Chain complexes and higher categories

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Summary

1. Construction of a functor

$$\nu \colon \mathbf{adc} \to \omega\text{-}\mathbf{cat},$$

   where **adc** is the category of augmented directed complexes (chain complexes with additional structure) and ω-**cat** is the category of (strict) ω-categories.
2. An important full subcategory Φ of **adc** equivalent under ν to a full subcategory of ω-**cat**.
3. Examples: simple ω-categories, cubes, orientals, opetopes.

1. The construction of ν

ω-categories

This entire section is based on [6].

An ω*-category* is a set C which serves as the morphism set for an infinite sequence of categories with composition operators $\#_0, \#_1, \ldots$ such that:

1. the category structures commute;
2. every identity with respect to $\#_n$ is also an identity with respect to $\#_{n+1}, \#_{n+2}, \ldots;$
3. every member of C is an identity with respect to $\#_n$ for some n.

Category objects in the category of abelian groups

In the category of abelian groups a category object is equivalent to a homomorphism.
In the category object corresponding to $\partial\colon K_1 \to K_0$:
the morphisms are the members of $K_0 \times K_1$;
the left and right identities of $x = (x_0, x_1)$ are given by

$$d^- x = (x_0, 0), \quad d^+ x = (x_0 + \partial x_1, 0);$$

composition is given by

$$(x_0, x_1) \# (y_0, y_1) = (x_0, x_1 + y_1) \quad (y_0 = x_0 + \partial x_1).$$

(Alternatively, there are objects $x_0 \in K_0$ and morphisms

$$x_1\colon x_0 \to x_0 + \partial x_1 \quad (x_0 \in K_0,\ x_1 \in K_1),$$

and composition is addition in K_1.)

ω-category objects in the category of abelian groups

An ω-category object in the category of abelian groups is equivalent to a chain complex

$$\dots \xrightarrow{\partial} K_1 \xrightarrow{\partial} K_0 \qquad (\partial\partial = 0).$$

The morphisms are the sequences

$$x = (x_0, x_1, \dots) \in \bigoplus_{i=0}^{\infty} K_i;$$

the left and right identities of x are given by

$$d_n^- x = (x_0, \dots, x_n, 0, \dots), \quad d_n^+ x = (x_0, \dots, x_{n-1}, x_n + \partial x_{n+1}, 0, \dots);$$

compositions where defined are given by

$$x \#_n y = (x_0, \dots, x_n, x_{n+1} + y_{n+1}, x_{n+2} + y_{n+2}, \dots).$$

The functor $\nu\colon$ **adc** $\to \omega$-**cat**

An *augmented directed complex* (adc) consists of an augmented chain complex of abelian groups

$$\ldots \xrightarrow{\partial} K_1 \xrightarrow{\partial} K_0 \xrightarrow{\epsilon} \mathbf{Z} \qquad (\partial\partial = 0,\ \epsilon\partial = 0),$$

together with a choice of distinguished submonoids K_i^* in K_i. The morphisms in the category **adc** of adcs are the augmentation-preserving chain maps taking distinguished submonoids into distinguished submonoids.
The ω-category νK associated to an adc K consists of the sequences $x = (x_0, x_1, \ldots) \in \oplus_i K_i$ such that

$$\epsilon x_0 = 1, \quad x_i \in K_i^*, \quad x_i + \partial x_{i+1} \in K_i^*,$$

with identities and compositions as before.

2. The full subcategory Φ of **adc**

Free adcs

An adc is *free* if the distinguished submonoids are free commutative monoids and if the chain groups are free abelian groups on the same bases.

If x is a chain in a free adc, then $\partial^+ x$ and $\partial^- x$ are the positive and negative parts of ∂x;

thus

$$\partial x = \partial^+ x - \partial^- x$$

such that $\partial^+ x$ and $\partial^- x$ are sums of basis elements with no common terms.

The full subcategory Φ of **adc**

The objects in Φ are the free adcs which are unital and loop-free in the following senses.
A free adc is *unital* if $\epsilon(\partial^-)^q a = \epsilon(\partial^+)^q a = 1$ for each basis element a, where $q = \dim a$.
A free adc is *loop-free* if the following condition holds: let q be a nonnegative integer and let $a_0, \ldots, a_k$ be basis elements with $\dim a_i = p(i) > q$ such that for $1 \leq i \leq k$ the sums of basis elements $(\partial^+)^{p(i-1)-q} a_{i-1}$ and $(\partial^-)^{p(i)-q} a_i$ have a common term; then $a_0, \ldots, a_k$ are distinct.

The main theorem

Theorem ([6])

*The restriction of ν to Φ is an equivalence between Φ and a full subcategory of ω-**cat**.*

Presentations

An *atom* in a free ω-category is a sequence of the form

$$\big((\partial^-)^q a, \ldots, \partial^- a, a, 0, \ldots\big),$$

where a is a q-dimensional basis element.

Theorem ([6])

Let K be an object of Φ, Then νK has a presentation such that:
the generators are the atoms;
the relations express the left and right identities of atoms as iterated composites of atoms.

Representations by geometric complexes and by sets

Let K be an object of Φ.

Theorem ([8])

If $(x_0, x_1, \dots)$ is a member of νK, then the chains x_i and $x_i + \partial x_{i+1}$ are sums of distinct basis elements.

One can therefore regard K as a geometric complex whose cells are the basis elements. The members of νK can be regarded as subcomplexes; the operators d_i^- and d_i^+ pick out parts of the boundary; the operators $\#_i$ are connected sums.
Alternatively one can regard the members of νK as subsets of the basis; the operators d_i^- and d_i^+ pick out subsets; the operators $\#_i$ are unions.

Representations by dual graphs

Let $(x_0, x_1, \ldots)$ be a member of νK, where K is an object of Φ, and let x_i or $x_i + \partial x_{i+1}$ be a sum of basis elements $a_1 + \ldots + a_k$.

Theorem ([8])

If $i = 0$ then $k = 1$; that is, x_0 and $x_0 + \partial x_1$ are single basis elements.
If $i > 0$ then the chain

$$x_{i-1} + \partial^+ a_1 + \ldots + \partial^+ a_k = \partial^- a_1 + \ldots + \partial^- a_k + (x_{i-1} + \partial x_i)$$

is a sum of distinct basis elements.

In the last part one can regard the basis elements as the edges in an 'open directed graph' with vertices $a_1, \ldots, a_k$; the terms of $\partial^+ a_i$ have source a_i; the terms of $\partial^- a_i$ have target a_i.

3. Examples

Simple chain complexes and simple ω-categories

This example is based on [7].

A *simple chain complex* is a free adc with a finite non-empty ordered basis such that

the initial and final basis elements are zero-dimensional,

the dimensions of consecutive basis elements differ by ± 1,

zero-dimensional basis elements have augmentation 1,

if a is a basis element of positive dimension q then $\partial^- a$ is the last $(q-1)$-dimensional basis element before a and $\partial^+ a$ is the first $(q-1)$-dimensional basis element after a.

A *simple ω-category* is the image under ν of a simple chain complex.

The corresponding full subcategories of **adc** and ω-**cat** are isomorphic to Joyal's category Θ of finite discs [4].

Globular description of ω-categories

In the definition of an ω-category, for $p = 0, 1, 2, \ldots$ there are:
p-dimensional elements together with unary operations and laws;
composition operations $x \#_q y$ for $q < p$, together with binary laws;
associative laws $(x \#_q y) \#_q z = x \#_q (y \#_q z)$ for $q < p$;
interchange laws $(x \#_q y) \#_r (z \#_q w) = (x \#_r z) \#_q (y \#_r w)$ for $r < q < p$.
These elements, operations and laws correspond to simple chain complexes in which the dimensions of the basis elements run up, down, up, ... from 0 to 0: via p; via p, q, p; via p, q, p, q, p; via p, q, p, r, p, q, p.
An ω-category is therefore equivalent to a contravariant set-valued functor on (a suitable subcategory of) Θ taking (a suitable class of) colimit diagrams to limit diagrams.

Cubes

Let I^q be the standard cellular chain complex of a q-cube with its standard basis; then I^q is a member of Φ.
One can show that ω-categories are equivalent to cubical sets with faces, degeneracies, compositions and connections satisfying suitable laws (Al-Agl, Brown, Steiner [1]).
The degeneracies are represented by projections onto single faces, the compositions by placing cubes side by side, the connections by projections onto unions of pairs of faces.
The tensor product $I^q \otimes I^r \cong I^{q+r}$ yields a closed monoidal structure on ω-**cat**.
This example has been applied to concurrency by Gaucher [3].

Orientals

Let Δ^q be the standard cellular chain complex of the q-simplex with its standard basis; then Δ^q is a member of Φ.
The corresponding ω-categories are Street's *orientals* [10].
One can show that ω-categories are equivalent to simplicial sets with wedge operations (projections onto unions of pairs of faces) satisfying suitable laws [9].
The q-simplexes in the simplicial set corresponding to an ω-category C are the morphisms $\nu\Delta^q \to C$.

Complicial sets

One can also characterise the simplicial sets corresponding to ω-categories as the *complicial sets* (Verity [11]).
These are simplicial sets together with distinguished classes of *thin* elements satisfying certain conditions, mainly that certain horns have unique thin fillers, and that the thin fillers of horns of this kind consisting of thin elements have their additional faces thin.
In the complicial set corresponding to an ω-category C, a q-simplex $\nu\Delta^q \to C$ is thin if $q > 0$ and the image of the morphism consists of elements of dimension less than q, i.e. of identities for $\#_{q-1}$.
Requiring existence but not uniqueness of thin fillers gives a possible approach to weak ω-categories, using directed topological analogues of Kan complexes.

Opetopes

Opetopes were introduced by Baez and Dolan in [2].
Roughly, by [8], an *opetope* is an atom in a member of Φ such that $\partial^{+}a$ is a single basis element for each positive-dimensional basis element a.
The basis elements in opetopes represent operations with many inputs and single outputs (contrast the single input single output operations in simple chain complexes and the many input many output operations in cubes and simplexes).
In particular one requires nullary operations.
Naively, these would lead to basis elements a with $\partial^{-}a = 0$, contradicting unitality.
To get round this, one requires thin basis elements; one then represents a nullary operation by a basis element a such that $\partial^{-}a$ is a thin basis element.

Opetopes and trees

Using the representations by dual graphs, one can show that opetopes are equivalent to suitably related sequences of ‘open rooted trees’ with certain constituents designated as thin, subject to certain conditions (Kock, Joyal, Batanin, Mascari [5]). The relationships between the trees essentially amount to the condition that $\partial^{+} a = 0$ for each basis element a.

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