Rewriting in shuffle operads and resolutions of operads

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Motivations from algebra

Shuffle operads and Gröbner bases

Polygraphic rewriting in shuffle operads

Higher dimensional rewriting in shuffle operads
Motivations from algebra
Why algebraic rewriting?

› Newman (1942) : rewriting is a **combinatorial** theory of equivalence

› Algebraic rewriting: a combinatorial theory of **congruence**

› In computer algebra: **ideal membership, resolutions, homological properties**

› In constructive mathematics: **cofibrant replacements**

Our algebraic structure of interest is the structure of **symmetric operads** (May 1972, Loday 1996), which are abstractions of multilinear maps.

**Example: symmetric operad Lie**

The symmetric operad Lie is generated by one antisymmetric operation $\mu$ of **arity 2**, satisfying the **Jacobi relation**

$$\mu_{123} + \mu_{231} + \mu_{312} = 0.$$

Compare with

$$[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = 0.$$
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**Example: symmetric operad Lie**

The symmetric operad Lie is generated by one antisymmetric **operation** $\mu$ of **arity** 2, satisfying the **Jacobi relation**

$$
\mu^{1,2,3} + \mu^{2,3,1} + \mu^{3,1,2} = 0.
$$

Compare with

$$
[[x_1, x_2], x_3] + [[x_2, x_3], x_1] + [[x_3, x_1], x_2] = 0.
$$

Due to the symmetric actions, there is no known way to do algebraic rewriting in symmetric operads: this motivates the study of **shuffle operads** [Dotsenko-Khoroshkin 2010].
Two ways of doing rewriting:

\( \langle \) with a monomial order and an algebraic formulation of confluence: **Gröbner bases**

\( \rangle \) in a higher dimensional setting: **polygraphs**

Our goal is to mix the two approaches.
Shuffle operads and Gröbner bases
If associative algebras are a linear version of words, then shuffle operads are a linear version of planar trees.
Shuffle operads [Dotsenko-Khoroshkin 2010]

- The category $\text{Coll}$ of collections is the presheaf category on $\text{Ord}$, the category of finite nonempty ordered sets with order-preserving bijections, with values in $\text{Vect}$, the category of vector spaces over $k$.

- A collection $V$ is determined by $V(k) := V(\{1 < \cdots < k\})$ for $k \geq 1$. An element of $V(k)$ is of arity $k$.

- The shuffle composition of two collections $V, W$ is

$$V \circ_\text{III} W(I) = \bigoplus_{k \geq 1} V(k) \otimes \left( \bigoplus_{f : I \rightarrow \{1, \ldots, k\}} W(f^{-1}\{1\}) \otimes \cdots \otimes W(f^{-1}\{k\}) \right)$$

where $I \in \text{Ord}$ and $f$ is a shuffle surjection, that is, $\min f^{-1}\{1\} < \cdots < \min f^{-1}\{k\}$. The unit for this composition is $1 := (k, 0, \ldots)$.

- $(\text{Coll}, \circ_\text{III}, 1)$ is a monoidal category. The category of shuffle operads, denoted by $\text{IIIOp}$, is the category of internal monoids in $(\text{Coll}, \circ_\text{III}, 1)$. 
Tree monomials

\[ \langle \text{Tree monomials} \rangle \]

\[ \text{Let } X = (X(k))_{k \geq 1} \text{ such that } X(k) \text{ is a basis of } V(k) \text{ for every } k \geq 1. \text{ In terms of planar trees, the collection } V \circ_{\text{III}} V \text{ has a basis of planar trees} \]

\[ \begin{array}{c}
  i_1 \quad \cdots \quad i_{n_1} \quad i_{n_1+\cdots+n_{k-1}+1} \quad \cdots \quad i_{n_1+\cdots+n_k} \\
  x_1 \quad \cdots \quad x_k \\
  x_0
\end{array} \]

\[ \text{For } j \in \{1, \ldots, k\}, \text{ the inputs of } x_j \text{ are } \{i_{n_1+\cdots+n_{j-1}+1}, \cdots < i_{n_1+\cdots+n_j}\}. \text{ The inputs of } x_0 \text{ are } \{i_1 < i_{n_1+1} < \cdots < i_{n_1+\cdots+n_{k-1}+1}\}. \text{ We always draw inputs in increasing order.} \]

\[ \text{By iterating this tree construction, we get the free shuffle operad on } X, \text{ denoted by } X^{\text{III}}, \text{ spanned by tree monomials. We refer to elements of } X^{\text{III}}(k) \text{ as polynomials of arity } k. \]
Example: shuffle operad \( \text{Lie}^b \)

The shuffle operad \( \text{Lie}^b \) is generated by one operation \( \mu \) of arity 2, and satisfies the shuffle Jacobi relation

\[
\begin{array}{c}
\mu_1 2 3 - \mu_1 3 2 - \mu_2 1 3 = 0.
\end{array}
\]
With the planar tree interpretation, we can define contexts:

**Contexts**

A **context of inner arity** $k$ is a tree monomial $C[\_]$ of the form

```
\begin{array}{c}
v_1 \quad \cdots \quad \square_k \quad \cdots \quad v_n \\
\vdots \\
u
\end{array}
```

where $\square_k$ is a symbol of arity $k$ and $u, \bar{v}, \bar{w}$ are tree monomials.

Given a polynomial $f = \sum \lambda_i u_i$ of arity $k$, we define the polynomial $C[f] := \sum \lambda_i C[u_i]$. 
Monomial orders

A monomial order is a total order on tree monomials that is compatible with contexts. For a polynomial $f$,

- its leading monomial $\text{lm}(f)$ is the greatest tree monomial that occurs,
- its leading coefficient $\text{lc}(f)$ is the coefficient in front of the leading monomial,
- For example, there exists a monomial order called path-lexicographic such that

$$\mu/3 > 1\mu > 1\mu/2 > 1\mu/3.$$
Gröbner bases for operads

Given two polynomials $f$ and $g$, if there exists a context $C$ such that $C[\text{lm}(g)] = \text{lm}(f)$, then we define the reduction of $f$ by $g$ as the polynomial $f - \frac{\text{lc}(f)}{\text{lc}(g)}C[g]$.

For example, the shuffle Jacobi relation induces the reductions

\[
\begin{align*}
1 & \mu 2 \mu 3 \mu 4 & \rightarrow & 1 & \mu 2 & \mu 3 & \mu 4 & + & 1 & \mu 2 \mu & \mu 3 & \mu 4 \\
1 & \mu 2 & \mu 3 & \mu 4 & \rightarrow & 1 & \mu 2 & \mu 4 & 3 & + & 1 & \mu 4 & \mu 3 & 1 & \mu 2 & \mu 4 & 3 \\
\end{align*}
\]

A Gröbner basis of an ideal $I$ of a free shuffle operad $X^{III}$ is a generating set $G$ such that every nonzero polynomial in $I$ can be reduced by an element of $G$. 
This approach allows us to obtain a homological result on operads:

**Koszulness**

Koszulness is a property on operads that ensures the existence of a minimal model, given by: in particular, the Koszul dual cooperad of a Koszul operad is a minimal model of the operad.

**Theorem [Dotsenko-Khoroshkin 2010]**

A quadratic operad with a Gröbner basis is Koszul.
Polygraphic rewriting in shuffle operads
Shuffle 1-operads

A shuffle 1-operad is an internal category in the category $\mathcal{O}_\text{op}$ of shuffle operads.

\[
P_0 \xleftarrow{s_0} i_1 \xrightarrow{t_0} P_1
\]

The elements of $P_0$ are called 0-cells, and those of $P_1$ are called 1-cells.
Shuffle 1-polygraphs

A shuffle 1-polygraph is a diagram

\[
\begin{array}{ccc}
X_0 & \xleftarrow{s_0} & X_1 \\
\uparrow & & \downarrow^{t_0} \\
X_0 & & X_1
\end{array}
\]

where

- \(X_0 = (X_0(k))_{k \geq 1}\) is the indexed set of generators
- \(X_1 = (X_1(k))_{k \geq 1}\) is the indexed set of rewriting rules
- the source and target maps \(s_0, t_0 : X_1 \to X_0^{\text{III}}\) are from rewriting rules to the free operad on the generators.
Shuffle 1-polygraphs

A shuffle 1-polygraph is a diagram

\[
\begin{array}{c}
X_0^{III} \\
\uparrow \\
X_0
\end{array}
\xleftarrow{s_0} \xrightarrow{i_1} \begin{array}{c}
X_1^{III} \\
\downarrow^{t_0} \\
X_1
\end{array}
\]

where

\[ X_0 = (X_0(k))_{k \geq 1} \text{ is the indexed set of generators} \]
\[ X_1 = (X_1(k))_{k \geq 1} \text{ is the indexed set of rewriting rules} \]
\[ \text{the source and target maps } s_0, t_0 : X_1 \to X_0^{III} \text{ are from rewriting rules to the free operad on the generators.} \]
\[ X^{III} = (X_0^{III}, X_1^{III}) \text{ is the free shuffle 1-operad where } X_0^{III} \text{ is the shuffle operad of 0-cells and } X_1^{III} \text{ is the shuffle operad of 1-cells.} \]
\[ \text{The shuffle operad presented by } X \text{ is the coequalizer } \overline{X} \text{ of } s_0, t_0 : X_1^{III} \rightrightarrows X_0^{III}. \]
Example: polygraphic presentation of $\text{Lie}^b$

The shuffle operad $\text{Lie}^b$ is presented by the shuffle 1-polygraph

$$X_{\text{Lie}^b} := \left\langle \mu \in X_0(2) \mid \alpha : \begin{array}{c}
\mu^3 \rightarrow \mu^2 + \mu^3
\end{array} \right\rangle.$$
Rewriting systems from 1-polygraphs

Let $X$ be a **left-monomial** 1-polygraph, that is, every source is a tree monomial.

A **rewriting step** is a 1-cell

$$\lambda C[\alpha] + i_1(b) : \lambda C[u] + b \rightarrow \lambda C[a] + b$$

of $X_{\text{III}}^1$, where $\alpha : u \rightarrow a$ is a rewriting rule, $C$ is a context, $\lambda$ is a nonzero scalar, and $b$ is a polynomial of $X_{\text{III}}^0$ such that $C[u] \notin \text{supp}(b)$.

$X$ is **terminating** if there are no infinite rewriting paths.
Branchings

- A **branching** is a pair of rewriting paths \((f, g)\) with the same source.
- A **local branching** is a branching \((f, g)\) where \(f\) and \(g\) are rewriting steps. We classify local branchings as:
  - **additive**
  - **multiplicative**
  - **intersecting**
  - **critical**
The 1-polygraph $X$ is (locally) confluent if, for every (local) branching $(f, g)$, there exist rewriting paths $h$ and $k$ and the confluent diagram

The 1-polygraph $X$ is convergent if it is confluent and terminating.

A Gröbner basis is equivalent to a convergent 1-polygraph whose rewriting rules reduce the leading term to the rest.
Cellular extension

Let $X$ be a 1-polygraph.

A cellular extension is an indexed set of generating 2-cells

where $s_0(A), t_0(A)$ are 0-cells and $s_1(A), t_1(A) : a \rightarrow b$ are 1-cells of $X^{\text{III}}$.

Let $\sim$ be the equivalence relation generated by $s_1(A) \sim t_1(A)$ for every element $A$ of the cellular extension. The cellular extension is acyclic if the equivalence relation $\sim$ has one equivalence class.

**Theorem (coherent critical branchings)**

Let \( X \) be a terminating 1-polygraph with a generating 2-cell for each critical branching \((f, g)\):

\[
\begin{aligned}
  a &\rightarrow f & b &\rightarrow h \\
  g &\rightarrow c & k &\rightarrow d \\
  A &\rightarrow & A
\end{aligned}
\]

Then the cellular extension is acyclic.

We can then consider compositions of generating 2-cells by gluing confluent diagrams: this leads to the notion of higher dimensional rewriting.
Example: coherent convergence of $X_{\text{Lie}}$\textsuperscript{b}

The 1-polygraph $X_{\text{Lie}}$\textsuperscript{b} only has one critical pair and is convergent. The cellular extension will have only one generating 2-cell:
Higher dimensional rewriting in shuffle operads
A **shuffle ω-operad** is an internal (strict) ω-category in $\text{IIIOp}$, that is, an object

\[
\begin{align*}
P_0 & \xleftarrow{s_0} i_1 \xrightarrow{t_0} P_1 \\
& \xleftarrow{s_1} i_2 \xrightarrow{t_1} P_2 \\
& \quad \cdots \\
& \xleftarrow{s_{n-1}} i_n \xrightarrow{t_{n-1}} P_n \\
& \xleftarrow{s_n} i_{n+1} \xrightarrow{t_n} \cdots
\end{align*}
\]

satisfying globularity, associativity, and identity axioms.

The interaction between the ω-category structure and the operad structure gives the **linear exchange relation**: for any $n$-cells $a$ and $b$, the two paths below are equal:

\[
\begin{align*}
s_0(a) & \xrightarrow{b} s_0(b) \\
& \xrightarrow{t_0(a)} t_0(a) \\
& \xrightarrow{t_0(a')} t_0(a')
\end{align*}
\]

\[
\begin{align*}
s_0(a) & \xrightarrow{a} s_0(a) \\
& \xrightarrow{t_0(b)} t_0(b) \\
& \xrightarrow{t_0(a')} t_0(a')
\end{align*}
\]
Shuffle $\omega$-polygraphs

The definition of 1-polygraphs extends to that of shuffle $\omega$-polygraphs:

\[
\begin{array}{c}
X_0^\mathcal{I} \\
\downarrow t_0 \\
X_0 \\

X_1^\mathcal{I} \\
\downarrow t_1 \\
X_1 \\

X_2^\mathcal{I} \\
\downarrow t_1 \\
X_2 \\

\vdots

X_n^\mathcal{I} \\
\downarrow t_n \\
X_n \\
\end{array}
\]

An $\omega$-polygraph is a polygraphic resolution if each cellular extension $X_{n+1}$ is acyclic.
Overlapping polygraphic resolution

Let $X$ be a convergent 1-polygraph. We can construct the **overlapping polygraphic resolution** $\text{Ov}(X)$ on $X$, where the elements of $\text{Ov}(X)_n$ correspond to certain overlappings of $n$ rewriting rules:

1-overlapping \hspace{1cm} 2-overlapping \hspace{1cm} 3-overlapping \hspace{1cm} 4-overlapping

and so on...
Theorem [Malbos-R. 2020]

A operad $P$ with a convergent quadratic polygraphic presentation $X$ is Koszul.

**Idea of proof.**

- Extend the 1-polygraph $X$ to the overlapping polygraphic resolution $Ov(X)$.
- Study the induced $P$-bimodule resolution $(P\langle Ov(X)_n \rangle)_n$, whose generators are concentrated on the superdiagonal.
- Calculate the **Quillen homology** of the operad $P$, which is concentrated on the diagonal, which gives a sufficient condition for Koszulness.
And now...

We have defined the notion of **polygraphic resolution** of an operad.

- How to construct a resolution/cofibrant replacement in the category of **differential graded operads**?
- Does the overlapping resolution give a **minimal** cofibrant replacement?
- Can shuffle operadic rewriting be generalized to **shuffle properads**?