Iterated parametricity and semi-cubical sets: interpreting the universe

(report on work in progress)

Hugo Herbelin

(based on joint work with Hugo Moeneclaey)

Logique, Homotopie, Catégories

5 February 2021
(online)

Context: cubical type theory and iterated parametricity

Cubical type theory and (iterated) parametricity type theory:

- equipped with a cubical heterogeneous equality: $(t =_{\lambda i.A} u) \triangleq \{f : \Pi i : \mathbb{I}.A \mid f(0) \equiv t \,\&\, f(1) \equiv u\}$ for $\mathbb{I}$ a formal interval $[0, 1]$
- supports computational extensional "equality" and higher inductive types
- usually modelled in set theory using cubical sets with at least faces

Context: cubical type theory and iterated parametricity

	Cubical type theory	*Parametric type theory*
name of equality	*path*	*bridge*
def. of $(A =_{\mathsf{U}_l} B)$	$A \simeq_{\mathsf{U}_l} B$ (i.e. equivalence)	$A \times B \to \mathsf{U}_l$
	provides univalence and Kan composition	not transportable
variants	*de Morgan* style in CCHM (2015) *Cartesian* style in ABCFHL (2019)	Bernardy, Coquand, Moulin (2014) Nuyts, Vezzosi, Devriese (2017) Cavallo, Harper (2020)
in addition to faces, equipped with	reflexivity/degeneracy & permutations connections diagonals/Cartesian optionally symmetry/reversion	internal if with reflexivity/degeneracy & permutations, external otherwise

CCHM = Cohen, Coquand, Huber, Mörtberg

ABCFHL = Angiuli, Brunerie, Coquand, Favonia, Harper, Licata

Context of this talk: external parametric type theory
(i.e. semi-cubical type theory with bridges)

Objective 1: modelling cubical equality in a bare type theory without universes

Objective 2: modelling cubical equality in a bare type theory with universes

Related work: Parametricity and Semi-Cubical Types, Moeneclaye (2021)

Objective 1: Modelling a bare cubical theory without universes

$$
\begin{array}{lcl}
\Gamma & ::= & \emptyset \mid \Gamma, a : A \mid \Gamma, i : \mathbb{I} \\
A, B & ::= & P \mid t =_{\lambda i.A} u \\
t, u, v, w, p, q & ::= & a \mid p\, i
\end{array}
$$

$$
\frac{}{\emptyset \text{ ok}} \qquad \frac{\Gamma \vdash A \text{ type}}{\Gamma, a : A \text{ ok}} \qquad \frac{\Gamma, a : A, \Gamma' \text{ ok}}{\Gamma, a : A, \Gamma' \vdash a : A} \qquad \frac{\Gamma \text{ ok}}{\Gamma \vdash P \text{ type}}
$$

$$
\frac{\Gamma \vdash p : t =_{\lambda i.A} u}{\Gamma, i \vdash p\, i : A} \qquad \frac{\Gamma, i : \mathbb{I} \vdash A \text{ type} \qquad \Gamma \vdash t : A_{\{0/i\}} \qquad \Gamma \vdash u : A_{\{1/i\}}}{\Gamma \vdash t =_{\lambda i.A} u \text{ type}}
$$

where $A_{\{0/i\}}$ and $A_{\{1/i\}}$ are defined compositionally with base case $(p\, i)_{\{0/i\}} \triangleq v$ and $(p\, i)_{\{1/i\}} \triangleq w$ whenever $p : v =_{\lambda i.B} w$.

Note: No reflexivity (= no weakening for i), no permutation (= no exchange for i), no diagonals (= no contraction for i), no connections, only a cubical equality

Digression: The correspondence between iterations of cubical equality and cubes

$t : X \qquad\qquad \cdot\, t \quad : \quad \cdot\, X$

$p : t =_{\lambda i.X} u \qquad\qquad t \xrightarrow{p} u \quad : \quad X \xrightarrow{=_{\lambda i.X}} X$

$\alpha : p =_{\lambda i.(r\,i =_{\lambda j.X} s\,i)} q$

$$\begin{array}{ccc} t & \xrightarrow{r} & v \\ {\scriptstyle p}\downarrow & \overset{\alpha}{\Longrightarrow} & \downarrow{\scriptstyle q} \\ u & \xrightarrow{s} & w \end{array} \quad : \quad \begin{array}{ccc} X & \xrightarrow{=_{\lambda i.X}} & X \\ {\scriptstyle =_{\lambda i.X}}\downarrow & \overset{=_{\lambda i.(\_ =_{\lambda j.X} \_)}}{\Longrightarrow} & \downarrow{\scriptstyle =_{\lambda i.X}} \\ X & \xrightarrow{=_{\lambda i.X}} & X \end{array}$$

and so on

Summary of objective 1

To provide a sound and complete interpretation of bare cubical theory without universes in Extensional Type Theory (ETT, with Π, Σ, strict identity, ...), i.e., basically, an interpretation of X (as a set of points), $t =_{\lambda i.X} u$ (as a set of lines), $p =_{\lambda i.(r\,i =_{\lambda j.X} s\,i)} q$ (as a set of squares), ... i.e., eventually, to interpret X as a semi-cubical set

Objective 2: Modelling a bare cubical theory with universes

$$
\begin{array}{lcl}
\Gamma & ::= & \emptyset \mid \Gamma, a : A \mid \Gamma, i : \mathbb{I} \\
t, u, v, w, p, q, A, B & ::= & P \mid t =_{\lambda i.A} u \mid a \mid p\,i \mid \mathsf{U}_l
\end{array}
$$

$$
\frac{}{\emptyset\ \mathsf{ok}} \qquad
\frac{\Gamma \vdash A : \mathsf{U}_l}{\Gamma, a : A\ \mathsf{ok}} \qquad
\frac{\Gamma, a : A, \Gamma'\ \mathsf{ok}}{\Gamma, a : A, \Gamma' \vdash a : A} \qquad
\frac{\Gamma\ \mathsf{ok}}{\Gamma \vdash P : \mathsf{U}_l} \qquad
\frac{\Gamma\ \mathsf{ok}}{\Gamma \vdash \mathsf{U}_l : \mathsf{U}_{l+1}}
$$

$$
\frac{\Gamma \vdash p : t =_{\lambda i.A} u}{\Gamma, i \vdash p\,i : A} \qquad
\frac{\Gamma, i : \mathbb{I} \vdash A : \mathsf{U}_l \qquad \Gamma \vdash t : A_0 \qquad \Gamma \vdash u : A_1}{\Gamma \vdash t =_{\lambda i.A} u : \mathsf{U}_l}
$$

Objective 2: To provide a sound and complete interpretation in ETT, as before, but additionally interpreting $A =_{\lambda i.\mathsf{U}_l} B$, $p =_{\lambda i.(r\,i =_{\lambda j.\mathsf{U}_l} s\,i)} q$ ..., i.e., eventually, to interpret U_l as a semi-cubical set whose points are themselves semi-cubical sets

Objective 1: modelling cubical equality in a bare type theory *without* universes

Grothendieck's construction at work: Fibered semi-cubical sets vs indexed semi-cubical sets

$$
\begin{array}{lll}
\vdots & & \vdots \\
A_2 : \mathsf{U}_l & & A_2 : \Pi a_{00}, a_{01}, a_{10}, a_{11} : A_0. \\
 & & \quad A_1(a_{00}, a_{01}) \times A_1(a_{10}, a_{11}) \times \\
 & & \quad A_1(a_{00}, A_{10}) \times A_1(a_{01}, A_{11}) \to \mathsf{U}_l \qquad \text{(squares)} \\
\downarrow\downarrow\downarrow\downarrow & & \\
A_1 : \mathsf{U}_l & & A_1 : A_0 \times A_0 \to \mathsf{U}_l \qquad \text{(segments)} \\
\downarrow\downarrow & & \\
A_0 : \mathsf{U}_l & & A_0 : \mathsf{U}_l \qquad \text{(points)}
\end{array}
$$

fibered view (presheaf) — indexed view (dependent types)

Comparison with iterated parametricity: translation of the universe

The parametricity translation associates to each type a pair of types and an heterogeneous relation over these types that expresses how to relate terms in the corresponding types.

Example: The context

$$A : \mathsf{U}_l$$

is translated to three contexts:

$$\begin{array}{rl} \text{2 points} & \begin{cases} A_0 : \mathsf{U}_l \\ A_1 : \mathsf{U}_l \end{cases} \\ \text{1 segment} & \{ \underbrace{A_0 : \mathsf{U}_l, A_1 : \mathsf{U}_l, A_\star : A_0 \times A_1 \to \mathsf{U}_l}_{\text{triplication of } A : \mathsf{U}_l} \end{array}$$

Graphically, the later context is a segment of "types"

$$A_0 \overset{A_\star}{\longrule} A_1$$

Comparison with iterated parametricity: translation of the universe

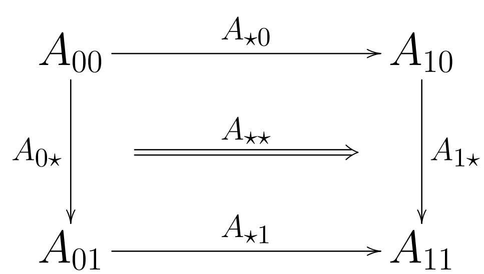
This can be iterated to produce nine contexts:

4 points
$$\begin{cases} A_{00} : \mathsf{U}_l \\ A_{01} : \mathsf{U}_l \\ A_{10} : \mathsf{U}_l \\ A_{11} : \mathsf{U}_l \end{cases}$$

4 segments
$$\begin{cases} A_{00} : \mathsf{U}_l, A_{01} : \mathsf{U}_l, A_{0\star} : A_{00} \times A_{01} \to \mathsf{U}_l \\ A_{10} : \mathsf{U}_l, A_{11} : \mathsf{U}_l, A_{1\star} : A_{10} \times A_{11} \to \mathsf{U}_l \\ A_{00} : \mathsf{U}_l, A_{10} : \mathsf{U}_l, A_{\star 0} : A_{00} \times A_{10} \to \mathsf{U}_l \\ A_{01} : \mathsf{U}_l, A_{11} : \mathsf{U}_l, A_{\star 1} : A_{01} \times A_{11} \to \mathsf{U}_l \end{cases}$$

1 square
$$\begin{cases} A_{00} : \mathsf{U}_l, A_{01} : \mathsf{U}_l, A_{0\star} : A_{00} \times A_{01} \to \mathsf{U}_l, A_{10} : \mathsf{U}_l, A_{11} : \mathsf{U}_l, A_{1\star} : A_{10} \times A_{11} \to \mathsf{U}_l, \\ A_{\star 0} : A_{00} \times A_{10} \to \mathsf{U}_l, A_{\star 1} : A_{01} \times A_{11} \to \mathsf{U}_l, \\ A_{\star\star} : \Pi a_{00}, a_{01}, a_{10}, a_{11}.\, A_{0\star}(a_{00}, a_{01}) \times A_{1\star}(a_{10}, a_{11}) \times A_{\star 0}(a_{00}, a_{10}) \times A_{\star 0}(a_{01}, a_{11}) \to \mathsf{U}_l \end{cases}$$

Graphically, the later context is a square of "types"



and so on...

Otherwise said

Observation: The recipe to build the set of n-cubes as a dependently-typed types over the cubes of smaller dimensions has the same structure as the recipe to build the type of $A_{\star \overset{n}{\dots} \star}$ in the parametricity translation.

Processus: build a dependently-typed stream A_0 (type), A_1 (equality on A_0), A_2 (squared equality on A_0 and A_1), ...

The definition in ETT of *semi-cubical sets* as a dependent stream of higher-dimensional cubical relations

Cubical sets

$$
\begin{array}{lll}
\mathsf{cubset}_l & : & \mathsf{U}_{l+1} \\
\mathsf{cubset}_l & \triangleq & \mathsf{cubset}_l^{\geq 0}(\star)
\end{array}
$$

$$
\begin{array}{llll}
\mathsf{cubset}_l^{\geq n} & (D : \mathsf{cubset}_l^{<n}) & : & \mathsf{U}_{l+1} \\
\mathsf{cubset}_l^{\geq n} & D & \triangleq & \Sigma R : \mathsf{cubset}_l^{=n}(D).\, \mathsf{cubset}_l^{\geq n+1}(D, R)
\end{array}
$$

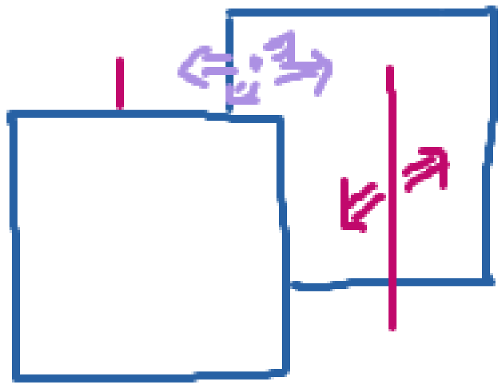
Truncated cubical sets

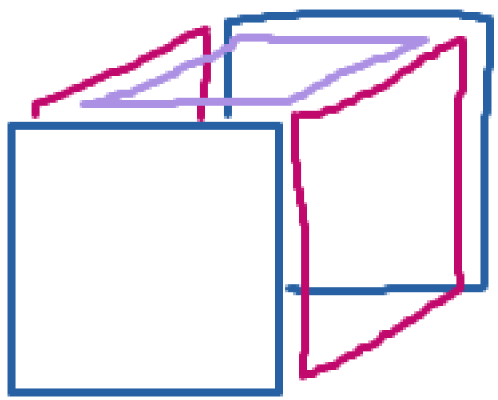
$$
\begin{array}{lll}
\mathsf{cubset}_l^{<n} & : & \mathsf{U}_{l+1} \\
\mathsf{cubset}_l^{<0} & \triangleq & \mathsf{unit} \\
\mathsf{cubset}_l^{<n'+1} & \triangleq & \Sigma D : \mathsf{cubset}_l^{<n'}.\, \mathsf{cubset}_l^{=n}(D))
\end{array}
$$

$$
\begin{array}{llll}
\mathsf{cubset}_l^{=n} & (D : \mathsf{cubset}_l^{<n}) & : & \mathsf{U}_{l+1} \\
\mathsf{cubset}_l^{=n} & D & \triangleq & \mathsf{fullbox}_l^n(D) \to \mathsf{U}_l
\end{array}
$$

where $\mathsf{fullbox}_l^n$ is defined by a technical mutual recursive construction

The recursive process used to build boxes and cubes



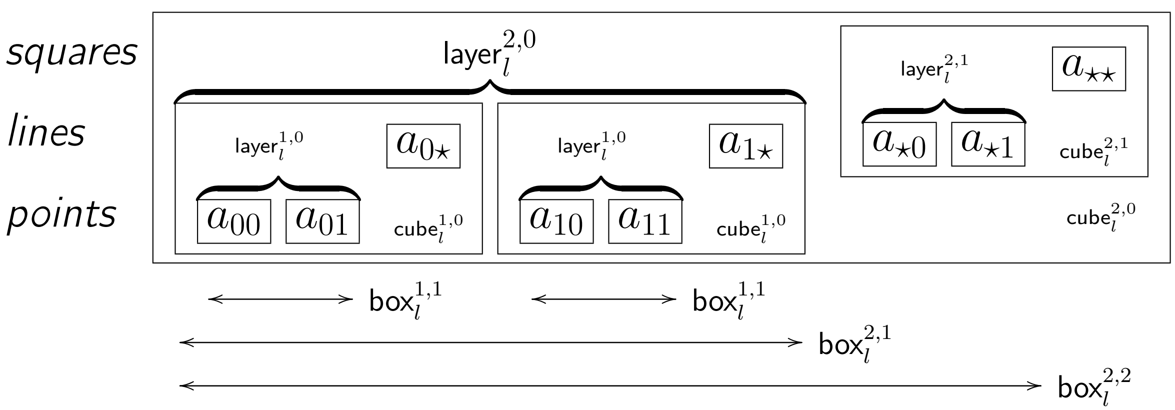


A n-box is made of n layers, each made of two opposite cubes of decreasing size and stretched to get the size of the box

The recursive construction, formally

$$
\begin{array}{lllll}
\mathsf{fullbox}_l^n & (D : \mathsf{cubset}_l^{<n}) & & : & \mathsf{U}_l \\
\mathsf{fullbox}_l^n & D & & \triangleq & \mathsf{box}_l^{n,n}(D) \\
\\
\mathsf{box}_l^{n,p,[p\leq n]} & (D : \mathsf{cubset}_l^{<n}) & & : & \mathsf{U}_l \\
\mathsf{box}_l^{n,0} & D & & \triangleq & \mathsf{unit} \\
\mathsf{box}_l^{n,p'+1} & D & & \triangleq & \Sigma d : \mathsf{box}_l^{n,p'}(D).\, \mathsf{layer}_l^{n,p'}(D)(d) \\
\\
\mathsf{layer}_l^{n,p,[p<n]} & (D : \mathsf{cubset}_l^{<n}) \ (d : \mathsf{box}_l^{n,p}(D)) & & : & \mathsf{U}_l \\
\mathsf{layer}_l^{n,p} & D \ d & & \triangleq & \mathsf{cube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}_{l,L}^{n,p}(D)(d)) \\
 & & & & \times\ \mathsf{cube}_l^{n-1,p}(\mathsf{hd}(D))(\mathsf{hd}(D))(\mathsf{subbox}_{l,R}^{n,p}(D)(d)) \\
\\
\mathsf{cube}_l^{n,p,[p\leq n]} & (D : \mathsf{cubset}_l^{<n}) \ (E : \mathsf{cubset}_l^{=n}(D)) \ (d : \mathsf{box}_l^{n,p}(D)) & & : & \mathsf{U}_l \\
\mathsf{cube}_l^{n,p,[p=n]} & D \ E \ d & & \triangleq & E.\, =_\star (d) \\
\mathsf{cube}_l^{n,p,[p<n]} & D \ E \ d & & \triangleq & \Sigma b : \mathsf{layer}_l^{n,p}(D)(d).\, \mathsf{cube}_l^{n,p+1}(D)(E)(d,b)
\end{array}
$$

which corresponds to the following organisation of the 3^n components of a n-cube (shown for $n = 2$), with $\mathbf{box}_l$ associating layers on the left and $\mathbf{cube}_l$ associating them on the right:



additionally, each atomic component at dimension n is a $\mathsf{cube}_l^{n,n}$

The recursive construction: restrictions ("faces")

$$\begin{array}{lllll}
\mathsf{subbox}^{n,q,p,[p\leq q<n]}_{l,\epsilon} & \begin{array}{l}(D : \mathsf{cubset}^{<n}_l)\\(d : \mathsf{box}^{n,p}_l(D))\end{array} & : & \mathsf{box}^{n-1,p}_l(\mathsf{hd}(D)) \\
\mathsf{subbox}^{n,q,0}_{l,\epsilon} & D\ \star & \triangleq & \star \\
\mathsf{subbox}^{n,q,p'+1}_{l,\epsilon} & D\ (d,b) & \triangleq & (\mathsf{subbox}^{n,q,p'}_{l,\epsilon}(D)(d), \mathsf{sublayer}^{n,q,p'}_{l,\epsilon}(D)(d)(b))
\end{array}$$

$$\begin{array}{llll}
\mathsf{sublayer}^{n,q,p,[p<q<n]}_{l,\epsilon} & \begin{array}{l}(D : \mathsf{cubset}^{<n}_l)\\(d : \mathsf{box}^{n,p}_l(D))\\(b : \mathsf{layer}^{n,p}_l(D)(d))\end{array} & : & \mathsf{layer}^{n-1,p}_l(\mathsf{hd}(D))(\mathsf{subbox}^{n,q,p}_{l,\epsilon}(D)(d)) \\
\mathsf{sublayer}^{n,q,p}_{l,\epsilon} & D\ d\ c & \triangleq & \begin{array}{l}(\overrightarrow{\mathsf{cohbox}^{n,p,q,p}_{l,\epsilon,L}(D)(d)}(\mathsf{subcube}^{n-1,q-1,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p,p}_{l,L}(D)(d))(c_L)),\\ \overrightarrow{\mathsf{cohbox}^{n,p,q,p}_{l,\epsilon,R}(D)(d)}(\mathsf{subcube}^{n-1,q-1,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p,p}_{l,bR}(D)(d))(c_R)))\end{array}
\end{array}$$

$$\begin{array}{llll}
\mathsf{subcube}^{n,q,p,[p\leq q<n]}_{l,\epsilon} & \begin{array}{l}(D : \mathsf{cubset}^{<n}_l)\\(E : \mathsf{cubset}^{=n}_l(D))\\(d : \mathsf{box}^{n,p}_l(D))\\(b : \mathsf{cube}^{n,p}_l(D)(E)(d))\end{array} & : & \mathsf{cube}^{n-1,p}_l(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,q,p}_{l,\epsilon}(D)(d)) \\
\mathsf{subcube}^{n,q,p,[p=q]}_{l,\epsilon} & D\ E\ d\ (b, _) & \triangleq & b_\epsilon \\
\mathsf{subcube}^{n,q,p,[p<q]}_{l,\epsilon} & D\ E\ d\ (b,c) & \triangleq & (\mathsf{sublayer}^{n,q,p}_{l,\epsilon}(D)(d)(b), \mathsf{subcube}^{n,q,p+1}_{l,\epsilon}(D)(E)(d,b)(c))
\end{array}$$

where **cohbox**$_{l,\epsilon,\epsilon'}$ is a coherence proof and we write $\overrightarrow{\mathsf{cohbox}_{l,\epsilon,\epsilon'}}$ for the rewriting of this proof from left to right

The recursive construction: coherences

$$
\begin{array}{lll}
\mathsf{cohbox}^{\substack{n,q,r,p \\ [p \leq r < q < n]}}_{l,\epsilon,\epsilon'} & (D : \mathsf{cubset}^{<n}_{l}) & : \; \mathsf{subbox}^{n-1,q-1,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{subbox}^{n,r,p}_{l,\epsilon'}(D)(d)) \\
 & (d : \mathsf{box}^{n,p}_{l}(D)) & \quad = \mathsf{subbox}^{n-1,r-1,p}_{l,\epsilon'}(\mathsf{hd}(D))(\mathsf{subbox}^{n,q,p}_{l,\epsilon}(D)(d)) \\
\mathsf{cohbox}^{n,q,r,0}_{l,\epsilon,\epsilon'} & D \; \star & \triangleq \; \mathsf{refl}\,\star \\
\mathsf{cohbox}^{n,q,r,p'+1}_{l,\epsilon,\epsilon'} & D \; (d,b) & \triangleq \; (\mathsf{cohbox}^{n,q,r,p'}_{l,\epsilon,\epsilon'}(D)(d), \mathsf{cohlayer}^{n,q,r,p'}_{l,\epsilon,\epsilon'}(D)(d)(b))
\end{array}
$$

$$
\begin{array}{lll}
 & (D : \mathsf{cubset}^{<n}_{l}) & \\
\mathsf{cohlayer}^{\substack{n,q,r,p \\ [p < r < q < n]}}_{l,\epsilon,\epsilon'} & (d : \mathsf{box}^{n,p}_{l}(D)) & : \; \mathsf{sublayer}^{n-1,q,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{subbox}^{n,r,p}_{l,\epsilon'}(D)(d))(\mathsf{sublayer}^{n,r,p}_{l,\epsilon'}(D)(d)(b)) \\
 & (b : \mathsf{layer}^{n,p}_{l}(D)(d)) & \quad = \mathsf{sublayer}^{n-1,r,p}_{l,\epsilon'}(\mathsf{hd}(D))(\mathsf{subbox}^{n,q,p}_{l,\epsilon}(D)(d))(\mathsf{sublayer}^{n,q,p}_{l,\epsilon}(D)(d)(b)) \\
\mathsf{cohlayer}^{n,q,r,p}_{l,\epsilon,\epsilon'} & D \; d \; c & \triangleq \; (\mathsf{cohcube}^{n-1,q-1,r-1,p}_{l,\epsilon,\epsilon'}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p,p}_{l,L}(D)(d))(c_L), \\
 & & \quad \mathsf{cohcube}^{n-1,q-1,r-1,p}_{l,\epsilon,\epsilon'}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p,p}_{l,R}(D)(d))(c_R))
\end{array}
$$

$$
\begin{array}{lll}
 & (D : \mathsf{cubset}^{<n}_{l}) & \\
\mathsf{cohcube}^{\substack{n,q,r,p \\ [p \leq r < q < n]}}_{l,\epsilon,\epsilon'} & (E : \mathsf{cubset}^{=n}_{l}(D)) & : \; \mathsf{subcube}^{n-1,q-1,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,r,p}_{l,\epsilon'}(D)(d))(\mathsf{subcube}^{n,r,p}_{l,\epsilon'}(D)(E)(d)(b)) \\
 & (d : \mathsf{box}^{n,p}_{l}(D)) & \quad = \mathsf{subcube}^{n-1,r-1,p}_{l,\epsilon'}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,q,p}_{l,\epsilon}(D)(d))(\mathsf{subcube}^{n,q,p}_{l,\epsilon}(D)(E)(d)(b)) \\
 & (b : \mathsf{cube}^{n,p}_{l}(D)(E)(d)) & \\
\mathsf{cohcube}^{n,q,r,p,[p=r]}_{l,\epsilon,\epsilon'} & D \; E \; d \; (b, \_) & \triangleq \; \mathsf{refl}\,\mathsf{subcube}^{n-1,q-1,p}_{l,\epsilon}(\mathsf{hd}(D))(\mathsf{tl}(D))(\mathsf{subbox}^{n,p,p}_{l,\epsilon'}(D)(d))(b_\epsilon) \\
\mathsf{cohcube}^{n,q,r,p,[p<r]}_{l,\epsilon,\epsilon'} & D \; E \; d \; (b, c) & \triangleq \; (\mathsf{cohlayer}^{n,q,r,p}_{l,\epsilon,\epsilon'}(D)(d)(b), \; \mathsf{cohcube}^{n,q,r,p+1}_{l,\epsilon,\epsilon'}(D)(E)(d,b)(c))
\end{array}
$$

where we rely on the strictness of equality in ETT to enforce irrelevance of proof of coherence (otherwise, coherences in higher dimensions are recursively needed)

Objective 2: modelling cubical equality in a bare type theory *with* universes

Bridge equality on universes

We now have to interpret the universe as a semi-cubical set of semi-cubical sets, where

$$\cdot A \quad : \quad \mathsf{U}_l \quad \triangleq \quad \mathsf{cubset}_l$$

$$A \xrightarrow{R} B \quad : \quad A =_{\mathsf{U}_l} B \quad \triangleq \quad A \times B \to \mathsf{cubset}_l$$

$$\begin{array}{ccc} A & \xrightarrow{T} & C \\ R \downarrow & \overset{P}{\Longrightarrow} & \downarrow S \\ B & \xrightarrow{U} & D \end{array} \quad : \quad R =_{\lambda i.(T i =_{\lambda j.\mathsf{U}_l} U\, i)} S \quad \triangleq \quad \Pi abcd.\, R(a,b) \times S(c,d) \times T(a,c) \times U(b,d) \to \mathsf{cubset}_l$$

$$\vdots$$

So, we have to define the sequence of

$$\begin{array}{lcl} \mathsf{univ}_{l,0} & \triangleq & \mathsf{cubset}_l \\ \mathsf{univ}_{l,1} & \triangleq & \lambda AB : \mathsf{cubset}_l . A \times B \to \mathsf{cubset}_l \\ \mathsf{univ}_{l,2} & \triangleq & \lambda ABCDRSTU.\, \Pi abcd.\, R(a,b) \times S(c,d) \times T(a,c) \times U(b,d) \to \mathsf{cubset}_l \\ & \ldots & \end{array}$$

which will inhabit cubset_{l+1}.

In particular, $\to$ has to build here a cubical set dependent on cubical sets! This goes through generalising homogeneous cubes into heterogeneous cubes and in defining dependent cubical sets.

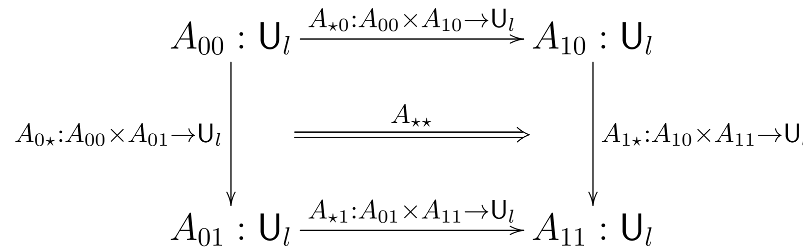
Homogeneous cubes of terms over cubical sets vs heterogenous cubes of terms over cubes of types

n-truncated (semi-)cubical sets

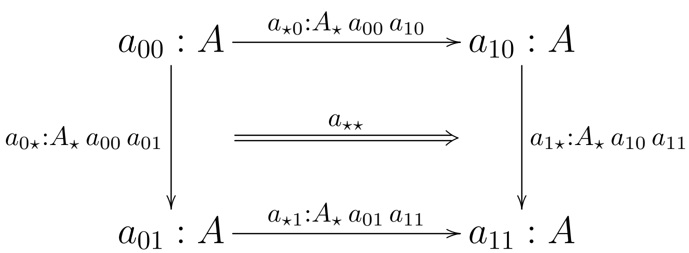
$$
\begin{array}{lcl}
A & : & \mathsf{U}_l \\
A_\star & : & A \times A \to \mathsf{U}_l \\
A_{\star\star} & & \\
\ldots & &
\end{array}
$$

$\hookrightarrow$

n-cubes of types

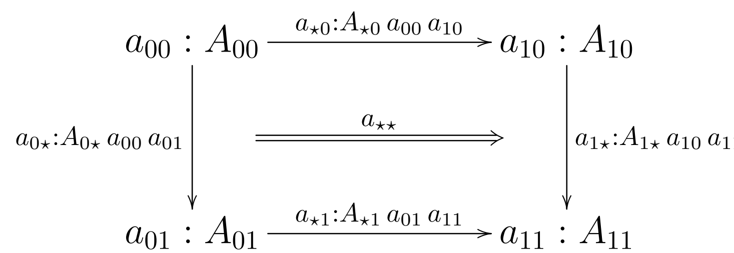


homogeneous n-cubes of terms over some n-cubes cubical set



$\subset$

heterogenous n-cubes of terms over some n-cube of types



Conclusions and open questions

- This work is one step towards setting the basis of a modular syntactic "indexed" model in ETT to both parametric type theory and (variants of) cubical type theory, realising a project imagined in Altenkirch-Kaposi 2014 (see also Polonsky 2014, Adams 2016).
- For instance, as a reminder, adding reflexivity, connections, diagonals, permutations, path equality would justify a cubical type theory with (well-behaved) univalence and higher inductive types.
- We expect to get a precise correspondence between models (indirect style) and appropriate type theories (direct style)
- We expect the syntactic model to make easy the comparison of reduction in the source and the target, inheriting properties of the source from those of the target, such as new definitional properties

Full n-cubes of types

$$\begin{array}{llcl}
\mathsf{fullCube}_l^n & & : & \mathsf{U}_{l+1} \\
\mathsf{fullCube}_l^n & & \triangleq & \Sigma D : \mathsf{fullBox}_l^n .\, \mathsf{Filler}_l^n
\end{array}$$

Full n-boxes of types

$$\begin{array}{llcl}
\mathsf{fullBox}_l^n & & : & \mathsf{U}_{l+1} \\
\mathsf{fullBox}_l^n & & \triangleq & \mathsf{Box}_l^{n,n}
\end{array}$$

Filler of a full n-box of types

$$\begin{array}{llcl}
\mathsf{Filler}_l^n & (D : \mathsf{fullBox}_l^n) & : & \mathsf{U}_{l+1} \\
\mathsf{Filler}_l^n & D & \triangleq & \mathsf{fullhetbox}_l^n(D) \to \mathsf{U}_l
\end{array}$$

Auxiliary definitions

$$\begin{array}{llcl}
\mathsf{Box}_l^{n,p,[p \le n]} & & : & \mathsf{U}_{l+1} \\
\mathsf{Box}_l^{n,0} & & \triangleq & \mathsf{unit} \\
\mathsf{Box}_l^{n,p'+1} & & \triangleq & \Sigma D : \mathsf{Box}_l^{n,p'} .\, \mathsf{Layer}_l^{n,p'}(D)
\end{array}$$

$$\begin{array}{llcl}
\mathsf{Layer}_l^{n,p,[p<n]} & (D : \mathsf{Box}_l^{n,p}) & : & \mathsf{U}_{l+1} \\
\mathsf{Layer}_l^{n,p} & D & \triangleq & \mathsf{Cube}_l^{n-1,p}(\mathsf{Subbox}_{l,L,p}^{n,p}(D)) \\
 & & & \times\ \mathsf{Cube}_l^{n-1,p}(\mathsf{Subbox}_{l,R,p}^{n,p}(D))
\end{array}$$

$$\begin{array}{llcl}
\mathsf{Cube}_l^{n,p,[p \le n]} & (D : \mathsf{Box}_l^{n,p}) & : & \mathsf{U}_{l+1} \\
\mathsf{Cube}_l^{n,p,[p=n]} & D & \triangleq & \mathsf{Filler}_l^n(D) \\
\mathsf{Cube}_l^{n,p,[p<n]} & D & \triangleq & \Sigma B : \mathsf{Layer}_l^{n,p}(D) .\, \mathsf{Cube}_l^{n,p+1}(D, B)
\end{array}$$

Figure 1: Full n-box and p-prefix of a partial n-box of higher-order relations

Full n-boxes of terms over a full n-boxes of types

$\mathsf{fullhetbox}_l^n$	$(D : \mathsf{fullBox}_l^n)$	$:$	U_l
$\mathsf{fullhetbox}_l^n$	D	$\triangleq$	$\mathsf{hetbox}_l^{n,n}(D)$

Filler of a full n-box of terms over some filled full n-box of types

$\mathsf{hetfiller}_l^n$	$(D : \mathsf{fullBox}_l^n)$ $(E : \mathsf{Filler}_l^{n,n}(D))$ $(d : \mathsf{fullhetbox}_l^n(D))$	$:$	U_l
$\mathsf{hetfiller}_l^n$	$D\ E\ d$	$\triangleq$	$E(d)$

Auxiliary definitions

$\mathsf{hetbox}_l^{n,p,[p\leq n]}$	$(D : \mathsf{Box}_l^{n,p})$	$:$	U_l
$\mathsf{hetbox}_l^{n,0}$	D	$\triangleq$	unit
$\mathsf{hetbox}_l^{n,p'+1,[p'<n]}$	(D, B)	$\triangleq$	$\Sigma d : \mathsf{hetbox}_l^{n,p'}(D).\ \mathsf{hetlayer}_l^{n,p'}(D)(B)(d)$
$\mathsf{hetlayer}_l^{n,p,[p<n]}$	$(D : \mathsf{Box}_l^{n,p})$ $(B : \mathsf{Layer}_l^{n,p}(D))$ $(d : \mathsf{hetbox}_l^{n,p}(D))$	$:$	U_l
$\mathsf{hetlayer}_l^{n,p}$	$D\ B\ d$	$\triangleq$	$\mathsf{hetcube}_l^{n-1,p}(\mathsf{Subbox}_{l,L,p}^{n,p}(D))(B_L)(\mathsf{subhetbox}_{l,L,p}^{n,p}(D)(d))$ $\times\ \mathsf{hetcube}_l^{n-1,p}(\mathsf{Subbox}_{l,R,p}^{n,p}(D))(B_R)(\mathsf{subhetbox}_{l,R,p}^{n,p}(D)(d))$
$\mathsf{hetcube}_l^{n,p,[p\leq n]}$	$(D : \mathsf{Box}_l^{n,p})$ $(C : \mathsf{Cube}_l^{n,p}(D))$ $(d : \mathsf{hetbox}_l^{n,p}(D))$	$:$	U_l
$\mathsf{hetcube}_l^{n,p,[p=n]}$	$D\ C\ d$	$\triangleq$	$\mathsf{hetfiller}_l^n(D)(C)(d)$
$\mathsf{hetcube}_l^{n,p,[p<n]}$	$D\ (B, C)\ d$	$\triangleq$	$\Sigma b : \mathsf{hetlayer}_l^{n,p}(D)(d).\ \mathsf{hetcube}_l^{n,p+1}(D, B)(C)(d, b)$

Figure 2: Full n-box and p-prefix of a partial n-box of terms over an n-box of higher-order relations

n-cube associated to the universe U_l

$$
\begin{array}{lcl}
\mathsf{UnivfullCube}_l^n & : & \mathsf{fullCube}_{l+1}^n \\
\mathsf{UnivfullCube}_l^n & \triangleq & (\mathsf{UnivfullBox}_l^n, \mathsf{UnivFiller}_l^n) \\
\\
\mathsf{UnivfullBox}_l^n & : & \mathsf{fullBox}_{l+1}^n \\
\mathsf{UnivfullBox}_l^n & \triangleq & \mathsf{UnivBox}_l^{n,n} \\
\\
\mathsf{UnivFiller}_l^n & : & \mathsf{Filler}_{l+1}^n \\
\mathsf{UnivFiller}_l^n & \triangleq & \lambda D : \mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n).\, \mathsf{depcubset}_l^n(D)
\end{array}
$$

---

incremental construction of the n-box associated to universe U_l

$$
\begin{array}{lcl}
\mathsf{UnivBox}_l^{n,p} & : & \mathsf{Box}_{l+1}^{n,p} \\
\mathsf{UnivBox}_l^{n,0} & \triangleq & \star \\
\mathsf{UnivBox}_l^{n,p'+1} & \triangleq & (\mathsf{UnivBox}_l^{n,p'}, \mathsf{UnivLayer}_l^{n,p'}) \\
\\
\mathsf{UnivLayer}_l^{n,p,[p<n]} & : & \mathsf{Layer}_{l+1}^{n,p}(\mathsf{UnivBox}_l^{n,p}) \\
\mathsf{UnivLayer}_l^{n,p} & \triangleq & (\mathsf{UnivCube}_l^{n-1,p}, \mathsf{UnivCube}_l^{n-1,p}) \\
\\
\mathsf{UnivCube}_l^{n,p,[p\leq n]} & : & \mathsf{Cube}_{l+1}^{n,p}(\mathsf{UnivBox}_l^{n,p}) \\
\mathsf{UnivCube}_l^{n,p,[p=n]} & \triangleq & \mathsf{UnivFiller}_l^n \\
\mathsf{UnivCube}_l^{n,p,[p<n]} & \triangleq & (\mathsf{UnivLayer}_l^{n,p}, \mathsf{UnivCube}_l^{n,p+1})
\end{array}
$$

Figure 3: The n-cube associated to the universe l

Full (non-truncated) cubical sets over an n-box of cubical sets

$$\begin{array}{lll} \mathsf{depcubset}_l^n & (D : \mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n)) & : \mathsf{U}_{l+1} \\ \mathsf{depcubset}_l^n & D & \triangleq \mathsf{depcubset}_l^{n,\geq 0}(D)(\star) \end{array}$$

Cubical set p-suffix over an n-box of cubical sets

$$\begin{array}{lll} \mathsf{depcubset}_l^{n,\geq p} & (D : \mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n)) \; (P : \mathsf{depcubset}_l^{n,<p}(D)) & : \mathsf{U}_{l+1} \\ \mathsf{depcubset}_l^{n,\geq p} & D \; P & \triangleq \Sigma R : \mathsf{depcubset}_l^{n,=p}(D)(P).\, \mathsf{depcubset}_l^{n,\geq p+1}(D)(P,R) \end{array}$$

p-truncated cubical sets over an n-box of cubical sets

$$\begin{array}{lll} \mathsf{depcubset}_l^{n,<p} & (D : \mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n)) & : \mathsf{U}_{l+1} \\ \mathsf{depcubset}_l^{n,<0} & D & \triangleq \mathsf{unit} \\ \mathsf{depcubset}_l^{n,<p'+1} & D & \triangleq \Sigma P : \mathsf{depcubset}_l^{n,<p'}(D).\, \mathsf{depcubset}_l^{n,=p'}(D)(P) \end{array}$$

Dependent structure carried at each dimension

$$\begin{array}{lll} \mathsf{depcubset}_l^{n,=p} & (D : \mathsf{fullhetbox}_l^n(\mathsf{UnivfullBox}_l^n)) \; (P : \mathsf{depcubset}_l^{n,<p}(D)) & : \mathsf{U}_{l+1} \\ \mathsf{depcubset}_l^{n,=p} & D \; P & \triangleq \mathsf{fulldepBox}_l^{n,p}(D)(P) \to \mathsf{U}_l \end{array}$$

Figure 4: Dependent cube set over a n-box of cubical sets