THE HIGHER-DIMENSIONAL ALGEBRAIC STRUCTURE OF PARTIAL ORDERS

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CHoCoLa MEETING
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Rewriting theory has proven to be very useful to study
▶ monoids (and groups)
▶ term algebras
Rewriting theory has proven to be very useful to study

- monoids (and groups)
- term algebras
- \textit{n-categories}

It can be generalized to higher dimensions!
I will be interested in what can be said about categories of

- relations
- partial orders
- increasing functions

The main result will be a “coherence theorem for commutative monads”.
Rewriting systems
A rewriting system consists of
  ▶ a set of terms generated by a free construction:
    ▶ free monoid: string rewriting systems
    ▶ free term algebra: term rewriting systems
  ▶ a set of rewriting rules: \( r : t \rightarrow u \)

Example
\[ \Sigma = \{a, b\} \quad \text{terms} = \Sigma^* \quad \text{rules} = \{ba \rightarrow ab\} \]
A **rewriting system** consists of

- a set of *terms* generated by a free construction:
  - free monoid: *string rewriting systems*
  - free term algebra: *term rewriting systems*
- a set of *rewriting rules*: $r : t \rightarrow u$

A term $t$ **rewrites** to a term $t'$ when there exists

- a rule $r : u \rightarrow u'$
- a context $C$ such that $t = C[u]$ and $t' = C[u']$

**Example**

$\Sigma = \{a, b\}$

- terms $= \Sigma^*$
- rules $= \{ba \rightarrow ab\}$

\[
aabaab \xrightarrow{aarrab} aaabab
\]
A rewriting system can be **terminating** when there is no infinite reduction path.

\[
\begin{align*}
t & \quad \downarrow \\
t_1 & \quad \downarrow \\
t_2 & \quad \downarrow \\
\vdots & \quad \quad \ddots
\end{align*}
\]
A rewriting system can be **terminating**

A rewriting can be **confluent** when

```
  t
   *   *
  /   \ /   \n u     v
```
A rewriting system can be **terminating**

A rewriting can be **confluent** when

\[ \begin{array}{c}
  t \\
  \downarrow * \quad \downarrow * \\
  u \\
  \quad \Downarrow * \\
  w \\
  \quad \Uparrow * \\
  v 
\end{array} \]

A rewriting system is **convergent** when both terminating and (locally) confluent.

In a convergent rewriting system, every term has a **normal form**:

- canonical representative of terms modulo rewriting.
A rewriting system can be **terminating**

A rewriting can be **confluent** when

![Diagram of confluent rewriting](image)

A rewriting system is **convergent**
when both terminating and (locally) confluent
A rewriting system can be **terminating**

A rewriting can be **confluent** when

\[
\begin{align*}
\ast & \downarrow & \ast \\
\ast & \downarrow & \ast \\
\ast & \downarrow & \ast
\end{align*}
\]

A rewriting system is **convergent** when both terminating and (locally) confluent

In a convergent rewriting system, every term has a **normal form**: canonical representative of terms modulo rewriting.
Why
are those properties interesting?
PRESENTATIONS OF MONOIDS

A presentation
\[ \langle G \mid R \rangle \]
of a monoid \( M \) consists of

- a set \( G \) of generators
- a set \( R \subseteq G^* \times G^* \) of relations

such that
\[ M \cong G^* / \equiv_R \]

Example

- \( \mathbb{N} \cong \langle a \mid \rangle \)
- \( \mathbb{N}/2\mathbb{N} \cong \langle a \mid aa = 1 \rangle \)
- \( \mathbb{N} \times \mathbb{N} \cong \langle a, b \mid ba = ab \rangle \)
- \( S_n \cong \langle \sigma_1, \ldots, \sigma_n \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i^2 = 1, \sigma_i\sigma_j = \sigma_j\sigma_i \rangle \)
- \( \ldots \)
How do we show that $M \cong \langle G \mid R \rangle$ i.e. $M \cong G^*/\equiv_R$?

1. Orient $R$ to get a string rewriting system.
2. Show that the rewriting system is terminating.
3. Show that the rewriting system is confluent.
4. Show that the normal forms are in bijection with $M$.

Example $N \times (N/2N) \cong \langle a, b \mid ba \rightarrow ab, bb \rightarrow 1 \rangle$.

Remark: we actually only need normal forms.
PRESENTATIONS OF MONOIDS

How do we show that $M \simeq \langle G \mid R \rangle$ i.e. $M \simeq G^* / \equiv_R$?

1. Orient $R$ to get a string rewriting system.
2. Show that the rewriting system is terminating.
3. Show that the rewriting system is confluent.
4. Show that the normal forms are in bijection with $M$. 
PRESENTATIONS OF MONOIDS

How do we show that $M \cong \langle G \mid R \rangle$ i.e. $M \cong G^\ast / \equiv_R$?

1. Orient $R$ to get a string rewriting system.
2. Show that the rewriting system is terminating.
3. Show that the rewriting system is confluent.
4. Show that the normal forms are in bijection with $M$.

Example $\mathbb{N} \times (\mathbb{N}/2\mathbb{N}) \cong \langle a, b \mid ba = ab, bb = 1 \rangle$
PRESENTATIONS OF MONOIDS

How do we show that $M \cong \langle G \mid R \rangle$ i.e. $M \cong G^* / \equiv_R$ ?

1. Orient $R$ to get a string rewriting system.
2. Show that the rewriting system is terminating.
3. Show that the rewriting system is confluent.
4. Show that the normal forms are in bijection with $M$.

Example $\mathbb{N} \times (\mathbb{N}/2\mathbb{N}) \overset{?}{\cong} \langle a, b \mid ba \rightarrow ab, \ bb \rightarrow 1 \rangle$
PRESENTATIONS OF MONOIDS

How do we show that $M \cong \langle G \mid R \rangle$ i.e. $M \cong G^*/\equiv_R$?

1. Orient $R$ to get a string rewriting system.
2. Show that the rewriting system is terminating.
3. Show that the rewriting system is confluent.
4. Show that the normal forms are in bijection with $M$.

Example: $\mathbb{N} \times (\mathbb{N}/2\mathbb{N}) \xrightarrow{R} \langle a, b \mid ba \rightarrow ab, \ bb \rightarrow 1 \rangle$

Critical pairs are:

\[
\begin{array}{c}
\text{bba} \\
\downarrow \quad \downarrow \\
a & bab
\end{array}
\quad \quad \quad
\begin{array}{c}
\text{bbb} \\
\downarrow \quad \downarrow \\
b & b
\end{array}
\]
PRESENTATIONS OF MONOIDS

How do we show that $M \cong \langle G \mid R \rangle$ i.e. $M \cong G^*/\equiv_R$?

1. Orient $R$ to get a string rewriting system.
2. Show that the rewriting system is terminating.
3. Show that the rewriting system is confluent.
4. Show that the normal forms are in bijection with $M$.

Example $\mathbb{N} \times (\mathbb{N}/2\mathbb{N}) \overset{?}{\Rightarrow} \langle a, b \mid ba \rightarrow ab, \ bb \rightarrow 1 \rangle$

Critical pairs are joinable:
PRESENTATIONS OF MONOIDS

How do we show that $M \cong \langle G \mid R \rangle$ i.e. $M \cong G^*/\equiv_R$?

1. Orient $R$ to get a string rewriting system.
2. Show that the rewriting system is terminating.
3. Show that the rewriting system is confluent.
4. Show that the normal forms are in bijection with $M$.

Example

$\mathbb{N} \times (\mathbb{N}/2\mathbb{N}) \overset{?}{\cong} \langle a, b \mid ba \rightarrow ab, \ bb \rightarrow 1 \rangle$

Normal forms are:

$a^n$ and $a^n b$

They are in bijection with $\mathbb{N} \times (\mathbb{N}/2\mathbb{N})$!
How do we show that $M \cong \langle G \mid R \rangle$ i.e. $M \cong G^* / \equiv_R$?

1. Orient $R$ to get a string rewriting system.
2. Show that the rewriting system is terminating.
3. Show that the rewriting system is confluent.
4. Show that the normal forms are in bijection with $M$.

**Example** $\mathbb{N} \times (\mathbb{N}/2\mathbb{N}) \triangleq \langle a, b \mid ba \rightarrow ab, bb \rightarrow 1 \rangle$

Normal forms are:

\[ a^n \quad \text{and} \quad a^n b \]

They are in bijection with $\mathbb{N} \times (\mathbb{N}/2\mathbb{N})$!

Remark: we actually only need normal forms.
How do we generalize this to present categories?
PRESENTING CATEGORIES

Presentation of a monoid \( M \cong \langle G \mid R \rangle \):
PRESENTING CATEGORIES

Presentation of a monoid $M \cong \langle G \mid R \rangle$:

$$
\begin{array}{c}
G \\
i \\
i \\
G^*
\end{array}
$$
Presentation of a monoid $M \cong \langle G \mid R \rangle$:

![Diagram](attachment:image.png)
PRESENTING CATEGORIES

Presentation of a monoid $M \cong \langle G \mid R \rangle$:

\[
\begin{array}{c}
G \\
i \downarrow \\
G^* \\
\end{array}
\begin{array}{c}
R \\
\downarrow s_R \\
\downarrow t_R \\
\end{array}
\]

can be generalized to presentation of a category:

\[
\begin{array}{c}
E \\
\downarrow s \\
V \\
\end{array}
\begin{array}{c}
\downarrow t \\
\end{array}
\]

a graph
Presentation of a monoid $M \cong \langle G \mid R \rangle$:

\[
\begin{array}{ccc}
G & \xrightarrow{s_R} & R \\
\downarrow i & \downarrow s_R & \downarrow t_R \\
G^* & \xleftarrow{i} & \xleftarrow{t_R} \xrightarrow{t^*_R} R \\
\end{array}
\]

can be generalized to presentation of a category:

\[
\begin{array}{ccc}
E & \xrightarrow{s} & V \\
\downarrow i & \downarrow s^*t & \downarrow \xleftarrow{t^*} i \\
E^* & \xleftarrow{s^*t} & \xleftarrow{t^*_R} R \\
\end{array}
\]

a free graph
**PRESENTING CATEGORIES**

Presentation of a monoid $M \cong \langle G \mid R \rangle$:

$$
\begin{array}{ccc}
G & \xrightarrow{s_R} & R \\
\downarrow i & & \downarrow t_R \\
G^* & \xleftarrow{s_R} & \\
\end{array}
$$

can be generalized to presentation of a category:

$$
\begin{array}{ccc}
E & \xrightarrow{s} & R \\
\downarrow i & & \downarrow t_R \\
V & \xleftarrow{s^*t} & E^* \\
\end{array}
$$

such that $s^*s_R = s^*t_R$ and $t^*s_R = t^*t_R$

a presentation of a category

$$
C \cong G^*/\equiv_R
$$
We see a pattern emerge!

[Burroni93, Street76, Power90]
A 0-polygraph:

\[ \Sigma^*_0 \]
A 1-polygraph:

\[ \sum_0 \rightarrow t_0 \rightarrow \sum_1 \rightarrow s_0 \rightarrow \sum^* \]

The 3-polygraph \( \Sigma \) presents a 2-category \( C \) when \( C \simeq \tilde{\Sigma} \).
A 1-polygraph generates a category:

\[
\begin{array}{ccc}
\Sigma_0 & \xleftarrow{s_0} & \Sigma_1 \\
& s_1 & \\
\Sigma^* & \xleftarrow{s_0^* t_0} & \Sigma^*_1 \\
& i_1 & \\
\end{array}
\]
A 2-polygraph:

\[
\begin{array}{c}
\Sigma_0 \\
\Sigma^*_0 \\
\Sigma^*_1 \\
\Sigma_1 \\
\Sigma_2
\end{array}
\]

such that \( s_0^* s_1 = s_0^* t_1 \) and \( t_0^* s_1 = t_0^* t_1 \)
A 2-polygraph generates a 2-category:

\[\begin{align*}
\Sigma_0 & \xleftarrow{s_0^* \ t_0} \Sigma_1 & \xleftarrow{s_1^* \ t_0} \Sigma_2 \\
\Sigma_0^* & \xleftarrow{s_0^* \ t_1} \Sigma_1^* & \xleftarrow{s_1^* \ t_1} \Sigma_2^*
\end{align*}\]

such that \(s_0^* s_1 = s_0^* t_1\) and \(t_0^* s_1 = t_0^* t_1\). 

The 3-polygraph \(\Sigma\) generates a 3-category \(\Sigma^*\).

We write \(\tilde{\Sigma}^*\) for the 2-category obtained from \(\Sigma^*\) by identifying two 2-cells \(f\) and \(g\) for which there exists a 3-cell \(\alpha: f \Rightarrow g\).

The 3-polygraph \(\Sigma\) presents a 2-category \(C\) when \(C \simeq \tilde{\Sigma}^*\).
A 3-polygraph:

\[ \Sigma^* \xleftarrow{0} \Sigma \xrightarrow{1} \Sigma \xrightarrow{2} \Sigma \xrightarrow{3} \]

\[ \Sigma^* \xleftarrow{0} \Sigma \xrightarrow{1} \Sigma \xrightarrow{2} \Sigma \xrightarrow{3} \]

such that \( s_1^* s_2 = s_1^* t_2 \) and \( t_1^* s_2 = t_1^* t_2 \)
A 3-polygraph . . .

such that $s_1^*s_2 = s_1^*t_2$ and $t_1^*s_2 = t_1^*t_2$
A 3-polygraph . . .

\[
\begin{array}{c}
\Sigma^* \\
\Sigma_0 \\
\Sigma_1 \\
\Sigma_2 \\
\Sigma_3
\end{array}
\]

such that \( s_1^*s_2 = s_1^*t_2 \) and \( t_1^*s_2 = t_1^*t_2 \)

▶ The 3-polygraph \( \Sigma \) generates a 3-category \( \Sigma^* \)
A 3-polygraph . . .

such that $s_1^*s_2 = s_1^*t_2$ and $t_1^*s_2 = t_1^*t_2$

- The 3-polygraph $\Sigma$ generates a 3-category $\Sigma^*$
- We write $\tilde{\Sigma}^*$ for the 2-category obtained from $\Sigma^*$ by identifying two 2-cells $f$ and $g$ for which there exists a 3-cell $\alpha : f \Rightarrow g$
A 3-polygraph . . .

such that $s_1^*s_2 = s_1^*t_2$ and $t_1^*s_2 = t_1^*t_2$

- The 3-polygraph $\Sigma$ generates a 3-category $\Sigma^*$
- We write $\tilde{\Sigma}^*$ for the 2-category obtained from $\Sigma^*$ by identifying two 2-cells $f$ and $g$ for which there exists a 3-cell $\alpha : f \Rightarrow g$
- The 3-polygraph $\Sigma$ presents a 2-category $C$ when $C \cong \tilde{\Sigma}^*$
Consider the simplicial category $\Delta$ whose

- objects are natural integers $[n] = \{0, 1, \ldots, n - 1\}$
- morphisms are increasing functions $f : [m] \to [n]$

For instance $f : 4 \to 3$

\[
\begin{array}{c}
[3] \\
\downarrow f \\
[4]
\end{array}
\quad
\begin{array}{cccc}
0 & 1 & 2 \\
\downarrow & \downarrow & \downarrow \\
0 & 1 & 2 & 3
\end{array}
\]
The category $\Delta$ is monoidal with $[0]$ as unit and $\otimes$ defined

- on objects: $[m] \otimes [n] = [m + n]$
- on morphisms:

$$
\begin{pmatrix}
0 & 1 & 2 \\
0 & 1 & 2
\end{pmatrix} \otimes \begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3
\end{pmatrix}
$$
The category $\Delta$ is monoidal with $[0]$ as unit and $\otimes$ defined

- on objects: $[m] \otimes [n] = [m + n]$
- on morphisms:

$$
\begin{pmatrix}
0 & 1 & 2 \\
0 & 1 & 2
\end{pmatrix}
\otimes
\begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 1 & 2 & 3
\end{pmatrix}
$$

A monoidal category is the same as a 2-category with only one 0-cell so we can (hope to) present it with a 3-polygraph! [MacLane, Burroni, Lafont]
PRESENTING THE SIMPLICIAL CATEGORY

We will show that the 2-category $\Delta$ is presented by the polygraph

$$\Sigma^*$$

whose generators are

- $\Sigma_0 = \{\ast\}$
PRESENTING THE SIMPLIFICIAL CATEGORY

We will show that the 2-category $\Delta$ is presented by the polygraph

\[
\begin{array}{ccc}
\Sigma_0 & \xleftarrow{t_0} & \Sigma_1 \\
& {s_0} & \downarrow \\
\Sigma^* & & \\
\end{array}
\]

whose generators are

- $\Sigma_0 = \{\ast\}$
- $\Sigma_1 = \{\mathbf{1} : \ast \to \ast\}$
We will show that the 2-category $\Delta$ is presented by the polygraph

$$\begin{array}{c}
\Sigma_0 \\
\Sigma^*_0 & \leftrightarrow & \Sigma^*_1 \\
\Sigma_1 \\
\end{array}$$

whose generators are

- $\Sigma_0 = \{*\}$
- $\Sigma_1 = \{1 : * \rightarrow *\}$ (so $\Sigma^*_1 \cong \mathbb{N}$)
PRESENTING THE SIMPLIFICIAL CATEGORY

We will show that the 2-category $\Delta$ is presented by the polygraph

$$
\begin{align*}
\Sigma_0 & \leftarrow \Sigma^* \\
\Sigma^* & \leftarrow \Sigma_1 \\
\Sigma_1 & \leftarrow \Sigma^* \\
\Sigma^* & \leftarrow \Sigma_2
\end{align*}
$$

whose generators are

- $\Sigma_0 = \{\ast\}$
- $\Sigma_1 = \{1 : \ast \to \ast\}$ (so $\Sigma^*_1 \cong \mathbb{N}$)
- $\Sigma_2 = \{\mu : (1 \otimes 1) \Rightarrow 1, \eta : 0 \Rightarrow 1\}$
PRESENTING THE SIMPLIFICIAL CATEGORY

We will show that the 2-category $\Delta$ is presented by the polygraph

$$
\begin{align*}
\Sigma_0 & \xleftarrow{s_0^* \circ t_0^*} \Sigma_1 \\
\Sigma_1 & \xleftarrow{s_1^* \circ t_1^*} \Sigma_2 \\
\Sigma_0 & \xrightarrow{t_0^*} \Sigma_1 \\
\Sigma_1 & \xrightarrow{t_1^*} \Sigma_2
\end{align*}
$$

whose generators are

- $\Sigma_0 = \{\ast\}$
- $\Sigma_1 = \{1 : \ast \to \ast\}$ (so $\Sigma_1^* \simeq \mathbb{N}$)
- $\Sigma_2 = \{\mu : (1 \otimes 1) \Rightarrow 1, \eta : 0 \Rightarrow 1\}$
We will show that the 2-category $\Delta$ is presented by the polygraph

$$
\begin{array}{cccc}
\Sigma_0 & \Sigma_1 & \Sigma_2 & \Sigma_3 \\
\downarrow s_0 & \downarrow s_1 & \downarrow s_2 & \\
\Sigma^* & \Sigma^* & \Sigma^* & \\
\uparrow s^*_0 & \uparrow s^*_1 & \uparrow s^*_2 & \\
\Sigma^* & \Sigma^* & \Sigma^* & \\
\downarrow t_0 & \downarrow t_1 & \downarrow t_2 & \\
\Sigma^* & \Sigma^* & \Sigma^* & \\
\end{array}
$$

whose generators are

- $\Sigma_0 = \{\ast\}$
- $\Sigma_1 = \{1 : \ast \to \ast\}$ (so $\Sigma^*_1 \cong \mathbb{N}$)
- $\Sigma_2 = \{\mu : (1 \otimes 1) \Rightarrow 1, \eta : 0 \Rightarrow 1\}$
- $\Sigma_3 = \{\begin{array}{l}
\alpha : \mu \circ (\mu \otimes 1) \Rightarrow \mu \circ (1 \otimes \mu), \\
\lambda : \mu \circ (\eta \otimes 1) \Rightarrow 1, \rho : \mu \circ (1 \otimes \eta) \Rightarrow 1
\end{array}\}$
The 2-generators can be drawn as string diagrams:
STRING DIAGRAMS

The 2-generators can be drawn as string diagrams:

\[\Rightarrow \quad \Rightarrow\]

and the 3-generators become

\[\alpha \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow\]

We recognize the laws for monoids!
PROVING THE PRESENTATION

We have to prove that we have a presentation

\[ \Delta \cong \Sigma^* \]

which means that diagrams built from the 2-generators

\[ \mu = \quad \text{and} \quad \eta = \]

by composition and tensoring, considered modulo the relations

are in bijection with increasing functions.
We have to prove that we have a presentation \( \Delta \cong \tilde{\Sigma}^* \).

- The generators can be interpreted as functions:

\[
\begin{array}{c}
\xrightarrow{0} \\
\xrightarrow{1}
\end{array}
\quad \xrightarrow{0} \\
\xrightarrow{1}
\]

Thus inducing a functor \([ \cdot ] : \partial \Sigma^* \to \Delta\).
We have to prove that we have a presentation $\Delta \cong \Sigma^*$. 

- The generators can be interpreted as functions. Thus inducing a functor $[-] : \partial \Sigma^* \rightarrow \Delta$.

- The left and right members of the 3-generators get interpreted as the same function ($[-]$ is compatible with relations):

  $$
  \begin{bmatrix}
  \begin{array}{c}
  \text{Diagram 1}
  \\
  \end{array}
  \end{bmatrix}
  =
  \begin{bmatrix}
  \begin{array}{c}
  0
  \\
  0
  \\
  1
  \\
  2
  \\
  \end{array}
  \end{bmatrix}
  =
  \begin{bmatrix}
  \begin{array}{c}
  \text{Diagram 2}
  \\
  \end{array}
  \end{bmatrix}
  $$

  Thus inducing a 2-functor $[-] : \Sigma^* \rightarrow \Delta$. 

PROVING THE PRESENTATION
We have to prove that we have a presentation $\Delta \cong \tilde{\Sigma}^*$. 

- The generators can be interpreted as functions. Thus inducing a functor $[-] : \partial \Sigma^* \to \Delta$. 

- The left and right members of the 3-generators get interpreted as the same function ($[-]$ is compatible with relations): Thus inducing a 2-functor $[-] : \tilde{\Sigma}^* \to \Delta$. 

- The functor $[-]$ is full.
We have to prove that we have a presentation $\Delta \cong \tilde{\Sigma}^*$.

- The generators can be interpreted as functions. Thus inducing a functor $\llbracket - \rrbracket : \partial \Sigma^* \to \Delta$.
- The left and right members of the 3-generators get interpreted as the same function ($\llbracket - \rrbracket$ is compatible with relations): Thus inducing a 2-functor $\llbracket - \rrbracket : \tilde{\Sigma}^* \to \Delta$.
- The functor $\llbracket - \rrbracket$ is full.
- The 2-functor $\llbracket - \rrbracket$ is faithful (more difficult), i.e. $\tilde{\Sigma}^* \cong \Delta$. 

PROVING THE PRESENTATION
To show that the 2-functor $[-] : \widetilde{\Sigma}^* \to \Delta$ is faithful, we have to show that if two diagrams get interpreted as the same function then they are equivalent modulo the 3-generators $\alpha$, $\lambda$ and $\rho$. 

\[
\left[ \begin{array}{c}
\vdots \\
\end{array} \right] = 0 \rightarrow 1 \rightarrow 2 \rightarrow 3 = \left[ \begin{array}{c}
\vdots \\
\end{array} \right]
\]
To show that the 2-functor $[\cdot] : \tilde{\Sigma}^* \rightarrow \Delta$ is faithful, we have to show that if two diagrams get interpreted as the same function then they are equivalent modulo the 3-generators $\alpha$, $\lambda$ and $\rho$.

\[
\begin{bmatrix}
\begin{bmatrix}
\begin{array}{c}
\end{array}
\end{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
\begin{bmatrix}
\begin{array}{c}
\end{array}
\end{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
\begin{bmatrix}
\begin{array}{c}
\end{array}
\end{bmatrix}
\end{bmatrix}
\]

We can use rewriting theory!
To show that the 2-functor \([−] : \tilde{Σ}^* \to Δ\) is faithful, we have to show that if two diagrams get interpreted as the same function then they are equivalent modulo the 3-generators \(α, λ\) and \(ρ\). We can use rewriting theory!

- The five critical pairs are joinable:
To show that the 2-functor $\widetilde{-} : \tilde{\Sigma}^* \to \Delta$ is faithful, we have to show that if two diagrams get interpreted as the same function then they are equivalent modulo the 3-generators $\alpha$, $\lambda$ and $\rho$. We can use rewriting theory!

- The five critical pairs are joinable:
To show that the 2-functor $[-] : \tilde{\Sigma}^* \to \Delta$ is faithful, we have to show that if two diagrams get interpreted as the same function then they are equivalent modulo the 3-generators $\alpha$, $\lambda$ and $\rho$. We can use rewriting theory!

- The five critical pairs are joinable:

- The rewriting system is terminating...
To show that the 2-functor $\widetilde{\Sigma}^* \to \Delta$ is faithful, we have to show that if two diagrams get interpreted as the same function then they are equivalent modulo the 3-generators $\alpha$, $\lambda$ and $\rho$. We can use rewriting theory!

- The five critical pairs are joinable:
- The rewriting system is terminating . . .
- The normal forms are tensor products of $M_i$ with $i \in \mathbb{N}$:

\[
M_i = \quad M_1 = \quad M_0 =
\]
To show that the 2-functor $[-] : \widetilde{\Sigma}^* \to \Delta$ is faithful, we have to show that if two diagrams get interpreted as the same function then they are equivalent modulo the 3-generators $\alpha$, $\lambda$ and $\rho$. We can use rewriting theory!

- The five critical pairs are joinable:
- The rewriting system is terminating.
- The normal forms are tensor products of $M_i$ with $i \in \mathbb{N}$:

$$M_i = \ldots$$

$$M_1 = \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \quad M_0 =$$

- Normal forms are in bijection with functions $f : [m] \to [n]$

$$f = [M_{|f^{-1}(0)|} \otimes M_{|f^{-1}(1)|} \otimes \ldots \otimes M_{|f^{-1}(n-1)|}]$$
To show that the 2-functor $[-] : \widetilde{\Sigma}^* \rightarrow \Delta$ is faithful, we have to show that if two diagrams get interpreted as the same function then they are equivalent modulo the 3-generators $\alpha$, $\lambda$ and $\rho$. We can use rewriting theory!

- The five critical pairs are joinable:
- The rewriting system is terminating...
- The normal forms are tensor products of $M_i$ with $i \in \mathbb{N}$:
- Normal forms are in bijection with functions $f : [m] \rightarrow [n]$

$$f = [M_{|f^{-1}(0)|} \otimes M_{|f^{-1}(1)|} \otimes \ldots \otimes M_{|f^{-1}(n-1)|}]$$
We have shown that

- we have a presentation \( \widetilde{\Sigma}^* \cong \Delta \)

CONSEQUENCES
We have shown that

- we have a presentation $\tilde{\Sigma}^* \cong \Delta$
- i.e. diagrams built from $\mu$ and $\eta$ modulo the relation generated by $\alpha$, $\lambda$ and $\rho$ are in bijection with functions
We have shown that

- we have a presentation \( \tilde{\Sigma}^* \cong \Delta \)
- i.e. diagrams built from \( \mu \) and \( \eta \) modulo the relation generated by \( \alpha \), \( \lambda \) and \( \rho \) are in bijection with functions
- the category \( \Sigma \) is the theory for monoids.
Since we have described $\Delta$ by generators and relations we know that a strict monoidal functor $M : \Delta \to C$ is uniquely determined by the images of the generators, which satisfy the relations:

- an object $M1 \in C$
Since we have described $\Delta$ by generators and relations we know that a strict monoidal functor $M : \Delta \to C$ is uniquely determined by the images of the generators, which satisfy the relations:

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- two morphisms $M\mu : M1 \otimes M1 \to M1$ and $M\eta : I \to M1$
\[ \Delta \text{ AS A THEORY FOR MONOIDS} \]

Since we have described \( \Delta \) by generators and relations we know that a strict monoidal functor \( M : \Delta \to C \) is uniquely determined by the images of the generators, which satisfy the relations:

- an object \( M1 \in C \)
- two morphisms \( M_\mu : M1 \otimes M1 \to M1 \) and \( M_\eta : I \to M1 \)
- such that

\[
\begin{array}{ccc}
M1 \otimes M1 \otimes M1 & \xrightarrow{M_\mu \otimes M1} & M1 \otimes M1 \\
M1 \otimes M1 & \xrightarrow{M_\mu} & M1 \\
M1 & \xrightarrow{M_\mu} & M1
\end{array}
\]

\[
\begin{array}{ccc}
M1 & \xrightarrow{M_\eta \otimes M1} & M1 \otimes M1 \\
M1 & \xrightarrow{M_\mu} & M1 \\
M1 & \xrightarrow{M_\mu} & M1
\end{array}
\]
\[ \Delta \text{ AS A THEORY FOR MONOIDS} \]

Since we have described \( \Delta \) by generators and relations we know that a strict monoidal functor \( \mathcal{M} : \Delta \to \mathcal{C} \) is uniquely determined by the images of the generators, which satisfy the relations:

- an object \( M_1 \in \mathcal{C} \)
- two morphisms \( M_\mu : M_1 \otimes M_1 \to M_1 \) and \( M_\eta : I \to M_1 \)
- such that

\[
\begin{array}{ccc}
M_1 \otimes M_1 \otimes M_1 & \xrightarrow{M_\mu \otimes M_1} & M_1 \otimes M_1 \\
\downarrow M_\mu & & \downarrow M_\mu \\
M_1 \otimes M_1 & \xrightarrow{M_\mu} & M_1 \\
\end{array}
\]

In other words, a monoidal functor \( \mathcal{M} : \Delta \to \mathcal{C} \) is a monoid in \( \mathcal{C} \)!

\[ \text{StrMonCat}(\Delta, \mathcal{C}) \cong \text{Mon}(\mathcal{C}) \]

\textbf{Ex:} in \textbf{Set}, \textbf{Cat}, \ldots
AN IMPORTANT EXAMPLE: MONADS

Given a category $C$, consider the 2-category with

- one 0-cell: $C$
- 1-cells: endofunctors $C \rightarrow C$
- 2-cells: natural transformations

It’s a 2-category with one 0-cell, i.e. a monoidal category.

Monoids in this category are precisely the monads on $C$. 
It is important to remark that we don’t really need to have a convergent rewriting system, we only need to provide a notion of canonical form.
It is important to remark that we don’t really need to have a convergent rewriting system, we only need to provide a notion of canonical form.

Actually, those higher-dimensional rewriting systems are much more complicated than usual (string/term) rewriting systems: a convergent rewriting system can have an infinite number of critical pairs!
Let’s see some more examples.
MORE EXAMPLES OF PROS

Definition

A **PRO** is a monoidal category whose objects are integers and tensor product is given on objects by addition (e.g. $\Delta$).
Definition

A **PRO** is a monoidal category whose objects are integers and tensor product is given on objects by addition (e.g. $\Delta$).

As for $\Delta$, a presentation of a PRO necessarily have

- $\Sigma_0 = \{\ast\}$: it is a 2-category with one 0-cell
- $\Sigma_1 = \{1\}$: the objects are $\Sigma_1^* \cong \mathbb{N}$
- it is thus enough to specify the 2-generators and the 3-generators (the relations)
A PRESENTATION OF $\Delta$

The simplicial category $\Delta$ admits a presentation with

- two 2-generators

\[ \mu : 2 \to 1 \quad \eta : 0 \to 1 \]

- three relations (3-generators)

\[ \alpha \text{ (associativity)} \quad \text{and} \quad \text{unitality} \]

- $\Delta$: theory of monoids
Dually, the category $\Delta^{\text{op}}$ admits a presentation with

- two 2-generators

$$\delta : 1 \to 2 \quad \varepsilon : 1 \to 0$$

- three relations (3-generators)

- $\Delta^{\text{op}}$: theory of comonoids
A PRESENTATION OF Bij

The PRO bij with \( \mathbb{N} \) as objects and bijections \( f : [n] \to [n] \) as morphisms admits a presentation with

- one 2-generator \( \gamma : 2 \to 2 \)

- two relations

\[
\begin{align*}
\xymatrix@C=2pc{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymatrix{ & \ast & } & \xymar
A PRESENTATION OF FinOrd

The PRO FinOrd with \( \mathbb{N} \) as objects and functions \( f : [m] \to [n] \) as morphisms admits a presentation with

- three 2-generators

\[
\begin{align*}
\mu : 2 & \to 1 \\
\eta : 0 & \to 1 \\
\gamma : 2 & \to 2
\end{align*}
\]
A PRESENTATION OF FinOrd

The PRO \textbf{FinOrd} with \( \mathbb{N} \) as objects and functions \( f : [m] \to [n] \) as morphisms admits a presentation with

- three 2-generators \( \mu : 2 \to 1, \eta : 0 \to 1, \gamma : 2 \to 2 \)
- relations expressing that
  - \((\mu, \eta)\) is a monoid + \( \gamma \) is a symmetry
  - compatibility between monoid and symmetry

\[ \begin{align*}
\mu & \equiv \eta \\
\gamma & \equiv \text{compatibility}
\end{align*} \]

- commutativity of \( \mu \)

\[ \begin{align*}
\text{commutativity of } \mu & \\
\text{compatibility} & \\
\end{align*} \]
A PRESENTATION OF FinOrd

The PRO FinOrd with $\mathbb{N}$ as objects and functions $f : [m] \to [n]$ as morphisms admits a presentation with

- three 2-generators $\mu : 2 \to 1$, $\eta : 0 \to 1$, $\gamma : 2 \to 2$
- relations expressing that
  - $(\mu, \eta)$ is a monoid + $\gamma$ is a symmetry
  - compatibility between monoid and symmetry

- commutativity of $\mu$

- FinOrd is thus the theory for commutative monoids
A PRESENTATION OF MRel

The PRO $\textbf{MRel}$ with $\mathbb{N}$ as objects and $m \times n$ matrices with coefficients in $\mathbb{N}$ as morphisms $[m] \to [n]$.

For instance, a morphism $[3] \to [2]$: $$\begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$
A PRESENTATION OF MRel

The PRO MRel with \( \mathbb{N} \) as objects and \( m \times n \) matrices with coefficients in \( \mathbb{N} \) as morphisms \([m] \rightarrow [n]\).

For instance, a morphism \([3] \rightarrow [2] \):

\[
\begin{pmatrix}
2 & 1 \\
0 & 0 \\
1 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\begin{pmatrix}
1 \\
2
\end{pmatrix}
\]
A PRESENTATION OF MRel

The PRO MRel with \( \mathbb{N} \) as objects and \( m \times n \) matrices with coefficients in \( \mathbb{N} \) as morphisms \([m] \to [n]\).

For instance, a morphism \([3] \to [2]\):

\[
\begin{pmatrix}
2 & 1 \\
0 & 0 \\
1 & 0
\end{pmatrix}
\sim\Rightarrow
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\sim\Rightarrow
\begin{pmatrix}
0 & 1 \\
1 & 2
\end{pmatrix}
\]
A PRESENTATION OF MRel

The PRO MRel with \( \mathbb{N} \) as objects and \( m \times n \) matrices with coefficients in \( \mathbb{N} \) as morphisms \([m] \to [n]\).

For instance, a morphism \([3] \to [2]\):

\[
\begin{pmatrix}
2 & 1 \\
0 & 0 \\
1 & 0
\end{pmatrix} \sim \begin{pmatrix}
0 & 1 \\
0 & 1 \\
0 & 2
\end{pmatrix} \sim \begin{array}{c}
\rotatebox{90}{$\Rightarrow$}
\end{array}
\]

It admits a presentation with

- five 2-generators

\[
\mu : 2 \to 1 \quad \eta : 0 \to 1 \quad \delta : 1 \to 2 \quad \varepsilon : 1 \to 0 \quad \gamma : 2 \to 2
\]
A PRESENTATION OF MRel

The PRO MRel with $\mathbb{N}$ as objects and $m \times n$ matrices with coefficients in $\mathbb{N}$ as morphisms $[m] \rightarrow [n]$.

It admits a presentation with

- five 2-generators

$$
\begin{align*}
\mu &: 2 \rightarrow 1 & \eta &: 0 \rightarrow 1 & \delta &: 1 \rightarrow 2 & \varepsilon &: 1 \rightarrow 0 & \gamma &: 2 \rightarrow 2
\end{align*}
$$

- relations

  - $(\mu, \eta, \gamma)$ is a commutative monoid
  - $(\delta, \varepsilon, \gamma)$ is cocommutative comonoid
  - bialgebra laws
A PRESENTATION OF Rel

The PRO Rel with $\mathbb{N}$ as objects and relations $R \subseteq [m] \times [n]$ as morphisms $m \rightarrow n$. 
A PRESENTATION OF Rel

The PRO \( \text{Rel} \) with \( \mathbb{N} \) as objects and relations \( R \subseteq [m] \times [n] \) as morphisms \( m \to n \).

It can be seen as a quotient of \( \text{MRel} \):

\[
\begin{pmatrix}
2 & 1 \\
0 & 0 \\
1 & 0
\end{pmatrix}
\approx
\begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix}
\approx
\begin{pmatrix}
1 & 1 \\
0 & 0 \\
1 & 0
\end{pmatrix}
\]
A PRESENTATION OF Rel

The PRO Rel with $\mathbb{N}$ as objects and relations $R \subseteq [m] \times [n]$ as morphisms $m \to n$.

It can be seen as a quotient of MRel:

$\begin{pmatrix} 2 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ \approx \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$

It admits the same presentation as MRel with the following extra relation:

Rel: theory for qualitative bialgebras
Given a morphism \( \phi = \cdots \phi \) : \( m \to n \) we define

\[
E\phi = \begin{array}{c}
\end{array} : m + 1 \to n
\]

\[
H\phi = \begin{array}{c}
\end{array} : m \to n + 1
\]

\[
W_i\phi = \begin{array}{c}
\end{array} : m \to n
\]

\[
Z = \begin{array}{c}
\end{array} 0 \to 0
\]
Given a morphism $\phi : m \to n$ we define

$$E\phi = \begin{array}{c} \vdots \\ \phi \\ \vdots \end{array} : m + 1 \to n$$

(add a line)

$$H\phi = \begin{array}{c} \vdots \\ \phi \\ \vdots \end{array} : m \to n + 1$$

(add a column)

$$W_i\phi = \begin{array}{c} \vdots \\ \phi \\ \vdots \end{array} : m \to n$$

(add a link)

$$Z = \begin{array}{c} \vdots \\ \phi \\ \vdots \end{array} : 0 \to 0$$

()
The Proof

Given a morphism $\phi : m \to n$ we define

<table>
<thead>
<tr>
<th>$E\phi$</th>
<th>$H\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m + 1 \to n$</td>
<td>$m \to n + 1$</td>
</tr>
</tbody>
</table>

$W_i\phi : m \to n$

Lemma

Every diagram is equivalent (modulo the relations) to a composite of those morphisms (called pre-canonical forms).
Given a morphism $\phi = \begin{array}{c}
\vdots
\phi
\vdots
\end{array} : m \to n$ we define

$E\phi = \begin{array}{c}
\vdots
\phi
\vdots
\end{array} : m + 1 \to n$

(add a line)

$W_i\phi = \begin{array}{c}
\vdots
\phi
\vdots
\end{array} : m \to n$

(add a link)

$H\phi = \begin{array}{c}
\vdots
\phi
\vdots
\end{array} : m \to n + 1$

(add a column)

$Z = \begin{array}{c}
\vdots
\phi
\vdots
\end{array} : 0 \to 0$

Lemma

$W_iW_j\phi = W_jW_i\phi \quad EH\phi = HE\phi \quad EW_i\phi = W_{i+1}E\phi$
THE PROOF

Given a morphism \( \phi = \cdots \) : \( m \to n \) we define

\[
E \phi = \begin{array}{c}
\vdots \\
\phi \\
\vdots
\end{array} : m + 1 \to n
\]

(\text{add a line})

\[
H \phi = \begin{array}{c}
\phi \cdots \\
\phi \cdots
\end{array} : m \to n + 1
\]

(\text{add a column})

\[
W_i \phi = \begin{array}{c}
\phi \\
\phi \\
\phi
\end{array} : m \to n
\]

(\text{add a link})

\[
Z = \begin{array}{c}
\phi \\
\phi
\end{array} 0 \to 0
\]

Lemma

\[
W_i W_j \Rightarrow W_j W_i \quad (i < j) \quad EH \Rightarrow HE \quad EW_i \Rightarrow W_{i+1}E
\]

normal forms are in bijection with multirelations.
Now begins the novel part: partial orders
THE CATEGORY OF FINITE POSETS

We write \textbf{FinPOSet} for the PRO whose

- objects are integers
- a morphism $f : [m] \to [n]$ is a finite poset $(f, \leq_f)$ with $m$ chosen minimal elements and $n$ chosen maximal elements (both sets being distinct)
THE CATEGORY OF FINITE POSETS

We write $\text{FinPOSet}$ for the PRO whose

- objects are integers
- a morphism $f : [m] \to [n]$ is a finite poset $(f, \leq_f)$ with $m$ chosen minimal elements and $n$ chosen maximal elements (both sets being distinct)

For instance:

```
[2]       0  1
|         |
|         |
f        |
|         | 0
[1]       0
```
\[(1) \xrightarrow{g} (2) \xrightarrow{f} (1)\]

\[g \quad \bullet \quad 0 \quad 1 \]

\[f \quad \bullet \quad 0 \quad 1\]

\[\text{and tensor product is juxtaposition as usual}\]
\[\begin{array}{ccc}
[1] & 0 & 0 \\
g & \bullet & \bullet \\
[2] & 0 & 1 \\
\end{array}\] = 
\[\begin{array}{ccc}
[1] & 0 & 0 \\
f & \bullet & \bullet \\
[2] & 0 & 1 \\
\end{array}\]
COMPOSITION

\[
\begin{array}{c}
\{1\} \\
g \\
\{2\}
\end{array}
\quad
\begin{array}{ccc}
0 & 0 & 0 \\
\bullet & 1 & \bullet \\
0 & 0 & 0
\end{array}
= \\
\begin{array}{ccc}
0 & 0 & 0 \\
\bullet & 1 & \bullet \\
0 & 0 & 0
\end{array}
\quad
\begin{array}{c}
\{1\} \\
f \\
\{2\}
\end{array}
\]

(and tensor product is juxtaposition as usual)
RELATIONS IN \textit{FinPOSet}

An element of a poset is \textit{internal} when it is not in the source or the target.
RELATIONS IN FinPOSet

An element of a poset is *internal* when it is not in the source or the target.

A relation can be seen as a poset with no internal elements: we have a faithful embedding \( \text{Rel} \hookrightarrow \text{FinPOSet} \).

\[
\begin{array}{ccc}
0 & \rightarrow & 1 \\
| & \downarrow & \\
0 & \rightarrow & 1 \\
& & \rightarrow & 2
\end{array}
\]
RELATIONS IN \text{FinPOSet}

An element of a poset is \textit{internal} when it is not in the source or the target.

A relation can be seen as a poset with no internal elements: we have a faithful embedding \( \text{Rel} \hookrightarrow \text{FinPOSet} \).

So, it makes sense to build a presentation extending the presentation for \( \text{Rel} \).
A PRESENTATION FOR FinPOSet

Theorem
The category FinPOSet is presented by the 3-polygraph with
- six 2-generators

\[
\begin{align*}
\mu &: 2 \to 1 & \eta &: 0 \to 1 & \delta &: 1 \to 2 & \varepsilon &: 1 \to 0 & \gamma &: 2 \to 2 & \sigma &: 1 \to 1
\end{align*}
\]

- relations
  - \((\mu, \eta, \delta, \varepsilon, \gamma)\) is a qualitative bialgebra (as for Rel)
  - dependencies are transitive
ABOUT THE PROOF

Notice that it cannot be done using a canonical rewriting system:

\[ \text{does not terminate} \]
ABOUT THE PROOF

Notice that it cannot be done using a canonical rewriting system:

- does not terminate

- does not allow to derive
What about presenting increasing functions between posets?
What about presenting increasing functions between posets?

We extend this to better understand commutative monads.
Definition
A **monad** $T$ on a category $C$ is an endofunctor $T : C \to C$ together with two natural transformations

$$\mu : TT \Rightarrow T \quad \quad \eta : \text{Id} \Rightarrow T$$

such that

$$
\begin{align*}
TTT & \xrightarrow{\mu_T} TT \\
TT & \xrightarrow{T\mu} T \\
T & \xrightarrow{T\eta} T
\end{align*}
$$
Definition
A **monad** \( T \) on a category \( C \) is an endofunctor \( T : C \to C \) together with two natural transformations

\[
\mu : TT \Rightarrow T \\
\eta : \text{Id} \Rightarrow T
\]

such that

\[
\begin{align*}
TTT & \xrightarrow{\mu_T} TT \\
T & \xrightarrow{\eta_T} TT & TT & \xleftarrow{T\eta} T
\end{align*}
\]

Example
The **stream monad** \( TA = A^R \) with

\[
\begin{align*}
\eta_A &: A \to TA \\
a &\mapsto \lambda t.a \\
\mu_A &: TTA \to TA \\
s &\mapsto \lambda t.s\text{tt}
\end{align*}
\]
**Definition**

A **strength** for a monad $T$ on a monoidal category $C$ is a natural transformation

$$\tau_{A,B} : A \otimes TB \to T(A \otimes B)$$

such that

\[
\begin{align*}
(A \otimes B) \otimes TC & \xrightarrow{T_{A\otimes B,C}} T((A \otimes B) \otimes C) \\
\alpha_{A,B,TC} & \downarrow \\
A \otimes (B \otimes TC) & \xrightarrow{A \otimes T_{B,C}} A \otimes T(B \otimes C) \xrightarrow{T \alpha_{A,B,C}} T((A \otimes (B \otimes C)))
\end{align*}
\]
**Definition**

A **strength** for a monad $T$ on a monoidal category $\mathcal{C}$ is a natural transformation

$$\tau_{A,B} : A \otimes TB \to T(A \otimes B)$$

such that

$$A \otimes TTB \xrightarrow{\tau_{A,TB}} T(A \otimes TB) \xrightarrow{T\tau_{A,B}} TT(A \otimes B)$$

$$A \otimes TB \xrightarrow{\mu_{A\otimes B}} T(A \otimes B)$$
Definition
A **strength** for a monad $T$ on a monoidal category $C$ is a natural transformation

$$\tau_{A,B} : A \otimes TB \to T(A \otimes B)$$

such that

$$I \otimes TA \xrightarrow{\tau_{I,A}} T(I \otimes A)$$

$$A \otimes B \xrightarrow{A \otimes \eta_B} A \otimes TB$$

$$\lambda_{TA} \downarrow \quad T \lambda_A \quad \eta_{A \otimes B} \downarrow \quad \tau_{A,B}$$

$$TA \quad T(A \otimes B)$$
Definition

A **strength** for a monad \( T \) on a monoidal category \( \mathcal{C} \) is a natural transformation

\[
\tau_{A,B} : A \otimes TB \to T(A \otimes B)
\]

such that

\[
\begin{align*}
I \otimes TA & \xrightarrow{\tau_{I,A}} T(I \otimes A) \\
\lambda_TA & \xrightarrow{T \lambda_A} TA
\end{align*}
\]

and

\[
\begin{align*}
A \otimes B & \xrightarrow{A \otimes \eta_B} A \otimes TB \\
\eta_A \otimes B & \xrightarrow{T \eta_{A \otimes B}} T(A \otimes B)
\end{align*}
\]

Definition

A **costrength** \( \tau_{A,B} : TA \otimes B \to T(A \otimes B) \) is defined dually.
Example
The stream monad is strong with

\[ \tau_{A,B} : A \times TB \rightarrow T(A \times B) \]

\[ (a, s) \mapsto \lambda t.(a, st) \]
Example

The stream monad is strong with

$$
\tau_{A,B} : A \times TB \rightarrow T(A \times B) \\
(a, s) \mapsto \lambda t.(a, st)
$$

where

$$
\begin{array}{ccc}
A \otimes TTB \xrightarrow{\tau_{A,TB}} T(A \otimes TB) \xrightarrow{T\tau_{A,B}} TT(A \otimes B) \\
A \otimes \mu_B \downarrow \quad \mu_A \otimes B \downarrow

A \otimes TB \xrightarrow{\tau_{A,B}} T(A \otimes B)
\end{array}
$$

means

$$
\lambda t.(\lambda t_1 t_2.(a, st_1 t_2))tt = \lambda t.(a, (\lambda t'.st't')t)
$$
Definition

A **commutative** monad \( T : C \to C \) is a monad together with a strength and a costrength

\[
\tau_{A,B} : A \otimes T B \to T (A \otimes B) \quad \nu_{A,B} : T A \otimes B \to T (A \otimes B)
\]

such that

\[
T (A \otimes TB) \xrightarrow{T \tau_{A,B}} TT (A \otimes B) \quad \nu_{A,TB} \quad \mu_{A \otimes B}
\]

\[
TA \otimes TB \quad \tau_{TA,B} \quad T (TA \otimes B) \xrightarrow{T \nu_{A,B}} TT (A \otimes B) \quad \mu_{A \otimes B}
\]
Example

The stream monad:

\[
\begin{align*}
T(A \otimes TB) & \xrightarrow{T \tau_{A,B}} TT(A \otimes B) \\
TA \otimes TB & \xrightarrow{\nu_{A, TB}} T(TA \otimes B) \\
& \xrightarrow{T \nu_{A, B}} TT(A \otimes B) \\
& \xrightarrow{\mu_{A \otimes B}} T(A \otimes B)
\end{align*}
\]

means

\[
\lambda t. (\lambda t_1 t_2. (s_1 t_1, s_2 t_2)) tt = \lambda t. (\lambda t_2 t_1. (s_1 t_1, s_2 t_2)) tt
\]
We can try to draw these laws using string diagrams:

- a monoidal category is a (pseudo-)monoid in $\textbf{Cat}$:

\[ \boxtimes : C \times C \to C \quad I : 1 \to C \]

satisfying associativity and unitality

\[ \begin{array}{ccc}
\text{\rotatebox{90}{$\boxtimes$}} & = & \begin{array}{c}
\text{\rotatebox{-270}{$\boxtimes$}}
\end{array} \\
\begin{array}{c}
\text{\rotatebox{-270}{$\boxtimes$}}
\end{array} & = & \text{\rotatebox{90}{$I$}} = \text{\rotatebox{90}{$\boxtimes$}} \\
\end{array} \]

(actually up to iso)
We can try to draw these laws using string diagrams:

- a monoidal category is a (pseudo-)monoid in $\text{Cat}$:

- a monad $T : C \rightarrow C$:

  together with
We can try to draw these laws using string diagrams:

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\[ \eta : \]
We can try to draw these laws using string diagrams:

- a monoidal category is a (pseudo-)monoid in \(\mathbf{Cat}\):

- a monad \(\mathcal{T} : \mathcal{C} \to \mathcal{C}\): together with

satisfying

(these define exactly functions between totally ordered sets)
the strength $\tau_{A,B} : A \otimes TB \to T(A \otimes B)$
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looks like an increasing function between posets:
IN STRING DIAGRAMS

and actually all the laws of commutative monads are compatible with this interpretation:

\[
A \otimes TTB \xrightarrow{\tau_{A,TB}} T(A \otimes TB) \xrightarrow{T\tau_{A,B}} TT(A \otimes B)
\]

\[
A \otimes \mu_B \downarrow \quad \quad \quad \quad T(A \otimes TB) \xrightarrow{\mu_{A \otimes B}} TT(A \otimes B)
\]

becomes

\[
A \otimes TB \xrightarrow{\tau_{A,B}} T(A \otimes B)
\]
and actually all the laws of commutative monads are compatible with this interpretation:

\[
T(A \otimes TB) \xrightarrow{T_{TA,B}} TT(A \otimes B)
\]

\[
TA \otimes TB \xrightarrow{\tau_{TA,B}} T(TA \otimes B) \xrightarrow{T\nu_{A,B}} TT(A \otimes B)
\]

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We define the PRO $\text{PTrees}$ as the monoidal subcategory of $\text{FinPOSet}$ whose morphisms $m \rightarrow n$ are posets with $m$ minimal and $n$ maximal chosen elements which are planar forests:
We define the PRO $\text{PTrees}$ as the monoidal subcategory of $\text{FinPOS} \text{Set}$ whose morphisms $m \to n$ are posets with $m$ minimal and $n$ maximal chosen elements which are planar forests:

- a poset is a forest when
  \[ a \leq c \land b \leq c \implies a \leq b \lor b \leq a \]

i.e.

\[
\begin{array}{c}
\text{c} \\
\downarrow \\
\text{a} & \text{b}
\end{array}
\quad \Rightarrow \quad
\begin{array}{c}
\text{c} \\
\downarrow \\
\text{a} & \text{b} & \text{c}
\end{array}
\quad \lor \quad
\begin{array}{c}
\text{c} \\
\downarrow \\
\text{a} & \text{b}
\end{array}
\]

\[
\begin{array}{c}
\text{c} \\
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\text{c} \\
\downarrow \\
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\end{array}
\]
We define the PRO **PTrees** as the monoidal subcategory of **FinPOSet** whose morphisms \( m \to n \) are posets with \( m \) minimal and \( n \) maximal chosen elements which are *planar forests*:

- a poset is a *forest* when

\[
a \leq c \land b \leq c \implies a \leq b \lor b \leq a
\]

- *planar* means that it can be drawn without crossings:

```
0 1
\[ \bullet \leftrightarrow \bullet \]
```

is forbidden
A PRESENTATION OF PTrees

Proposition

The PRO PTrees is presented by the 3-polygraph with

- three 2-generators

\[ \begin{align*}
\text{\Upsilon} &= \text{\Upsilon} \\
\text{\Upsilon} &= \text{\Upsilon} \\
\text{\Upsilon} &= \text{\Upsilon}
\end{align*} \]

- three relations
We define the monoidal 2-category **IncPTrees** by considering **PTrees** (planar forests) together with increasing functions between them, which preserve the number of trees.

**Theorem**

*The category **IncPTrees** is presented by the 3-polygraph with three 2-generators:*
We define the monoidal 2-category \textbf{IncP\textsc{Trees}} by considering \textbf{P\textsc{Trees}} (planar forests) together with increasing functions between them, which preserve the number of trees.

**Theorem**

The category \textbf{IncP\textsc{Trees}} is presented by the 3-polygraph with

- three 2-generators:

- 3-generators:
MAKING THIS PRECISE

We define the monoidal 2-category **IncPTrees** by considering **PTrees** (planar forests) together with increasing functions between them, which preserve the number of trees.

**Theorem**

*The category **IncPTrees** is presented by the 3-polygraph with*

- **three 2-generators:**

- **3-generators:**

\[
\begin{align*}
\Rightarrow & \Rightarrow \\
\Rightarrow & \Rightarrow \\
\Rightarrow & \Rightarrow \\
\end{align*}
\]
We define the monoidal 2-category $\text{IncPTrees}$ by considering $\text{PTrees}$ (planar forests) together with increasing functions between them, which preserve the number of trees.

**Theorem**

The category $\text{IncPTrees}$ is presented by the 3-polygraph with

- three 2-generators:
- 3-generators
- relations: the axioms of commutative monads
MAKING THIS PRECISE

Theorem

A strong monoidal functor $\text{IncPTrees} \rightarrow \text{Cat}$

is the same as

a category together with a commutative monad
Theorem (MacLane)

In a monoidal category, “all diagrams” commute.

\[ ((A \otimes I) \otimes B) \otimes C \rightarrow (A \otimes B) \otimes C \]

\[ (A \otimes (I \otimes B)) \otimes C \rightarrow A \otimes (B \otimes C) \]

\[ A \otimes ((I \otimes B) \otimes C) \rightarrow A \otimes (I \otimes (B \otimes C)) \]
Theorem

Given a monoidal category $\mathcal{C}$ with a strong monad there are as many canonical morphisms in $\mathcal{C}(A, B)$ as there are functions from $A$ to $B$ seen as posets:

$$T(TTA \otimes (TI \otimes B)) \rightarrow T(A \otimes TB)$$
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Theorem

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$$T\left( TTA \otimes (TI \otimes B) \right) \rightarrow T(\mathcal{A} \otimes TB)$$
In a programming language, if \( s : T\mathbb{N} \) is a stream of integers, one would like to automatically make sense of programs such as

\[
s : T\mathbb{N} \vdash 3 + s : T\mathbb{N}
\]
TOWARDS MONADIC COERCIONS?

In a programming language, if \( s : T\mathbb{N} \) is a stream of integers, one would like to automatically make sense of programs such as

\[
s : T\mathbb{N} \vdash 3 + s : T\mathbb{N}
\]

A monad is characterized by:

- its return (or unit): \( \rho_A : A \rightarrow TA \)
- its bind: \( \beta_A : (A \rightarrow TB) \rightarrow (TA \rightarrow TB) \)

We would like to implicitly use those as coercions, but it would have to be done in a coherent way!
We have shown that higher-dimensional rewriting methods can be helpful to better understand algebraic structures.

But lots remains to be done...