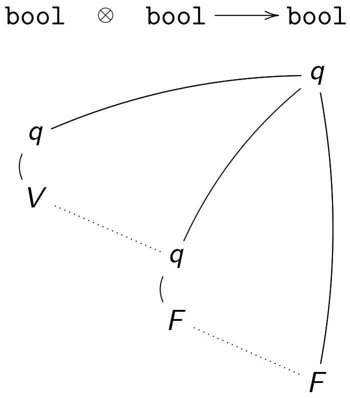


Solution: add a scheduling criterion

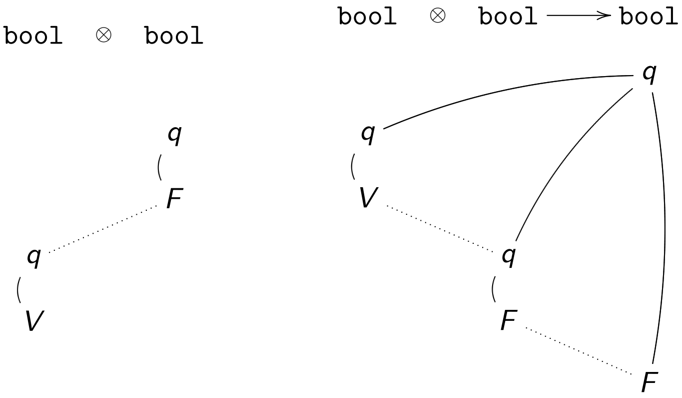
The scheduling criterion

the left conjunction:



The scheduling criterion

The right boolean composed with the left conjunction:



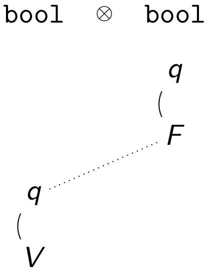
The scheduling criterion

Two kinds of tensors: $\otimes$ and $\parr$.

$$\texttt{bool} \otimes \texttt{bool} \multimap \texttt{bool} \quad = \quad \texttt{bool}^* \parr \texttt{bool}^* \parr \texttt{bool}$$

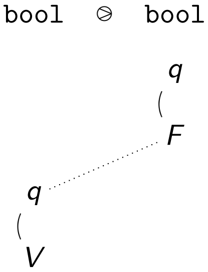
The scheduling criterion

Two kinds of tensors: $\otimes$ and $\parr$.



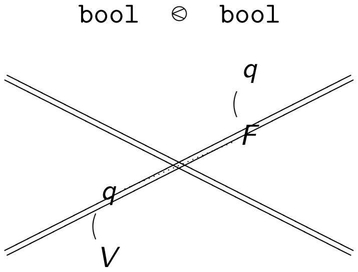
The scheduling criterion

Two kinds of tensors: $\otimes$ and $⅋$.



The scheduling criterion

Two kinds of tensors: $\otimes$ and $⅋$.



Functoriality

Definition

A strategy $\sigma : A$ is **receptive** when for every path $s : * \longrightarrow\!\!\!\!\!\rightarrow x$ in σ and for every Opponent move $m : x \longrightarrow y$ the path $s \cdot m : * \longrightarrow\!\!\!\!\!\rightarrow y$ is also in σ.

Functoriality

Definition

A strategy $\sigma : A$ is **receptive** when for every path $s : * \twoheadrightarrow x$ in σ and for every Opponent move $m : x \longrightarrow y$ the path $s \cdot m : * \twoheadrightarrow y$ is also in σ.

Theorem

Ingenuous strategies which satisfy the scheduling criterion and are receptive compose and satisfy

$$(\sigma ; \tau)^{\circ} \quad = \quad \sigma^{\circ} ; \tau^{\circ}$$

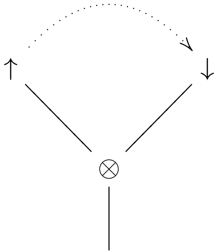
This defines a monoidal functor
(realizing the *Timeless Games* programme initited by
Baillot,Danos,Ehrard,Regnier 1998).

Part VI

Full completeness

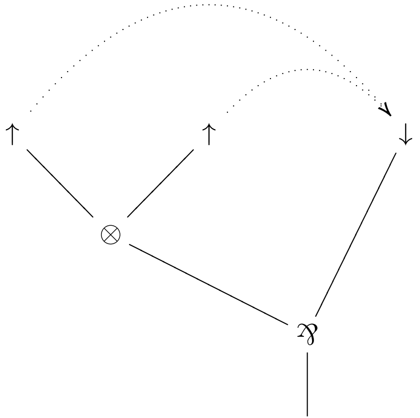
Innocence

The scheduling criterion detects directed cycles.



Innocence

The scheduling criterion does not detect non-directed cycles.



We thus elaborate a more subtle scheduling criterion.

Part VII

Thank you!