# Hypergraphs for knowledge

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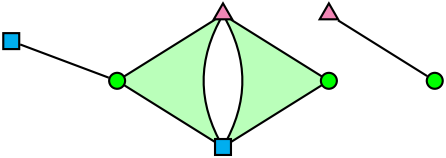
# Simplicial sets

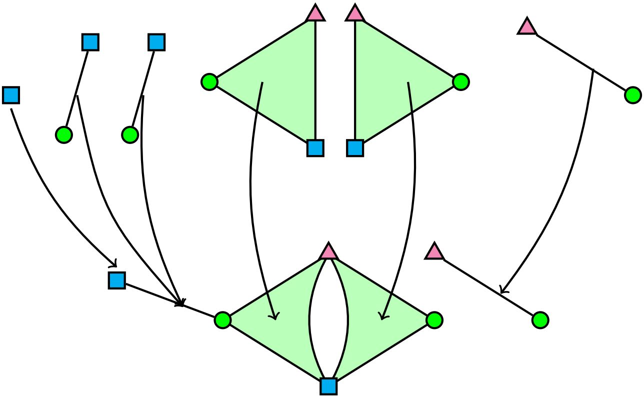
Recall the semi-simplicial sets semantics of epistemic logic:



$\varphi ::= \mathsf{p} \mid \mathsf{true} \mid \neg\varphi \mid \varphi \wedge \psi \mid D_U \varphi \qquad U \subseteq \{$●, ■, ▲$\}, \; \mathsf{p} \in \mathsf{Ap}$

$M, w \models D_U \varphi$ iff for all $U$-adjacent simplices $w'$, $M, w' \models \varphi$

# Coverings of simplicial sets



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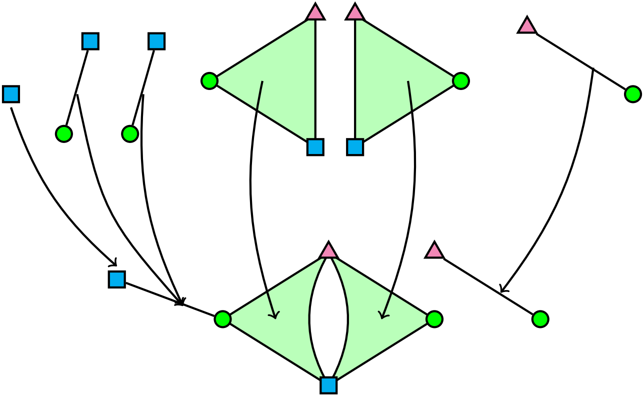

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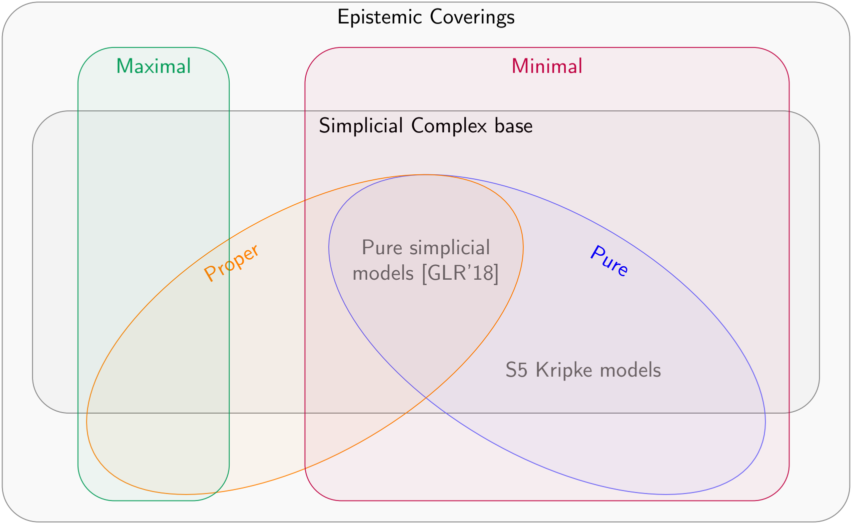
LICS'23: Goubault, K., Ledent, Rajsbaum
"Semi-simplicial Set Models for Distributed Knowledge"

# "Semi-simplicial Set Models for Distributed Knowledge"

## Theorem

*Epistemic coverings are isomorphic to epistemic Kripke frames*



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Here is an idea!

# Model: chromatic hypergraphs

## Definition

A chromatic hypergraph $H$ is a tuple $(E, \{V_a, p_a\}_{a \in \mathcal{A}})$, where:

- $E$ is a set of hyperedges,
- $V_a$ is a set of *views* of agent $a$,
- $p_a : E \to V_a$ is a surjective partial function for each agent $a$.

Additionally, we require that for each $e \in E$, $p_a(e)$ is defined for at least one $a \in \mathcal{A}$.

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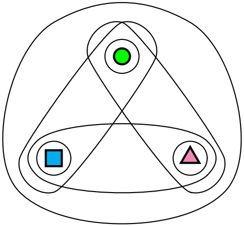

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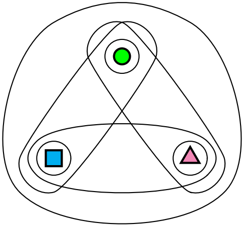

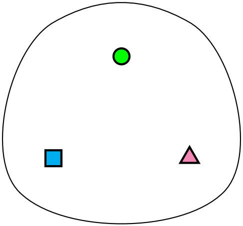

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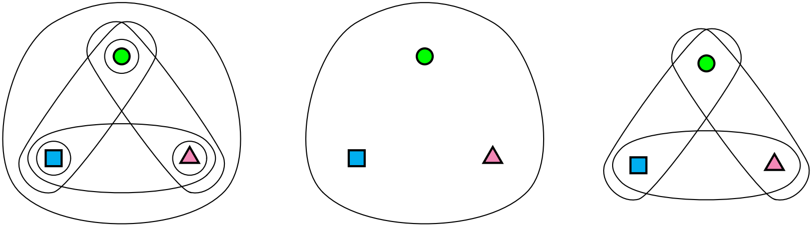

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$$\varphi_a ::= \mathsf{p}_a \mid \mathsf{true}_a \mid \neg\varphi \mid \varphi \wedge \psi \mid \widehat{\mathsf{K}}_a\Phi \mid \mathsf{K}_a\Phi \qquad a \in \mathcal{A},\; \mathsf{p}_a \in \mathsf{Ap}_a$$

$$\Phi ::= \mathsf{p}_E \mid \mathsf{true}_E \mid \neg\Phi \mid \Phi \wedge \Psi \mid \faUser_a\varphi \mid \faUser[regular]_a\varphi \qquad a \in \mathcal{A}, \mathsf{p}_E \in \mathsf{Ap}_E$$

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Rules:

$$\frac{\vdash \varphi \qquad \vdash \varphi \to \psi}{\vdash \psi}\ \text{MP} \qquad \frac{\vdash_E \Phi}{\vdash_a \mathsf{K}_a\Phi}\ \text{Nec-a} \qquad \frac{\vdash_a \varphi}{\vdash_E \faUser[regular]_a\varphi}\ \text{Nec-E}$$

$$\frac{\vdash \Phi \to \Psi}{\vdash \heartsuit\Phi \to \heartsuit\Psi}\ \text{Mono} \qquad \frac{\vdash \Phi \to \faUser[regular]_a\psi}{\vdash \widehat{\mathsf{K}}_a\Phi \to \psi}\Updownarrow\ \text{Adj-1} \qquad \frac{\vdash \varphi \to \mathsf{K}_a\Psi}{\vdash \faUser_a\varphi \to \Psi}\Updownarrow\ \text{Adj-2}$$

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(The Adj-1 and Adj-2 rules are drawn with double lines.)

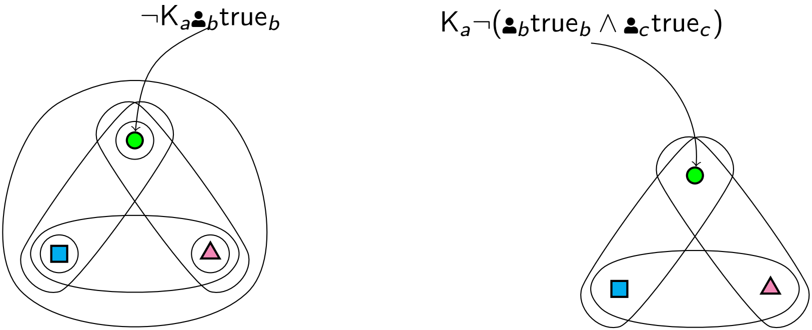
Axioms:

- $\vdash_a \varphi \to \widehat{\mathsf{K}}_a\bullet_a\varphi$ (surjectivity)
- $\vdash_a \widehat{\mathsf{K}}_a\bullet_a\varphi \to \varphi$ (functionality)
- $\vdash_E \Phi \to \bigvee_{a\in\mathcal{A}} \bullet_a\widehat{\mathsf{K}}_a\Phi$ (joint totality)

# Semantics

$$M, v_a \models \widehat{\mathsf{K}}_a \Phi \iff \exists (e \ni v_a). M, e \models \Phi$$

$$M, e \models \faUser_a \varphi \iff \exists (v_a \in e). M, v_a \models \varphi$$



# Some properties: derivable

- self-awareness: $\mathsf{K}_a(\faUser_a\varphi \to \faUser[regular]_a\varphi)$
- "positive" introspection: $\mathsf{K}_a\Phi \to \mathsf{K}_a\faUser_a\mathsf{K}_a\Phi$
- "negative" introspection: $\Phi \to \faUser[regular]_a\widehat{\mathsf{K}}_a\Phi$
- locality by definition: $\mathsf{K}_a\faUser_a\mathsf{p} \vee \mathsf{K}_a\faUser_a\neg\mathsf{p}$

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- locality by definition: $K_a\faUser_a p \lor K_a\faUser_a\neg p$

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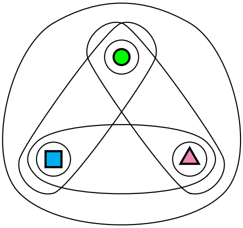
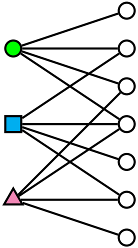
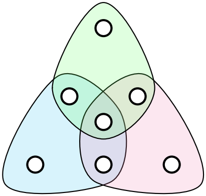
- "safe" knowledge: $\faUser_a K_a\Phi$ – not normal
- "unsafe" knowledge: $\faUser[regular]_a K_a\Phi$ – normal

# Dual hypergraphs

| Chromatic hypergraphs | Bipartite graph | Frames |
|---|---|---|
|  |  |  |

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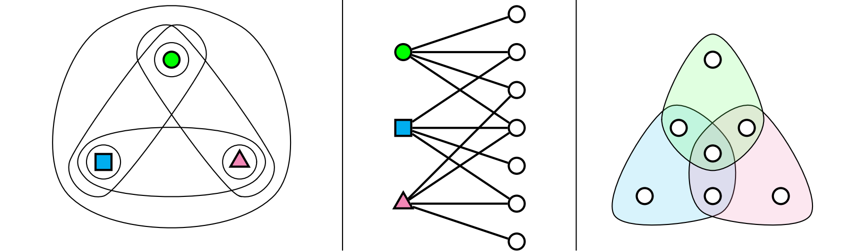
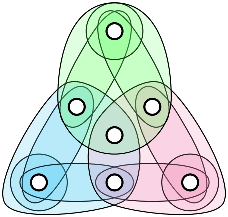


### Theorem

*Category of chromatic hypergraphs and category of partial epistemic frames are isomorphic.*

# Extensions and dualities

Neighborhood frames:
several points of view per world



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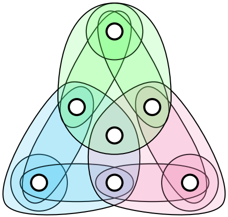
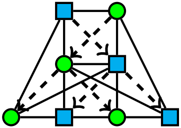


Dynamics à la Panangaden-Taylor'92:
extension with temporality



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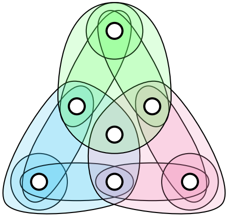

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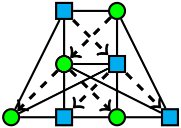
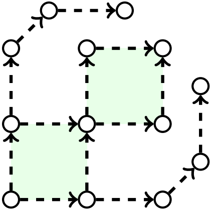


Directed semantics:
concurrency as ignorance



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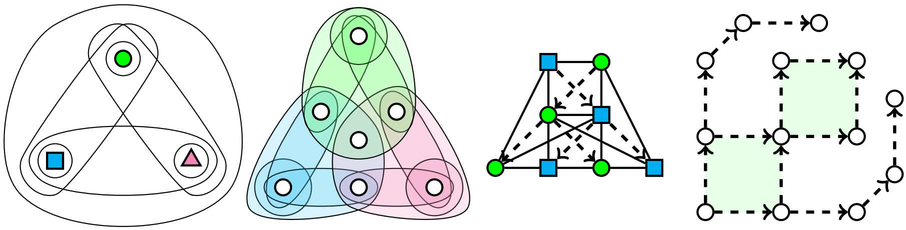

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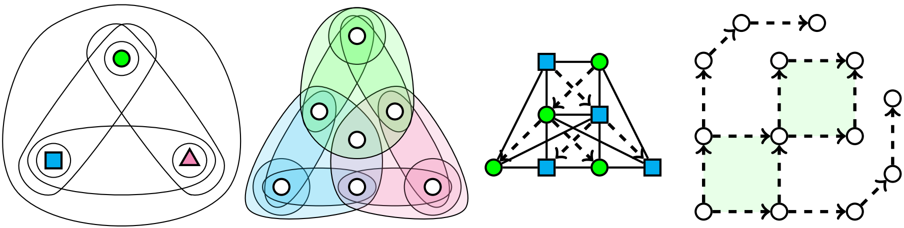

Thank you!