Hypergraphs for knowledge

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Recall the **semi-simplicial sets** semantics of epistemic logic:

\[
\varphi ::= p \mid \text{true} \mid \neg \varphi \mid \varphi \land \psi \mid D_U \varphi \quad U \subseteq \{\bullet, \square, \triangle\}, \ p \in Ap
\]

\[
M, w \models D_U \varphi \text{ iff for all } U\text{-adjacent simplices } w', \ M, w' \models \varphi
\]
Coverings of simplicial sets

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LICS’23: Goubault, K., Ledent, Rajsbaum
"Semi-simplicial Set Models for Distributed Knowledge"
Theorem

Epistemic coverings are isomorphic to epistemic Kripke frames
“Problems”:

awkward model? topology almost lost

dead agents know everything:

\[ \text{if } \mathbf{k} \not\in \mathbf{w}, \text{ then for all } \phi, M, \mathbf{w} | = K \phi \]
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  if \( \bullet \notin w \), then for all \( \varphi \), \( M, w \models K\bullet \varphi \)
“Problems”:
- awkward model? topology almost lost
- dead agents know everything:
  if $\bullet \not\in w$, then for all $\varphi$, $M, w \models K_\bullet \varphi$

Here is an idea!
Model: chromatic hypergraphs

Definition
A chromatic hypergraph $H$ is a tuple $(E, \{V_a, p_a\}_{a \in A})$, where:

- $E$ is a set of hyperedges,
- $V_a$ is a set of views of agent $a$,
- $p_a : E \rightarrow V_a$ is a surjective partial function for each agent $a$.

Additionally, we require that for each $e \in E$, $p_a(e)$ is defined for at least one $a \in A$. 
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![Diagram of chromatic hypergraph](image-url)
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Logic: two-level syntax

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$$\varphi_a ::= p_a \mid \text{true}_a \mid \neg \varphi \mid \varphi \land \psi \mid \hat{K}_a \Phi \mid K_a \Phi \quad a \in A, \ p_a \in Ap_a$$

$$\Phi ::= p_E \mid \text{true}_E \mid \neg \Phi \mid \Phi \land \Psi \mid \frak{a}_a \varphi \mid \frak{g}_a \varphi \quad a \in A, \ p_E \in Ap_E$$
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\[\varphi_a ::= p_a \mid \text{true}_a \mid \neg \varphi \mid \varphi \land \psi \mid \hat{K}_a \Phi \mid K_a \Phi \quad a \in \mathcal{A}, \ p_a \in \text{Ap}_a\]

\[\Phi ::= p_E \mid \text{true}_E \mid \neg \Phi \mid \Phi \land \Psi \mid \Diamond_a \varphi \mid \Box_a \varphi \quad a \in \mathcal{A}, \ p_E \in \text{Ap}_E\]
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\[
\varphi_a ::= p_a \mid \text{true}_a \mid \lnot \varphi \mid \varphi \land \psi \mid \hat{K}_a \Phi \mid K_a \Phi \quad a \in \mathcal{A}, \ p_a \in \text{Ap}_a
\]

\[
\Phi ::= p_E \mid \text{true}_E \mid \lnot \Phi \mid \Phi \land \psi \mid \sharp_a \varphi \mid \flat_a \varphi \quad a \in \mathcal{A}, \ p_E \in \text{Ap}_E
\]

Rules:

\[
\frac{\vdash \varphi}{\vdash \varphi \rightarrow \psi} \quad \text{MP} \quad \frac{\vdash E \Phi}{\vdash_a K_a \Phi} \quad \text{Nec-a} \quad \frac{\vdash a \varphi}{\vdash E \flat_a \varphi} \quad \text{Nec-E}
\]

\[
\frac{\vdash \Phi \rightarrow \psi}{\vdash \Box \Phi \rightarrow \Box \psi} \quad \text{Mono} \quad \frac{\vdash \Phi \rightarrow \flat_a \psi}{\vdash \Phi \rightarrow \flat_a \psi} \quad \text{Adj-1} \quad \frac{\vdash \varphi \rightarrow K_a \psi}{\vdash \varphi \rightarrow K_a \psi} \quad \text{Adj-2}
\]
Logic: two-level syntax

\[ \varphi_a ::= p_a | true_a | \neg \varphi | \varphi \land \psi | \hat{K}_a \Phi | K_a \Phi \quad a \in A, \ p_a \in Ap_a \]

\[ \Phi ::= p_E | true_E | \neg \Phi | \Phi \land \Psi | \vartriangleleft_a \varphi | \vartriangleleft_\Phi \quad a \in A, \ p_E \in Ap_E \]

Rules:

\[ \frac{\vdash \varphi}{\vdash \varphi \rightarrow \psi} \quad \text{MP} \]

\[ \frac{\vdash E \Phi}{\vdash \hat{K}_a \Phi} \quad \text{Nec-a} \]

\[ \frac{\vdash \hat{K}_a \Phi}{\vdash E \vartriangleleft_\Phi} \quad \text{Nec-E} \]

\[ \frac{\vdash \Phi \rightarrow \Psi}{\vdash \Diamond \Phi \rightarrow \Diamond \Psi} \quad \text{Mono} \]

\[ \frac{\vdash \Phi \rightarrow \vartriangleleft_\Phi}{\vdash \vartriangleleft_\Diamond \Phi \rightarrow \Diamond \Psi} \quad \text{Adj-1} \]

\[ \frac{\vdash \Phi \rightarrow \vartriangleleft_\Phi}{\vdash \hat{K}_a \Phi \rightarrow \psi} \]

\[ \frac{\vdash E \Phi}{\vdash \vartriangleleft_\Phi \rightarrow \Psi} \quad \text{Adj-2} \]

Axioms:

\[ \vdash a \varphi \rightarrow \hat{K}_a \vartriangleleft_\Phi \quad (\text{surjectivity}) \]

\[ \vdash a \hat{K}_a \vartriangleleft_\Phi \rightarrow \varphi \quad (\text{functionality}) \]

\[ \vdash_E \Phi \rightarrow \bigvee_{a \in A} \vartriangleleft_a \hat{K}_a \Phi \quad (\text{joint totality}) \]
Semantics

\[
M, v_a \models \hat{K}_a \Phi \iff \exists (e \ni v_a). M, e \models \Phi \\
M, e \models \hat{\alpha}_a \phi \iff \exists (v_a \in e). M, v_a \models \phi
\]
Some properties: derivable

- **self-awareness**: $K_a(\blacklozenge_a \varphi \rightarrow \Box_a \varphi)$
- “positive” introspection: $K_a \Phi \rightarrow K_a \blacklozenge_a K_a \Phi$
- “negative” introspection: $\Phi \rightarrow \Box_a \hat{K}_a \Phi$
- locality by definition: $K_a \blacklozenge_a p \lor K_a \blacklozenge_a \neg p$
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- **self-awareness**: $K_a (\Diamond_a \varphi \rightarrow \Box_a \varphi)$
- **“positive” introspection**: $K_a \Phi \rightarrow K_a \Diamond_a K_a \Phi$
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- **locality by definition**: $K_a \Diamond_a p \lor K_a \Diamond_a \neg p$

- **“safe” knowledge**: $\Diamond_a K_a \Phi$ – not normal
- **“unsafe” knowledge**: $\Box_a K_a \Phi$ – normal
Theorem: The category of chromatic hypergraphs and the category of partial epistemic frames are isomorphic.

<table>
<thead>
<tr>
<th>Chromatic hypergraphs</th>
<th>Bipartite graph</th>
<th>Frames</th>
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Dual hypergraphs

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**Theorem**

*Category of chromatic hypergraphs and category of partial epistemic frames are isomorphic.*
Extensions and dualities

Neighborhood frames:
several points of view per world
Extensions and dualities

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several points of view per world

Dynamics à la Panangaden-Taylor’92:
extension with temporality
Extensions and dualities

Neighborhood frames:
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Directed semantics:
concurrency as ignorance
Conclusion

Main takeaway:
structures for (epistemic) logic are geometric in nature
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structures for (epistemic) logic are geometric in nature
▶ Beyond standard epistemic logic: many-sorted logic for explicit reasoning about worlds and points of view
▶ Beyond simplicial complexes: hypergraph semantics
Conclusion

- Main takeaway: structures for (epistemic) logic are geometric in nature
- Beyond standard epistemic logic: many-sorted logic for explicit reasoning about worlds and points of view
- Beyond simplicial complexes: hypergraph semantics

Thank you!