Towards Efficient Computation of Trace Spaces of Concurrent Programs

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CEA, LIST

CHOCO Party

Goal

When verifying a concurrent program,
there is a priori a large number of possible interleavings to check
(exponential in the number of processes)

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Joint work with **M. Raussen**, L. Fajstrup, É. Goubault,
E. Haucourt and A. Lang

Programs generate trace spaces

Consider the program

$$\texttt{x:=1;y:=2} \quad | \quad \texttt{y:=3}$$

It can be scheduled in three different ways:

`y:=3;x:=1;y:=2` `x:=1;y:=3;y:=2` `x:=1;y:=2;y:=3`

Giving rise to the following graph of traces:

$$\begin{tikzcd}
{} \arrow[r, "\texttt{x:=1}"] & {} \arrow[r, "\texttt{y:=2}"] & {} \\
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Concurrent access to shared variables should be protected using **mutexes** $a, b, \ldots$:

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$$P_b\,;\mathtt{x:=1};\,V_b\,;\,P_a\,;\mathtt{y:=2};\,V_a \mid P_a\,;\mathtt{y:=3};\,V_a$$

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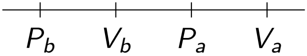
$$P_b.V_b.P_a.V_a \quad | \quad P_a.V_a$$

Geometric semantics

A program will be interpreted as a **directed space**:

- $P_b.V_b.P_a.V_a$



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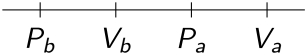

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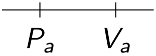

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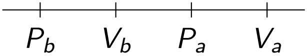

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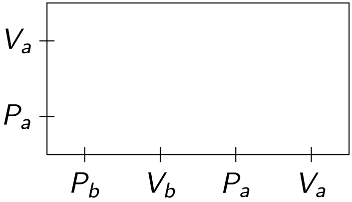

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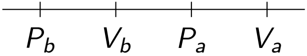

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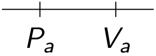

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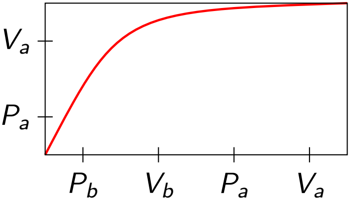

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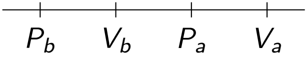

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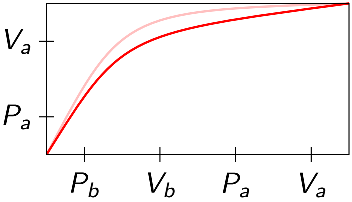
Homotopy



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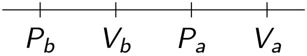

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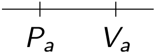

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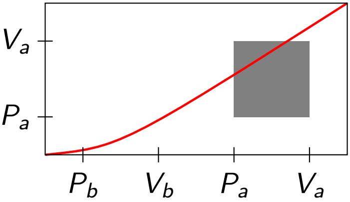
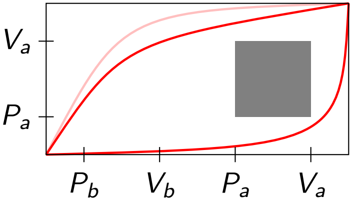


$P_b.V_b.P_a.P_a.V_a.V_a$ Forbidden region

Schedulings

A **scheduling** is the homotopy class of a path.



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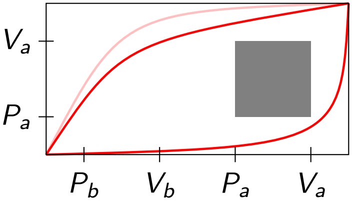

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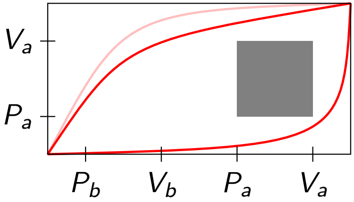
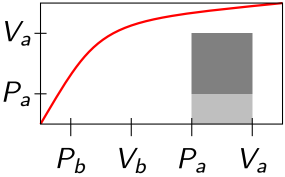
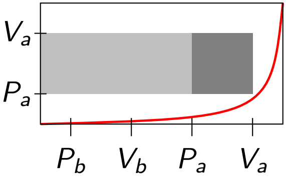


We want to compute *a path in every scheduling*

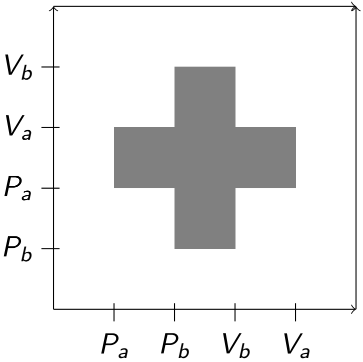
We do this by testing possible ways to go around forbidden regions:





The Swiss flag

$$P_a.P_b.V_b.V_a | P_b.P_a.V_a.V_b$$



A forbidden region

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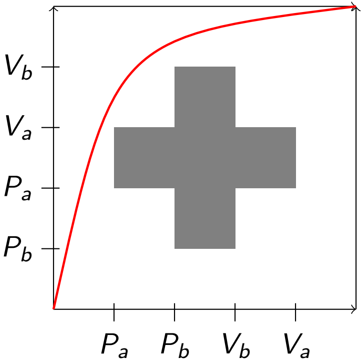

A trace: $P_b.P_a.V_a.P_a.V_b.P_b.V_b.V_a$

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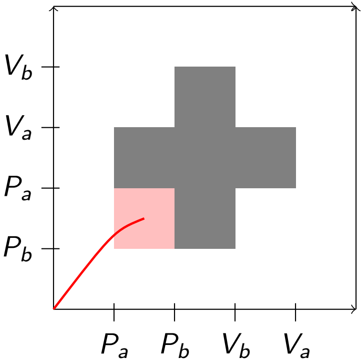

A deadlock: $P_b.P_a$

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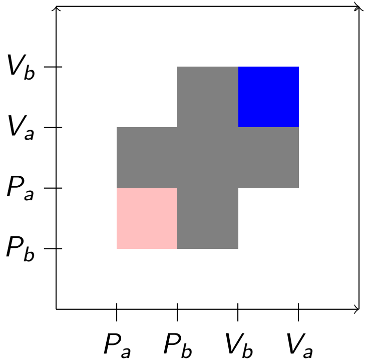
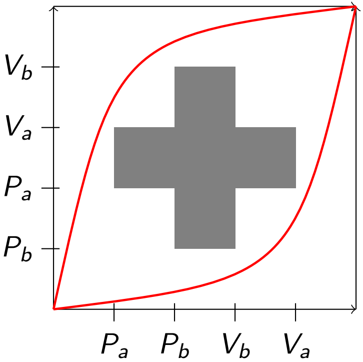


An unreachable region

The Swiss flag

$$P_a.P_b.V_b.V_a | P_b.P_a.V_a.V_b$$



Here we are interested in maximal paths modulo homotopy

Plan

1. Trace semantics of programs
2. Geometric semantics of programs
3. Computation of the trace space

Resources

We suppose fixed a set $\mathcal{R}$ of **resources** a with capacity $\kappa_a \in \mathbb{N}$.

The execution of programs are such that

1. a resource a cannot be locked (V_a) more than κ_a times
2. a resource a cannot be freed if it has not been locked

Example

A mutex is a resource of capacity 1.

Programs

We consider programs of the form:

$$p \quad ::= \quad \mathbf{1} \quad | \quad P_a \quad | \quad V_a \quad | \quad p.p \quad | \quad p|p \quad | \quad p{+}p \quad | \quad p^*$$

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We omit non-deterministic choice, loops, thread creation an join:

$$\begin{array}{cclr} A & ::= & P_a \;|\; V_a & \textit{actions} \\ t & ::= & A.t \;|\; \mathbf{1} & \textit{threads} \\ p & ::= & t|t|\ldots|t & \textit{programs} \end{array}$$

Trace semantics

The trace semantics of a program will be an **asynchronous graph**:

- a graph $G = (V, E)$ labeled by actions
- with an *independence relation* I

$$
\begin{tikzcd}
y_1 \arrow[r, "B"] & z \\
x \arrow[u, "A"] \arrow[r, "B"'] & y_2 \arrow[u, "A"']
\end{tikzcd}
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relating paths of length 2

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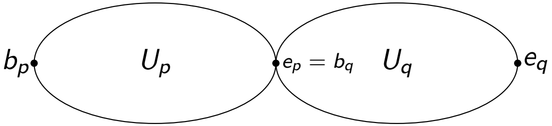
relating paths of length 2

Homotopy is the smallest congruence on paths containing I.

To every program p we associate (U_p, b_p, e_p) defined by:

- $U_{\mathbf{1}}$: terminal graph
- U_{P_a}: $b_{P_a} \xrightarrow{P_a} e_{P_a}$ $\qquad$ U_{V_a}: $b_{P_a} \xrightarrow{V_a} e_{V_a}$
- $U_{p.q}$:



- $U_{p|q}$ is the "cartesian product" of U_p and U_q:

$$(x, y) \xrightarrow{A} (x', y) \qquad \text{when } x \xrightarrow{A} x' \in U_p$$

$$(x, y') \xrightarrow{B} (x, y') \qquad \text{when } y \xrightarrow{B} y' \in U_q$$

$$\begin{tikzcd}
(y, x') \arrow[r, "B"] & (y, y') \\
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\end{tikzcd}$$

Trace semantics

Example:

$$P_b.V_b.P_a.V_a \quad | \quad P_a.V_a$$

$$\begin{tikzcd}
{} \arrow[r, "P_b"] & {} \arrow[r, "V_b"] & {} \arrow[r, "P_a"] & {} \arrow[r, "V_a"] & e_p \\
{} \arrow[u, "V_a"] \arrow[r, "P_b"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"] \arrow[r, "V_b"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"] \arrow[r, "P_a"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"] \arrow[r, "V_a"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"'] \\
b_p \arrow[u, "P_a"] \arrow[r, "P_b"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"] \arrow[r, "V_b"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"] \arrow[r, "P_a"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"] \arrow[r, "V_a"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"']
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The **resource function** r_a associates to every vertex x:

number of releases of a - number locks of a

Example:

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{} \arrow[u, "V_a"] \arrow[r, "P_b"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"] \arrow[r, "V_b"] \arrow[ur, phantom, "\sim"] & \color{red}{x} \arrow[u, "V_a"] \arrow[r, "P_a"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"] \arrow[r, "V_a"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"'] \\
b_p \arrow[u, "P_a"] \arrow[r, "P_b"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"] \arrow[r, "V_b"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"] \arrow[r, "P_a"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"] \arrow[r, "V_a"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"']
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Ex: $r_a(x) = -1$, $r_b(x) = 0$

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$$\begin{tikzcd}
\cdot \arrow[r, "P_b"] & \cdot \arrow[r, "V_b"] & \cdot \arrow[r, "P_a"] & \cdot \arrow[r, "V_a"] & e_p \\
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Ex: $r_a(y) = -2$, $r_b(y) = 0$

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\end{tikzcd}$$

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number of releases of a - number locks of a

Ex: $r_a(y) = -2 < -1 = \kappa_a$

Trace semantics T_p:
U_p where we remove vertices x which do not satisfy

$$0 \leqslant r_a(x) + \kappa_a \leqslant \kappa_a$$

Example:

$$P_b.V_b.P_a.V_a \quad | \quad P_a.V_a$$

$$\begin{tikzcd}
{} \arrow[r, "P_b"] & {} \arrow[r, "V_b"] & {} \arrow[r, "P_a"] & {} \arrow[r, "V_a"] & {} \\
{} \arrow[u, "V_a"] \arrow[r, "P_b"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"] \arrow[r, "V_b"] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "V_a"] & & {} \arrow[u, "V_a"'] \\
{} \arrow[u, "P_a"] \arrow[r, "P_b"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"] \arrow[r, "V_b"'] \arrow[ur, phantom, "\sim"] & {} \arrow[u, "P_a"] \arrow[r, "P_a"'] & {} \arrow[r, "V_a"'] & {} \arrow[u, "P_a"']
\end{tikzcd}$$

Geometric semantics

The trace semantics is difficult to use to build intuitions...

In a similar way, one can define a **geometric semantics** where programs are interpreted by *directed spaces*.

Geometric semantics

A **path** in a topological space X is a continuous map $I = [0, 1] \to X$.

Definition

A **d-space** (X, dX) consists of

- a topological space X
- a set dX of paths in X, called *directed paths*, such that
  - *constant paths*: every constant path is directed,
  - *reparametrization*: dX is closed under precomposition with increasing maps $I \to I$, which are called *reparametrizations*,
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Example

$(X, \leqslant)$ space with a partial order, $dX = \{\text{increasing maps } I \to X\}$

$\vec{I}$: d-space induced by $[0,1]$

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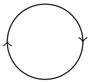
Example

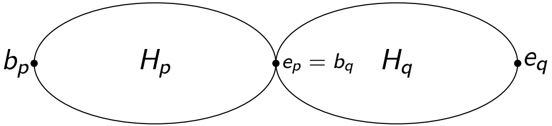
$S^1 = \{e^{i\theta}\} 0 \leqslant \theta < 2\pi$
dS^1: $p(t) = e^{i f(t)}$ for some increasing function $f : I \to \mathbb{R}$



Geometric semantics

To each program p we associate a d-space (H_p, b_p, e_p):

- $H_{\mathbf{1}}$: $\bullet$
- $H_{P_a} = \vec{I}$ $\qquad$ $H_{V_a} = \vec{I}$
- $H_{p.q}$:



- $H_{p|q}$: $H_p \times H_q$, $b_{p|q} = (b_p, b_q)$, $e_{p|q} = (e_p, e_q)$

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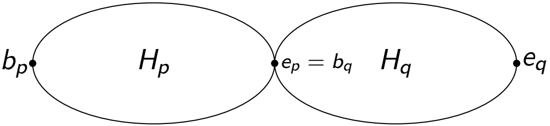

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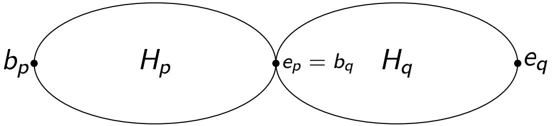

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$F_p = \{x \in H_p \ / \ \exists a \in \mathcal{R}, \quad r_a(x) + \kappa_a < 0 \quad \text{or} \quad r_a(x) > 0\}$

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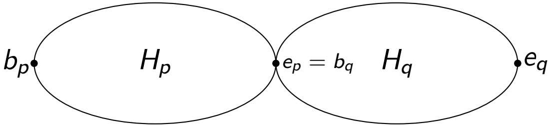

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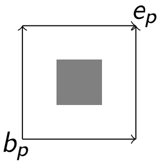
Geometric semantics: $G_p = H_p \setminus F_p$

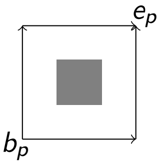
Examples of geometric semantics

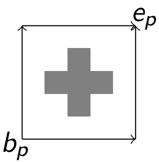
$P_a.V_a | P_a.V_a$



Examples of geometric semantics

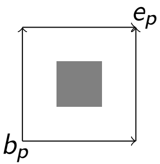
$P_a.V_a | P_a.V_a$ $\quad$ $P_a.P_b.V_b.V_a | P_b.P_a.V_a.V_b$

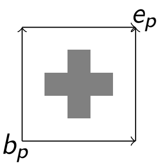


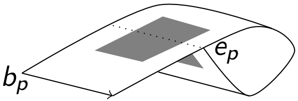


Examples of geometric semantics

$P_a.V_a|P_a.V_a$ $\quad$ $P_a.P_b.V_b.V_a|P_b.P_a.V_a.V_b$ $\quad$ $P_a.(V_a.P_a)^*|P_a.V_a$

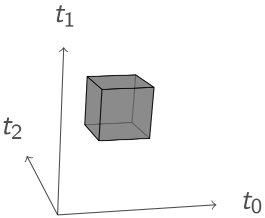




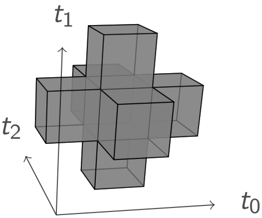


Examples of geometric semantics

$P_a.V_a|P_a.V_a|P_a.V_a$
$(\kappa_a = 2)$



$P_a.V_a|P_a.V_a|P_a.V_a$
$(\kappa_a = 1)$



Geometric realization

The two semantics are “essentially the same”: the geometric semantics is the **geometric realization** of a *cubical set*

$$G_p \quad = \quad \int^{n \in \square} T_p(n) \cdot \vec{I}^n$$

Proposition

Given a program p, with T_p as trace semantics and G_p as geometric semantics,

- *every path $\pi : b \to e$ in T_p induces a path $\overline{\pi} : b \to e$ in G_p,*
- *$\pi \sim \rho$ in T_p implies $\overline{\pi} \sim \overline{\rho}$ in G_p*
- *every path ρ of G_p is homotopic to a path $\overline{\pi}$ (π path in G_p)*

Computing the trace space

Goal

Given a program p, we describe an algorithm to compute a trace in each equivalence class of traces $\pi : b_p \to e_p$ *up to homotopy in* G_p.

The proposition before ensures that it is the same to compute this in the trace semantics or in the geometric semantics.

The algorithm

Suppose given a program

$$p \quad = \quad p_0 | p_1 | \ldots | p_{n-1}$$

with n *threads*.

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$$p \quad = \quad p_0|p_1|\dots|p_{n-1}$$

with n *threads*.

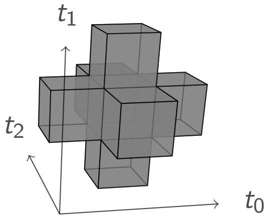
Under mild assumptions, the geometric semantics is of the form

$$G_p \quad = \quad \vec{I}^n \setminus \bigcup_{i=0}^{l-1} R^i$$

where

$$R^i = \prod_{j=0}^{n-1}]x_j^i, y_j^i[$$

are l *open rectangles*.



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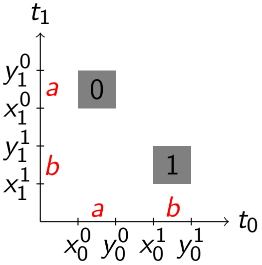
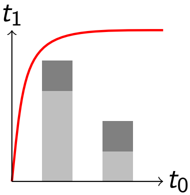
Example

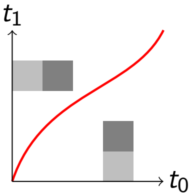
$$P_a.V_a.P_b.V_b | P_b.V_b.P_a.V_a$$

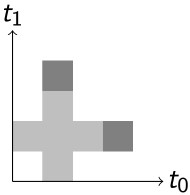


The algorithm

The main idea of the algorithm is to extend the forbidden cubes downwards in various directions and look whether there is a path from b to e in the resulting space.







By combining those information, we will be able to compute traces modulo homotopy.

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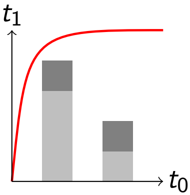

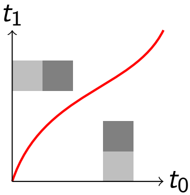

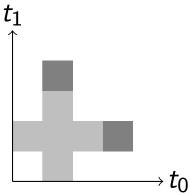

By combining those information, we will be able to compute traces modulo homotopy.

The directions in which to extend the holes will be coded by boolean matrices M.

The index poset

$\mathcal{M}_{l,n}$: boolean matrices with l rows and n columns.

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X_M: space obtained by *extending*
for every (i,j) such that $M(i,j) = 1$
the forbidden cube i downwards
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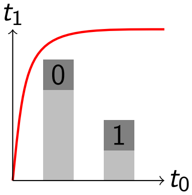
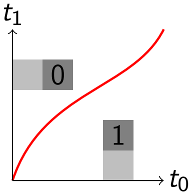


$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$



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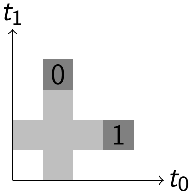

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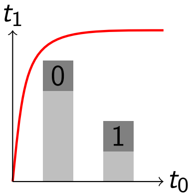

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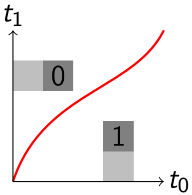

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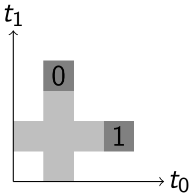

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Psi : \mathcal{M}_{l,n} \to \{0,1\}$:

- $\Psi(M)=0$ if there is a path $b \to e$: M is alive
- $\Psi(M)=1$ if there is no path $b \to e$: M is dead

The index poset

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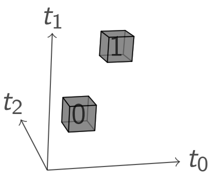

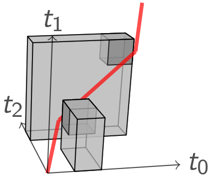

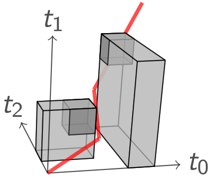

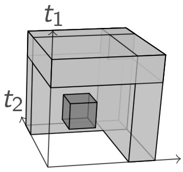

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

alive *alive* *alive* *dead*

The index poset

- $\mathcal{M}_{l,n}$ is equipped with the pointwise ordering
- Ψ is increasing: more 1 $\Rightarrow$ more obstructions
- $\mathcal{M}^R_{l,n}$: matrices with non-null rows
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The **dead poset** $D(X) = \{M \in \mathcal{M}_{l,n}^{C} \ / \ \Psi(M) = 1\}$.

$D(X) \quad \rightsquigarrow \quad \mathcal{C}(X) \quad \rightsquigarrow \quad$ homotopy classes of traces

The dead poset

Proposition

A matrix $M \in \mathcal{M}^C_{l,n}$ is in $D(X)$ iff it satisfies

$$\forall (i,j) \in [0:l[\times[0:n[, \qquad M(i,j) = 1 \quad \Rightarrow \quad x^i_j < \min_{i' \in R(M)} y^{i'}_j$$

where $R(M)$: indexes of non-null rows of M.

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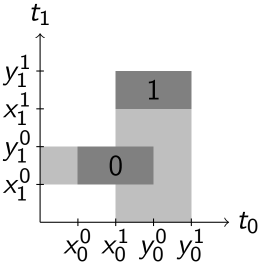
Example

M is dead:



$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{array}{l} x_1^0 = 1 < 2 = \min(y_1^0, y_1^1) \\ x_0^1 = 2 < 3 = \min(y_0^0, y_0^1) \end{array}$$

The index poset

Proposition

A matrix M is in $\mathcal{C}(X)$ iff for every $N \in D(X)$, $N \nleq M$.

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Remark

$N \nleq M$: *there exists* (i,j) *s.t.* $N(i,j) = 1$ *and* $M(i,j) = 0$.

Remark

Since $\mathcal{C}(X)$ is downward closed it will be enough to compute the set $\mathcal{C}_{\max}(X)$ of maximal alive matrices.

Connected components

$M \wedge N$: pointwise min of M and N

Definition

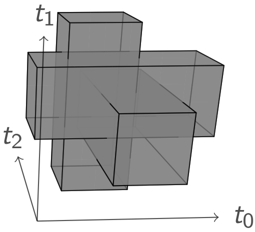
Two matrices M and N are **connected** when $M \wedge N$ does not contain any null row.

Proposition

The connected components of $\mathcal{C}(X)$ are in bijection with homotopy classes of traces $b \to e$ in X.

Dining philosophers

n processes p_k in parallel:



$$p_k = P_{a_k}.P_{a_{k+1}}.V_{a_k}.V_{a_{k+1}}$$

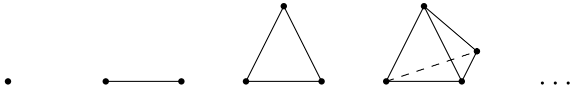
n	sched.	ALCOOL (s)	ALCOOL (MB)	SPIN (s)	SPIN (MB)
8	254	0.1	0.8	0.3	12
9	510	0.8	1.4	1.5	41
10	1022	5	4	8	179
11	2046	32	9	42	816
12	4094	227	26	313	3508
13	8190	1681	58	∞	∞
14	16382	13105	143	∞	∞

What is exactly the geometric information contained in the index poset?

The geometry of the trace space

The n-dimensional standard simplex:

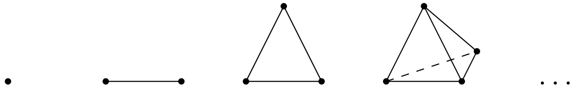
0	1	2	3	$\dots$



The geometry of the trace space

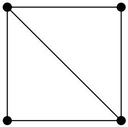
The n-dimensional standard simplex:

0 1 2 3 ...



Definition

A **simplicial** set is a sequence (X_n) of sets of n-simplices together with face maps.



The geometry of the trace space

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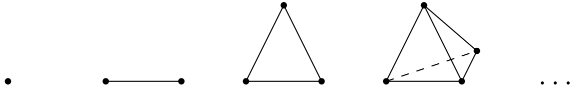
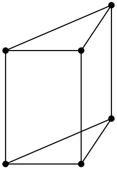


Definition

A **prodsimplicial** set is a sequence (X_n) of sets of products of k-uples n_i-simplices $(n = n_1 + \ldots + n_k)$ together with face maps.



The geometry of the trace space

Proposition

The index poset $\mathcal{C}(X)$ is a prodsimplicial set, a matrix $M \in \mathcal{C}(X)$ representing a prodsimplex

$$\Delta_{k_0} \times \Delta_{k_2} \times \ldots \times \Delta_{k_{l-1}}$$

where $k_i + 1$ is the number of 1 on the i-th line of M.

Proposition

The geometric realization of the prodsimplicial set $\mathcal{C}(X)$ is homotopy equivalent to the trace space.

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$\Rightarrow$ Yes: computers are much better at manipulating numbers than complex algebraic structures

Future works

We compute **one execution trace in each homotopy class**.

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What remains to do:

- use these trace to do static analysis
  (e.g. abstract interpretation)
- extend the methodology to program with loops
- compute schedulings compositionally
- relate this the component category
- ...