# Deep Inference for Graphical Theorem Proving

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Formath seminar

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# Context

**Goal:** Make proof assistants *easier* to use

- **Intuitive** and **discoverable** for newcomers
- **Productive** and **beautiful** for experts

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For now, focus on common logical heart:

*Intuitionistic First-Order Logic (iFOL)*

# Outline of this talk

**Part I: Symbolic Manipulations**

Proof-by-Action

Integration with Coq

Deep Inference Semantics of DnD

**Part II: Iconic Manipulations**

The Bubble Calculus

The Flower Calculus

The Flower Prover

Part I

# Symbolic Manipulations

# Proof-by-Action

*“A demo is worth a thousand words…”*

# Paradigm

- Fully graphical: no textual proof language
- Both spatial and temporal:

$$\text{proof} \quad = \quad \text{gesture sequence}$$

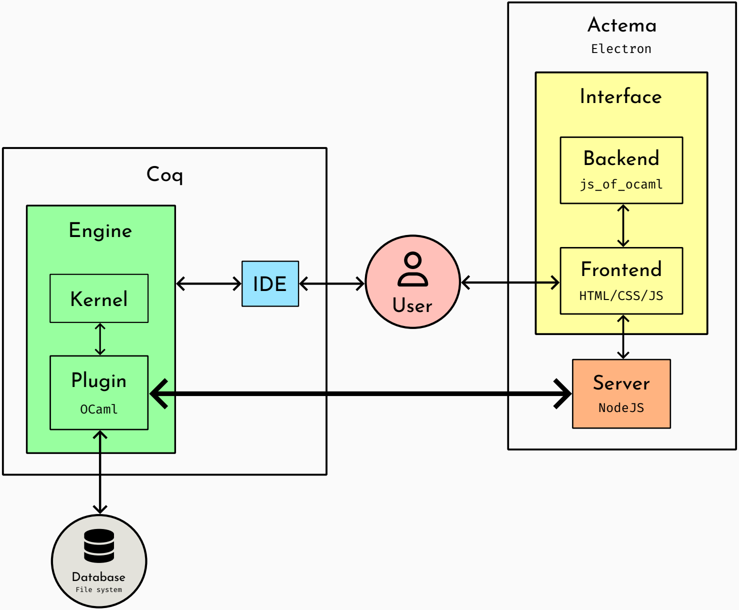
- **Different modes** of reasoning with a **single “syntax”**:

$$\text{Click} \iff \text{introduction/elimination}$$
$$\text{Drag-and-Drop} \iff \text{backward/forward}$$

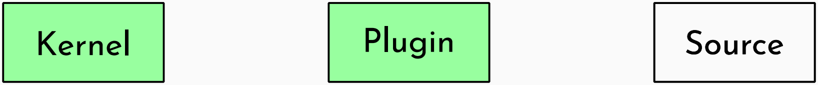
*Sound and **complete** for iFOL!*

# Integration with Coq

# Architecture



# Protocol



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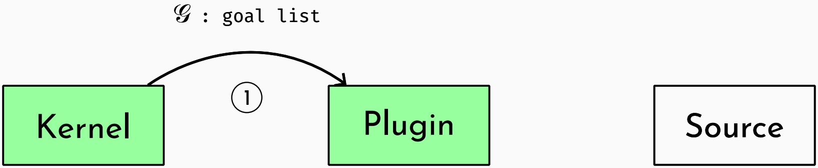

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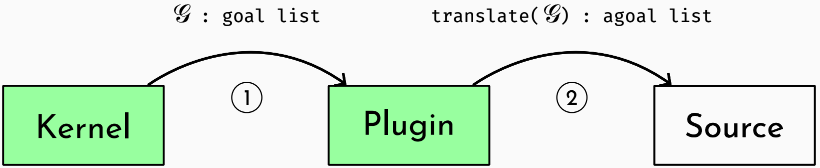
$\mathscr{G}$ : goal list
translate($\mathscr{G}$) : agoal list
1
2
Kernel
Plugin
Source

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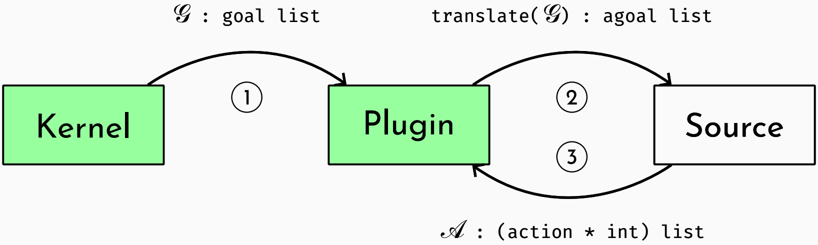
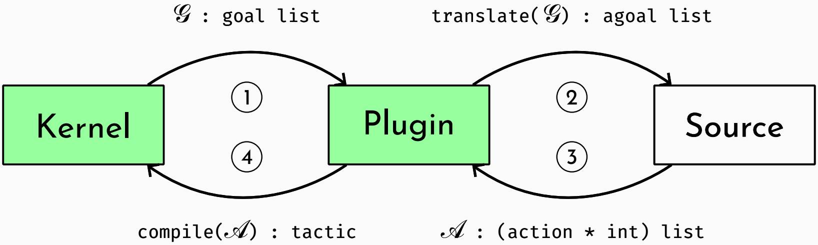


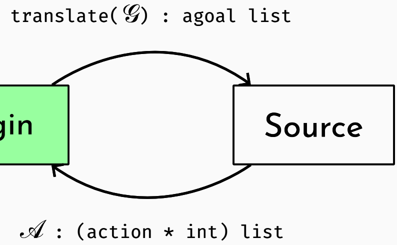
𝒢 : goal list
translate(𝒢) : agoal list
Kernel
1
Plugin
2
3
Source
𝒜 : (action * int) list

# Protocol



𝒢 : goal list
translate(𝒢) : agoal list
Kernel
1
4
Plugin
2
3
Source
compile(𝒜) : tactic
𝒜 : (action * int) list

# Protocol


translate($\mathscr{G}$) : agoal list
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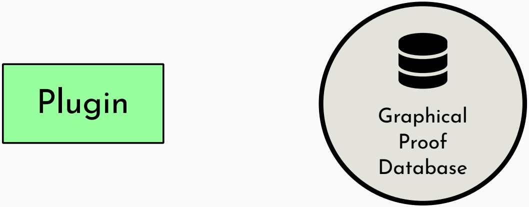
$\mathscr{A}$ : (action * int) list

# Protocol (non-interactive)



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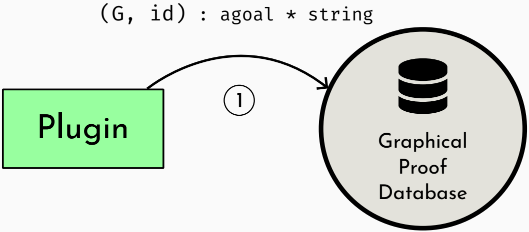

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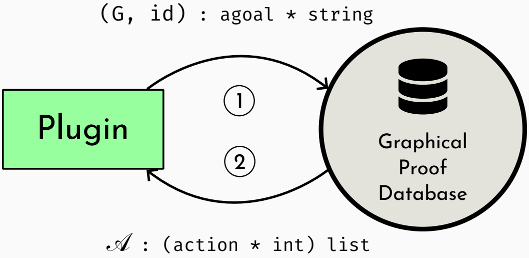

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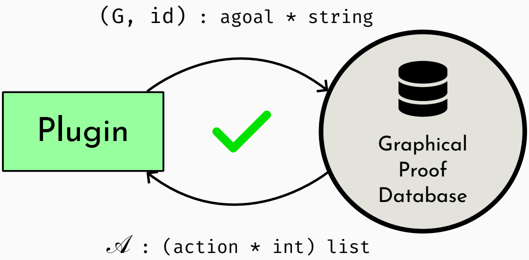

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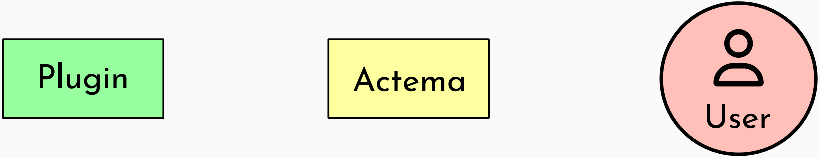

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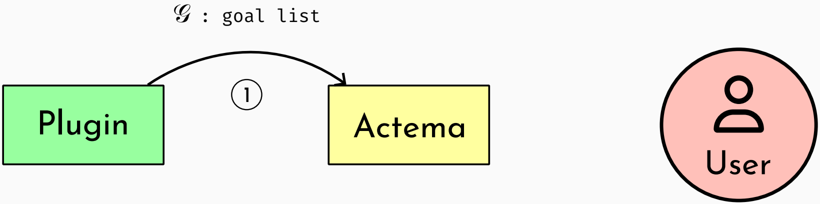

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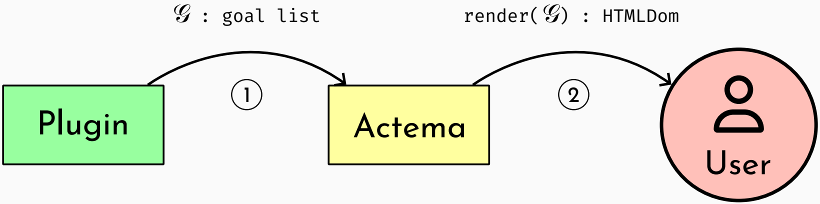

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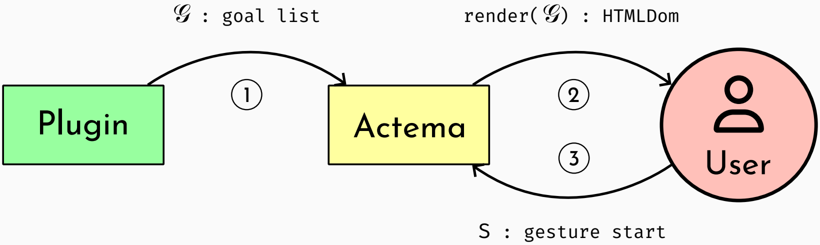

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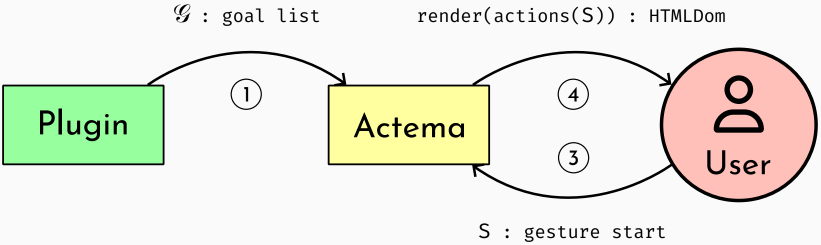

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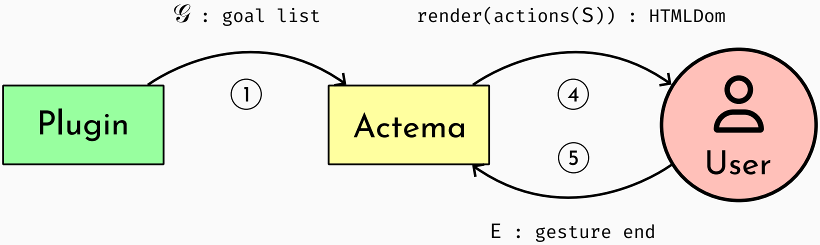

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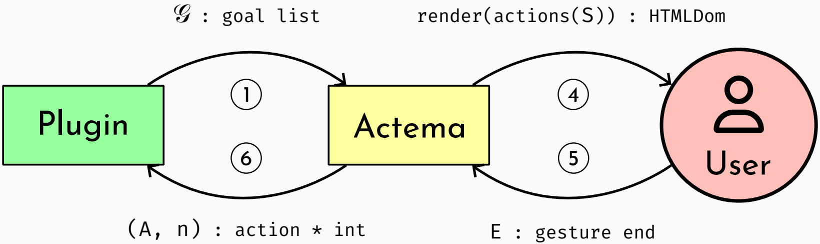

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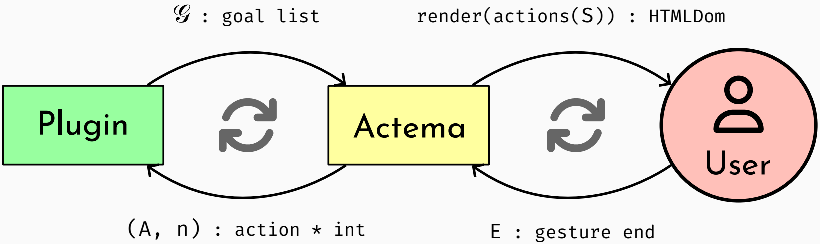

# Deep Inference Semantics of DnD

# Proof-by-Pointing

Bertot et al. (1994)

- Socrates example:

$$\text{Backward} \iff \texttt{apply H1}$$

$$\text{Forward} \iff \texttt{apply H1 in H2}$$

- $\underline{A} \wedge B \ogreaterthan B \wedge (\underline{A} \vee C) \wedge D$ is trickier...

$$\dfrac{A, B \vdash B \qquad \dfrac{\dfrac{\dfrac{}{A, B \vdash A}}{A, B \vdash A \vee C}\vee R_1 \qquad A, B \vdash D}{A, B \vdash (A \vee C) \wedge D}\wedge R}{\dfrac{A, B \vdash B \wedge (A \vee C) \wedge D}{A \wedge B \vdash B \wedge (A \vee C) \wedge D}\wedge L}\wedge R$$

```
destruct H as [HA HB].
split.
* admit.
* split.
  - left. assumption.
  - admit.
```

**Idea:** instead of *destroying* connectives, switch them

$$
\begin{aligned}
\text{switch} \quad & \quad \underline{A} \wedge B \otimes \boxed{B \wedge} (\underline{A} \vee C) \wedge D \\
& \rhd\ B \wedge (\underline{A} \wedge B \otimes (\underline{A} \vee C) \boxed{\wedge D}) \\
& \rhd\ B \wedge (\underline{A} \wedge B \otimes \underline{A} \boxed{\vee C}) \wedge D \\
& \rhd\ B \wedge ((\underline{A} \boxed{\wedge B} \otimes \underline{A}) \vee C) \wedge D \\
\text{identity} \quad & \rhd\ B \wedge ((B \Rightarrow (\boxed{\underline{A} \otimes \underline{A}})) \vee C) \wedge D \\
\text{unit elimination} \quad & \rhd\ B \wedge ((\boxed{B \Rightarrow \top}) \vee C) \wedge D \\
& \rhd\ B \wedge (\boxed{\top \vee C}) \wedge D \\
& \rhd\ B \wedge \boxed{\top \wedge} D \\
& \rhd\ B \wedge D
\end{aligned}
$$

**Rewrite rules** inspired by the *Calculus of Structures* (Guglielmi (1999)).

Add the following rules:

- Init $C^{+}\boxed{A \Rightarrow B} \ \rhd \ C^{+}\boxed{A \ogreaterthan B}$ $\quad C^{-}\boxed{A \wedge B} \ \rhd \ C^{-}\boxed{A \circledast B}$
- Release $C^{+}\boxed{A \ogreaterthan B} \ \rhd \ C^{+}\boxed{A \Rightarrow B}$ $\quad C^{-}\boxed{A \circledast B} \ \rhd \ C^{-}\boxed{A \wedge B}$
- Contraction $C^{-}\boxed{A} \ \rhd \ C^{-}\boxed{A \wedge A}$

**Theorem (Completeness)**

*If $\Gamma \vdash A$ is provable in the sequent calculus* LJ, *then* $\bigwedge \Gamma \Rightarrow A \ \rhd^{*} \ \top$.

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**Theorem (Completeness)**

*If $\Gamma \vdash A$ is provable in the sequent calculus* LJ, *then* $\bigwedge \Gamma \Rightarrow A \ \triangleright^{*} \ \top$.

**Conjecture (me):** release rules are *admissible*.
$\Longrightarrow$ would (almost) entail completeness of **DnD actions**

- Unify linked subformulas
- **Instantiate** unified variables
- **Switch** uninstantiated quantifiers

$$\exists y. \forall x. R(x,y) \oslash \forall x'.\exists y'.R(x',y')$$
$$\rhd \quad \forall y.(\forall x.R(x,y) \oslash \forall x'.\exists y'.R(x',y'))$$
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$$\rhd \quad \forall y.\forall x'.(\forall x.R(x,y) \oslash R(x',y))$$
$$\rhd \quad \forall y.\forall x'.(R(x',y) \oslash R(x',y))$$
$$\rhd \quad \forall y.\forall x'.\top$$
$$\rhd^* \quad \top$$

$x \longmapsto x'$

$y \longleftarrow\!\dashv y'$

✓

- Unify linked subformulas
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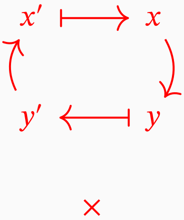
$$\forall x'.\exists y'.\underline{R(x',y')} \ogreaterthan \exists y.\forall x.\underline{R(x,y)}$$



- Unify linked subformulas
- **Check** for ∀∃ dependency cycles
- **Instantiate** unified variables
- **Switch** uninstantiated quantifiers

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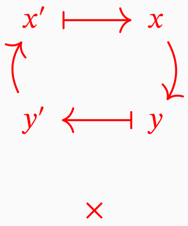

Add 4 rules $\Longrightarrow$ `rewrite` for free!

$$\underline{t} = u \oslash \underline{A} \;\triangleright\; A\{t := u\} \qquad t = \underline{u} \oslash \underline{A} \;\triangleright\; A\{u := t\}$$

$$\underline{t} = u \circledast \underline{A} \;\triangleright\; A\{t := u\} \qquad t = \underline{u} \circledast \underline{A} \;\triangleright\; A\{u := t\}$$

Compositional with semantics of **connectives**:

- **Quantifiers:** rewrite modulo *unification*
- **Implication:** *conditional* rewrite
- **Arbitrary** combinations are possible:

$$\forall x. x \neq 0 \Rightarrow \underline{f(x)} = \mathrm{g}(x) \oslash \exists y. A(\underline{f(y)}) \vee B(y)$$

$$\triangleright^* \;\; \exists y. (y \neq 0 \wedge A(\mathrm{g}(y))) \vee B(y)$$

# Compiling actions (B. Werner, K. Chaudhuri)

- **Click** actions: standard Coq tactics
- **Drag-and-Drop** actions: $\sim$ 3000 lines of Coq/Ltac
  - **Deep embedding** of goal $\Gamma \vdash C$ in FOL
  - Subterm selection as **paths**, i.e. `list nat`
  - **Computational reflection** for *deep inference* semantics [Donato et al. (2022b)]
    - Backward: new conclusion $C'$
    - Forward: new hypothesis $A$
  - Final tactic = apply **soundness** theorem
    - Backward: $\Gamma \Rightarrow C' \Rightarrow C$
    - Forward: $\Gamma \Rightarrow A$

# Part II

# Iconic Manipulations

# The Bubble Calculus

# The chemical metaphor

| | | |
|---|---|---|
| Item | $\Longleftrightarrow$ | Ion |
| Color | $\Longleftrightarrow$ | Polarity |
| Logical connective | $\Longleftrightarrow$ | Chemical bond |
| Click | $\Longleftrightarrow$ | Heating |
| DnD | $\Longleftrightarrow$ | Bimolecular reaction |

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- Works well for $\Rightarrow$ and $\wedge$ only!
- Problem: **context-scoping** through *premisses/tabs*

# Adding bubbles

**Idea:** Internalize subgoals by nesting sequents

$$\sigma ::= \Gamma \; \textcircled{\sigma_1} \; \dots \; \textcircled{\sigma_n} \; \Delta \qquad \Gamma ::= A_1 \; \dots \; A_n \qquad \Delta ::= \varnothing \mid A$$

$$(A \vee B \Rightarrow C) \Rightarrow (A \Rightarrow C) \wedge (B \Rightarrow C)$$

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$A \vee B \Rightarrow C$ $\qquad$ $(A \Rightarrow C) \wedge (B \Rightarrow C)$

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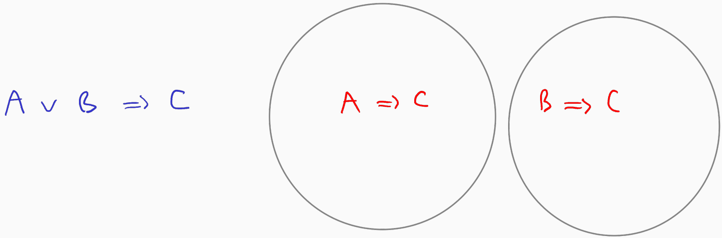

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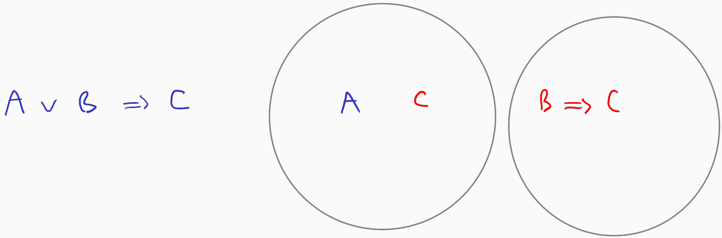

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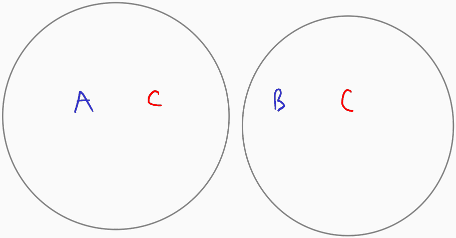
$A \vee B \Rightarrow C$



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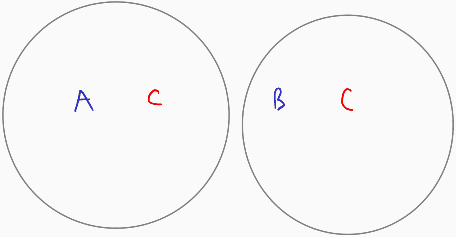

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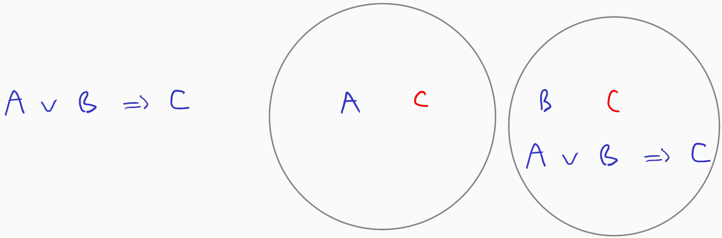

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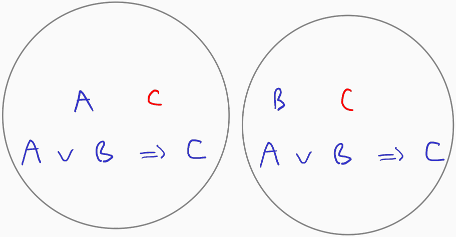

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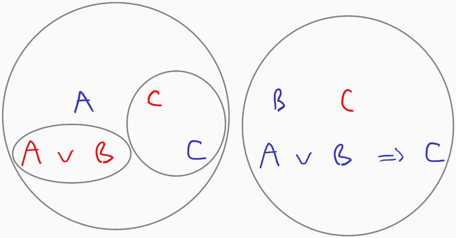

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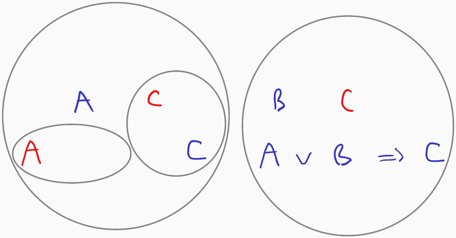

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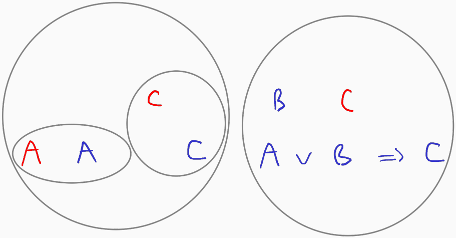

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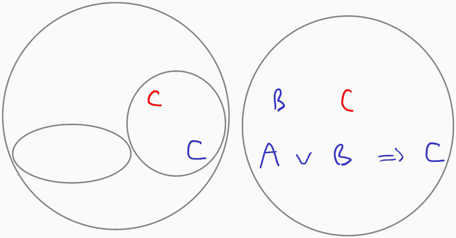

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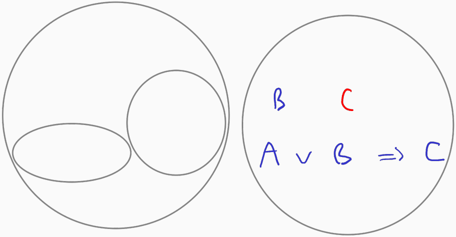

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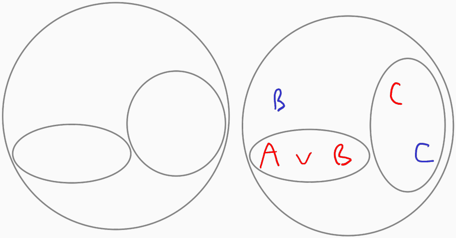

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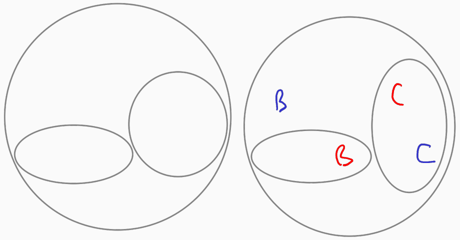

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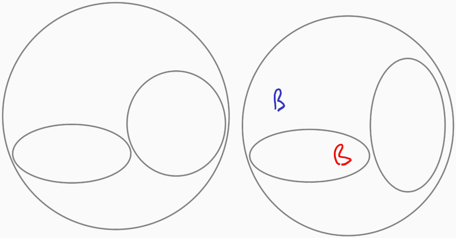

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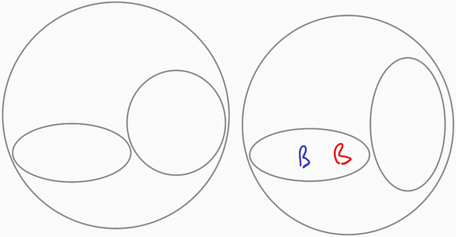
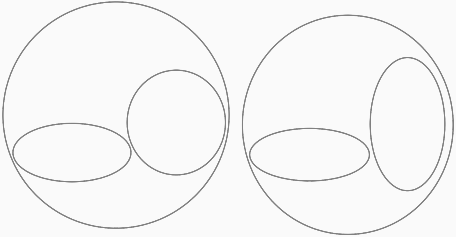


# Adding bubbles

**Idea:** Internalize subgoals by nesting sequents

$\sigma ::= \Gamma \; \textcircled{\sigma_1} \; \dots \; \textcircled{\sigma_n} \; \Delta \qquad \Gamma ::= A_1 \; \dots \; A_n \qquad \Delta ::= \varnothing \mid A$



# Adding bubbles

**Idea:** Internalize subgoals by nesting sequents

$$\sigma ::= \Gamma \; \enclose{circle}{\sigma_1} \; \dots \; \enclose{circle}{\sigma_n} \; \Delta \qquad \Gamma ::= A_1 \; \dots \; A_n \qquad \Delta ::= \emptyset \mid A$$

# Reducing non-determinism

**Moto:** Non-reversibility reduces *freedom*

$$A \vee B \;\triangleright\; A$$

$$A \vee B \;\triangleright\; B$$

$$A \Rightarrow B \quad C \;\triangleright\; \boxed{A} \quad \boxed{B \quad C}$$

- Hack: use only DnD
- New objective: **full formula decomposition** property
  $\Longrightarrow$ ability to reason without formulas
- (Guenot, 2013): only classical $\{\wedge, \vee\}$ and intuitionistic $\{\Rightarrow\}$

$\Delta ::= \iota_1 \ \ldots \ \iota_n$ $\qquad \iota ::= A \mid \sigma$

$A \vee B \ \triangleright \ A \ \ B$ $\qquad A \Rightarrow B \ \triangleright \ (A \ \ B)$

$B \vee (A \Rightarrow C)$ $\qquad A \Rightarrow (B \vee C)$

$\Delta ::= \iota_1 \ \dots \ \iota_n \qquad\qquad \iota ::= A \mid \sigma$

$A \vee B \ \triangleright \ A \ \ B \qquad\qquad A \Rightarrow B \ \triangleright \ \left( A \ \ B \right)$

$B \vee (A \Rightarrow C) \qquad \left( A \qquad B \vee C \right)$

$$\Delta ::= \iota_1 \ \ldots \ \iota_n \qquad\qquad \iota ::= A \mid \sigma$$

$$A \vee B \;\triangleright\; A \;\; B \qquad\qquad A \Rightarrow B \;\triangleright\; (A \;\; B)$$

$$B \vee (A \Rightarrow C) \qquad (A \quad B \quad C)$$

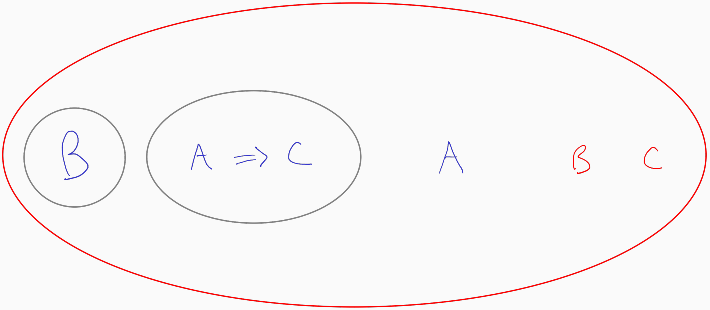
$$\Delta ::= \iota_1 \ \dots \ \iota_n \qquad\qquad \iota ::= A \mid \sigma$$

$$A \vee B \ \triangleright \ A \ \ B \qquad\qquad A \Rightarrow B \ \triangleright \ (A \ \ B)$$

$B \vee (A \Rightarrow C) \quad A \quad B \quad C$

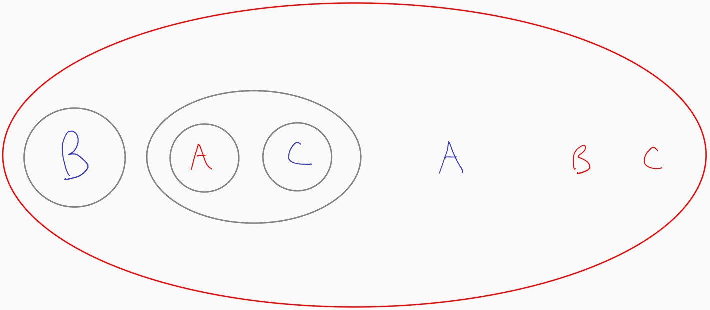
$$\Delta ::= \iota_1 \ \ldots \ \iota_n \qquad \iota ::= A \mid \sigma$$

$$A \vee B \ \triangleright \ A \ \ B \qquad A \Rightarrow B \ \triangleright \ A \ \ B$$



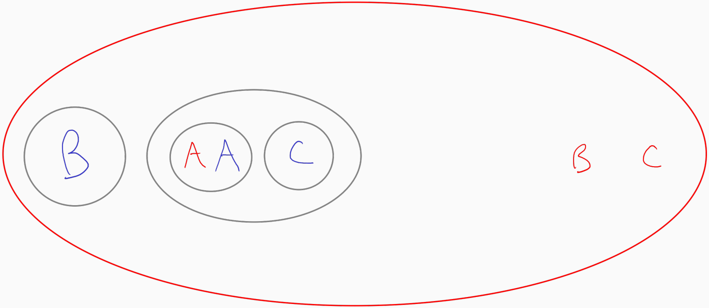
$$\Delta ::= \iota_1 \ \dots \ \iota_n \qquad \iota ::= A \mid \sigma$$

$$A \vee B \ \triangleright \ A \ \ B \qquad A \Rightarrow B \ \triangleright \ \bigcirc(A \ \ B)$$



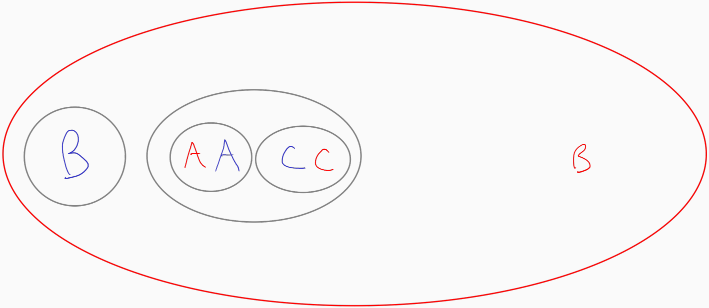
$$\Delta ::= \iota_1 \ \ldots \ \iota_n \qquad \iota ::= A \mid \sigma$$

$$A \vee B \ \triangleright \ A \ \ B \qquad A \Rightarrow B \ \triangleright \ \textcircled{A \ \ B}$$



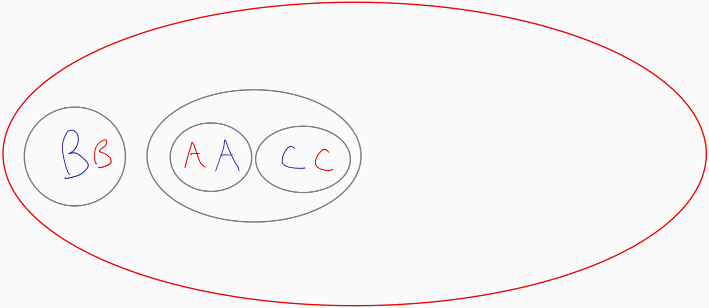
$\Delta ::= \iota_1 \;\ldots\; \iota_n$ $\qquad$ $\iota ::= A \mid \sigma$

$A \vee B \;\triangleright\; A \;\; B$ $\qquad$ $A \Rightarrow B \;\triangleright\; (A \;\; B)$



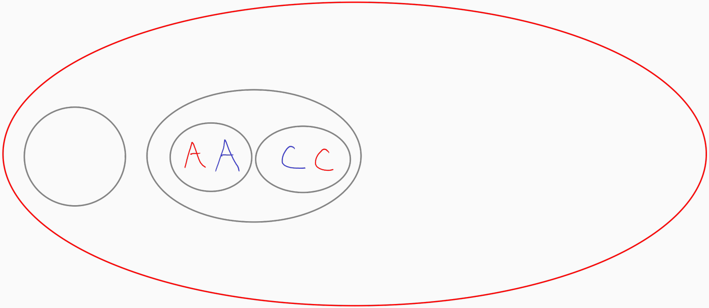
$\Delta ::= \iota_1 \;\dots\; \iota_n$ $\qquad$ $\iota ::= A \mid \sigma$

$A \vee B \;\triangleright\; A \;\; B$ $\qquad$ $A \Rightarrow B \;\triangleright\; (A \;\; B)$



$$\Delta ::= \iota_1 \ \ldots \ \iota_n \qquad \iota ::= A \mid \sigma$$

$$A \vee B \ \triangleright \ A \ \ B \qquad A \Rightarrow B \ \triangleright \ \bigcirc\!\!(A \ \ B)$$

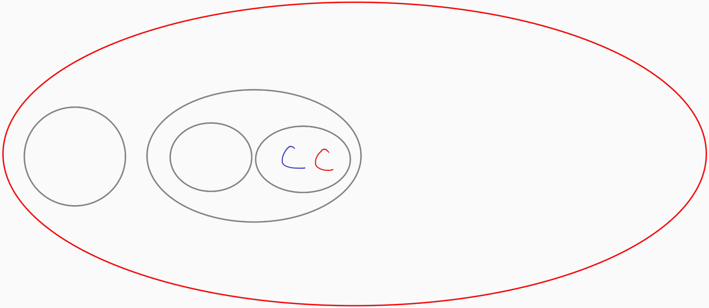


# Coloring bubbles



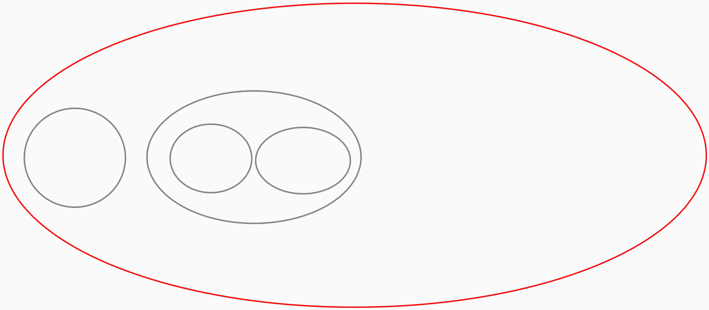
$$\Delta ::= \iota_1 \;\ldots\; \iota_n \qquad\qquad \iota ::= A \mid \sigma$$

$$A \vee B \;\triangleright\; A \;\; B \qquad\qquad A \Rightarrow B \;\triangleright\; A \;\; B$$



$$\Delta ::= \iota_1 \ \dots \ \iota_n \qquad \iota ::= A \mid \sigma$$

$$A \vee B \quad \triangleright \quad A \ \ B \qquad\qquad A \Rightarrow B \quad \triangleright \quad (A \ \ B)$$



$\Delta ::= \iota_1 \; \dots \; \iota_n$ $\qquad \iota ::= A \mid \sigma$

$A \vee B \quad \triangleright \quad A \;\; B$ $\qquad A \Rightarrow B \quad \triangleright \quad (A \;\; B)$

$A \Rightarrow (B \vee C)$ $\qquad B \vee (A \Rightarrow C)$

# Red bubbles



$$\Delta ::= \iota_1 \;\dots\; \iota_n \qquad\qquad \iota ::= A \mid \sigma$$

$$A \vee B \;\; \triangleright \;\; A \;\; B \qquad\qquad A \Rightarrow B \;\; \triangleright \;\; \bigcirc\!\left(A \;\; B\right)$$

$$A \Rightarrow (B \vee C) \qquad\qquad B \quad A \Rightarrow C$$

$$\Delta ::= \iota_1 \;\ldots\; \iota_n \qquad\qquad \iota ::= A \mid \sigma$$

$$A \vee B \;\triangleright\; A \;\; B \qquad\qquad A \Rightarrow B \;\triangleright\; \bigcirc\!(A \;\; B)$$

$$A \Rightarrow (B \vee C) \qquad\qquad B \;\; \bigcirc\!(A \;\; C)$$

# Red bubbles



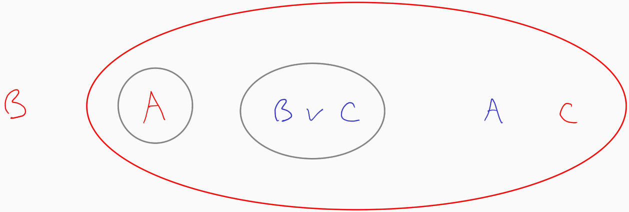
$\Delta ::= \iota_1 \;\ldots\; \iota_n \qquad\qquad \iota ::= A \mid \sigma$

$A \vee B \;\; \triangleright \;\; A \;\; B \qquad\qquad A \Rightarrow B \;\; \triangleright \;\; A \;\; B$

$B \qquad A \Rightarrow (B \vee C) \qquad A \qquad C$

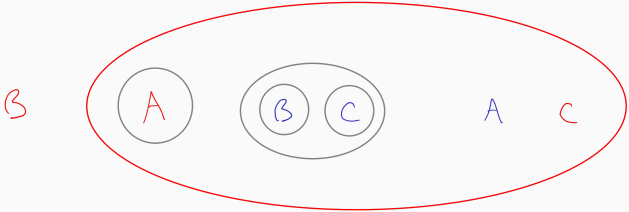
$$\Delta ::= \iota_1 \;\ldots\; \iota_n \qquad\qquad \iota ::= A \mid \sigma$$

$$A \vee B \;\;\triangleright\;\; A \;\; B \qquad\qquad A \Rightarrow B \;\;\triangleright\;\; (A \;\; B)$$



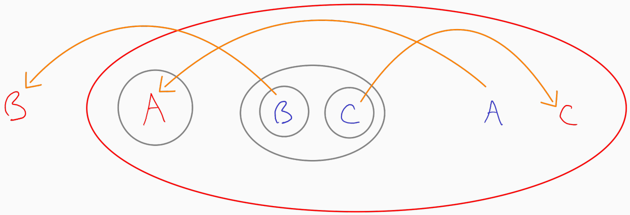
$$\Delta ::= \iota_1 \;\ldots\; \iota_n \qquad\qquad \iota ::= A \mid \sigma$$

$$A \vee B \quad \rhd \quad A \;\; B \qquad\qquad A \Rightarrow B \quad \rhd \quad \bigcirc\!\!\!\!\!\!\!\quad A \;\; B$$



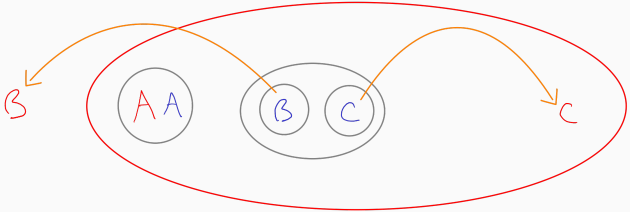
$\Delta ::= \iota_1 \; \ldots \; \iota_n$ $\qquad$ $\iota ::= A \mid \sigma$

$A \vee B \;\; \triangleright \;\; A \;\; B$ $\qquad$ $A \Rightarrow B \;\; \triangleright \;\; (A \;\; B)$



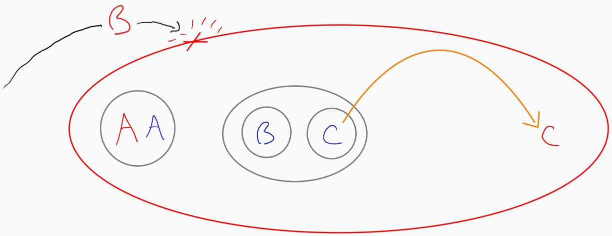
$$\Delta ::= \iota_1 \ \dots \ \iota_n \qquad\qquad \iota ::= A \mid \sigma$$

$$A \vee B \quad \triangleright \quad A \quad B \qquad\qquad A \Rightarrow B \quad \triangleright \quad A \quad B$$



$\Delta ::= \iota_1 \ \ldots \ \iota_n$ $\qquad$ $\iota ::= A \mid \sigma$

$A \vee B \quad \triangleright \quad A \ \ B$ $\qquad$ $A \Rightarrow B \quad \triangleright \quad (A \ \ B)$



# Distribution semantics

$$\left\llbracket \Gamma \; \textcircled{$\Gamma_1 \;\; \mathcal{S}_1 \;\; \Delta_1$} \; \ldots \; \textcircled{$\Gamma_n \;\; \mathcal{S}_n \;\; \Delta_n$} \; \Delta \right\rrbracket$$

$$=$$

$$\bigwedge_i \llbracket \Gamma \; \Gamma_i \;\; \mathcal{S}_i \;\; \Delta_i \; \Delta \rrbracket$$

Need for a (non-trivializing) base case:

$$\sigma ::= \underbrace{\Gamma \vdash \Delta}_{\text{subgoal}} \mid \underbrace{\Gamma \;\; \mathcal{S} \;\; \Delta}_{\text{branching}}$$

# Blue bubbles — Clouston et al. (2013)

- Allow nesting in hypotheses $\Longrightarrow$ dual-intuitionistic logic

$$\Gamma ::= \iota_1 \;\; \dots \;\; \iota_n$$

- Rules for **subtraction** $-$ dual to $\Rightarrow$:

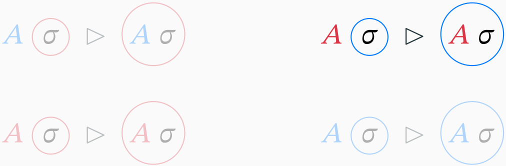
$$A - B \;\triangleright\; \bigcirc(A \;\; B) \qquad\qquad A - B \;\triangleright\; \bigcirc(A) \;\; \bigcirc(\mathrm{B})$$

- Blue bubbles **hermetic** to blue items

# A new view on classical VS intuitionistic

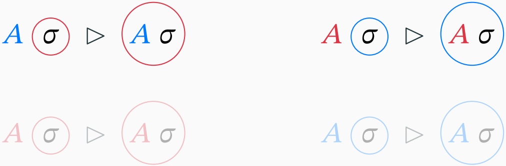

Intuitionistic logic

# A new view on classical VS intuitionistic




Dual-intuitionistic logic

# A new view on classical VS intuitionistic



Bi-intuitionistic logic

# A new view on classical VS intuitionistic

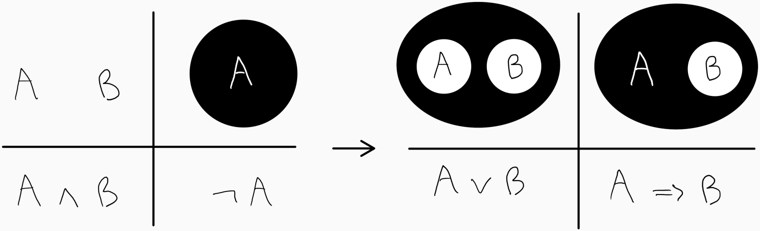

Classical logic

# A new view on classical VS intuitionistic


*Intuitionism = same polarities* ***repel*** *eachother*

# The Flower Calculus

- $\vee$ solved, but $\Rightarrow$ still irreversible!
- Key idea: space is **polarized**, *not objects*
- In classical logic:



Only 3 edition principles!

- **(De-)Iteration** (*copy/cut-paste*):


- **Insertion**: 

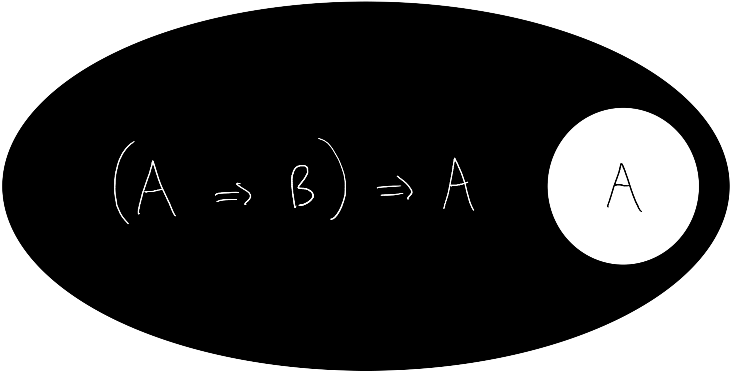
- **Deletion**: 


And a space principle, the **double-cut** law:

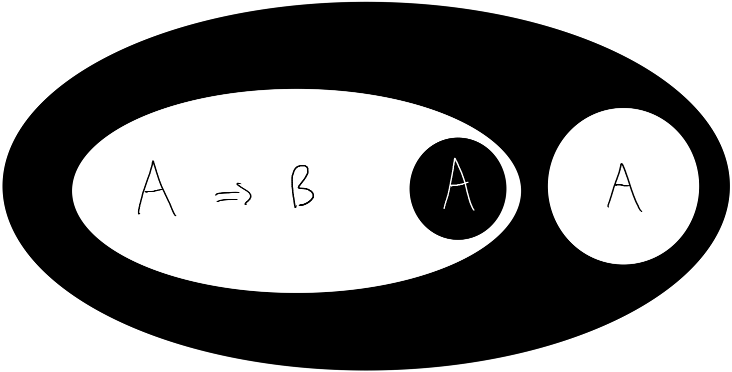
# Example: Peirce's law

$$((A \Rightarrow B) \Rightarrow A) \Rightarrow A$$

# Example: Peirce's law



# Example: Peirce's law



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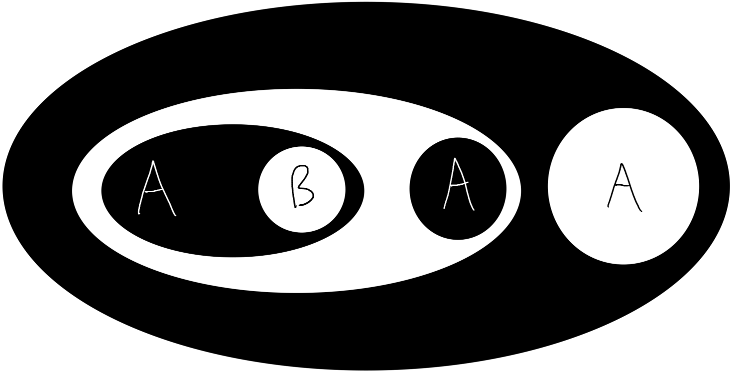

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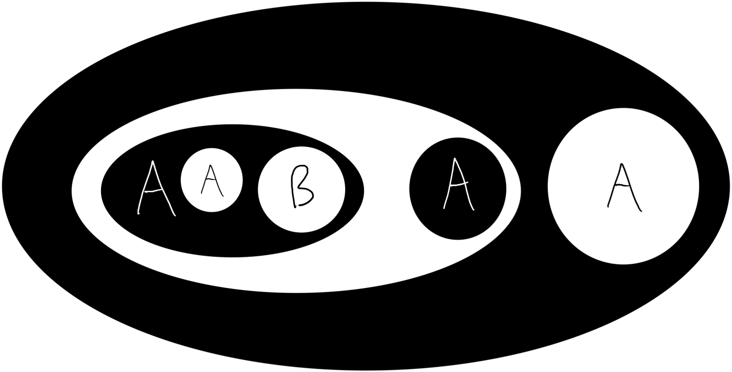

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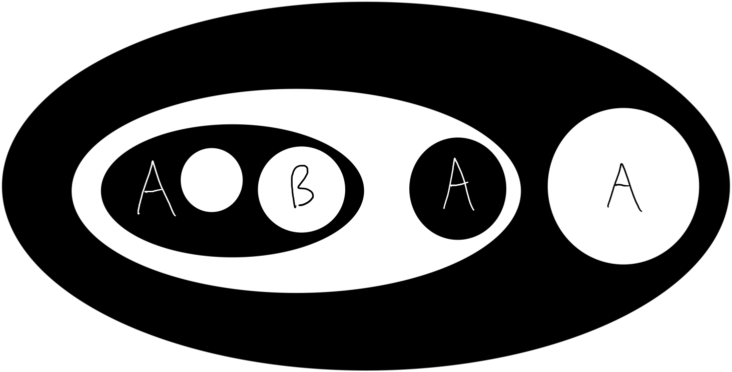

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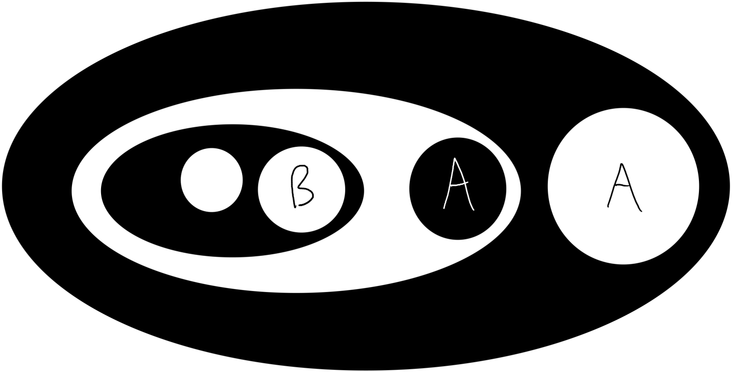
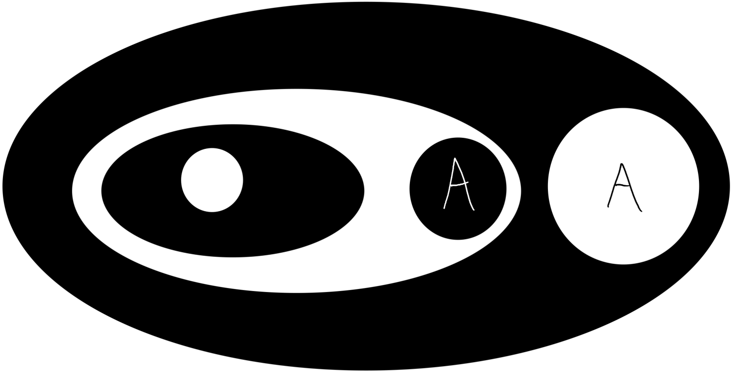


# Example: Peirce's law



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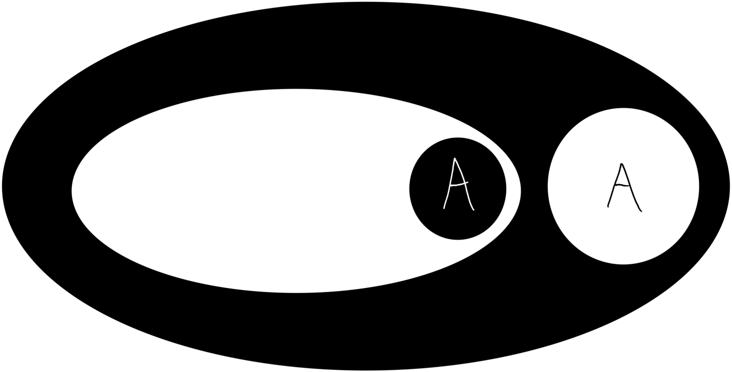
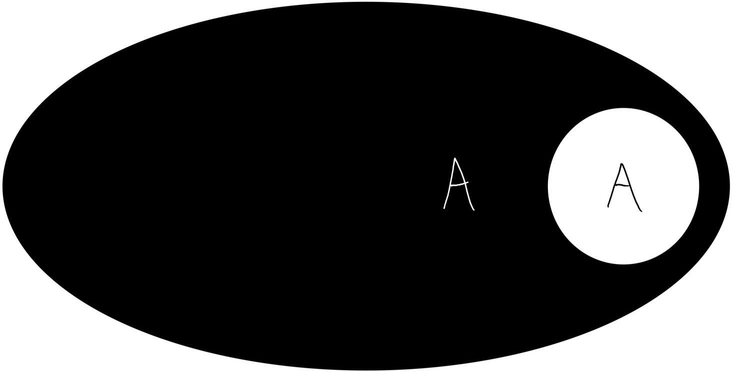
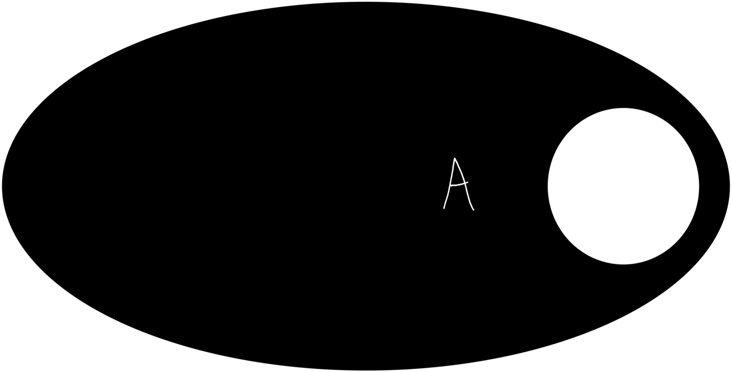


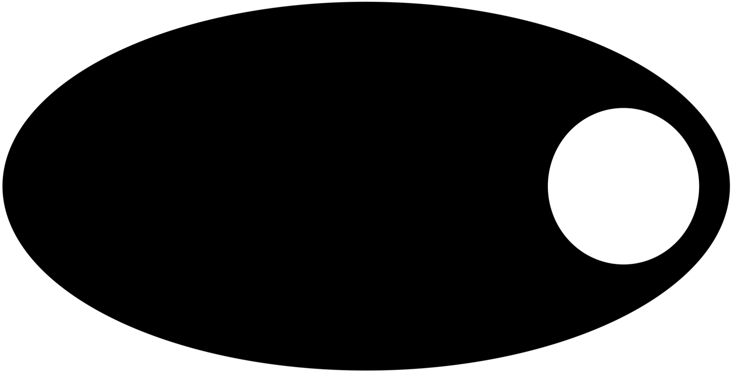
# Example: Peirce's law



# Example: Peirce's law

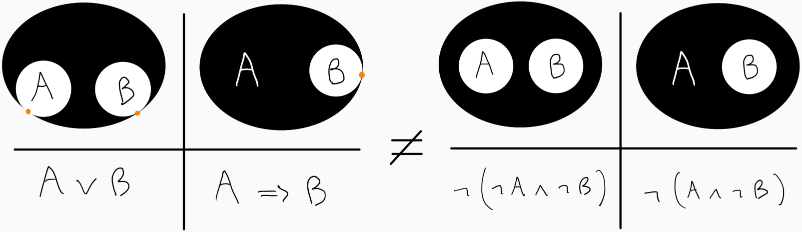


# Example: Peirce's law



# Example: Peirce's law

Officialize Peirce's scroll



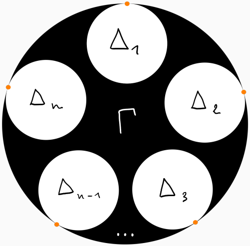
# Flowers

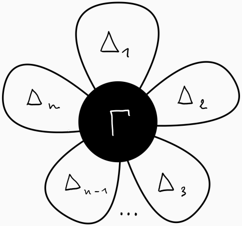
Turn *inloops* into petals

$$\phi, \psi ::= \Gamma \supset \mathcal{C} \qquad \text{(Flowers)}$$

$$\Gamma, \Delta ::= \phi_1, \dots, \phi_n \qquad \text{(Gardens)}$$

$$\mathcal{C} ::= \Delta_1 ; \dots ; \Delta_n \qquad \text{(Coronas)}$$




$$[\![\phi]\!] = \bigwedge [\![\Gamma]\!] \Rightarrow \bigvee_i \bigwedge [\![\Delta_i]\!]$$

# Identity and Space

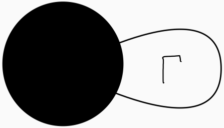
**(De-)iteration** splits in two:

$$\phi, \Delta \boxed{\phantom{\phi}} \equiv \phi, \Delta \boxed{\phi} \qquad \text{(Wind Pollination)}$$

$$\Gamma, \phi \supset \Delta \boxed{\phantom{\phi}} ; \mathcal{C} \equiv \Gamma, \phi \supset \Delta \boxed{\phi} ; \mathcal{C} \qquad \text{(Self Pollination)}$$

Decomposition law:

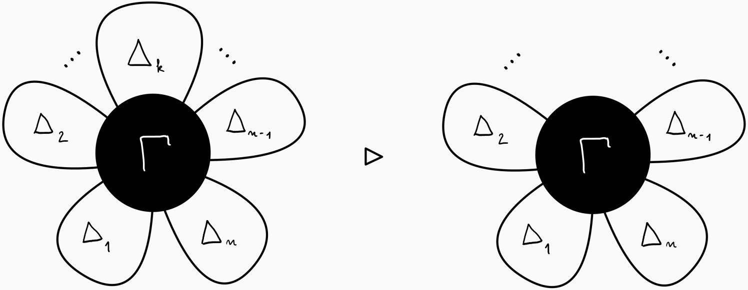

$\equiv$ $G$ becomes



$\equiv$

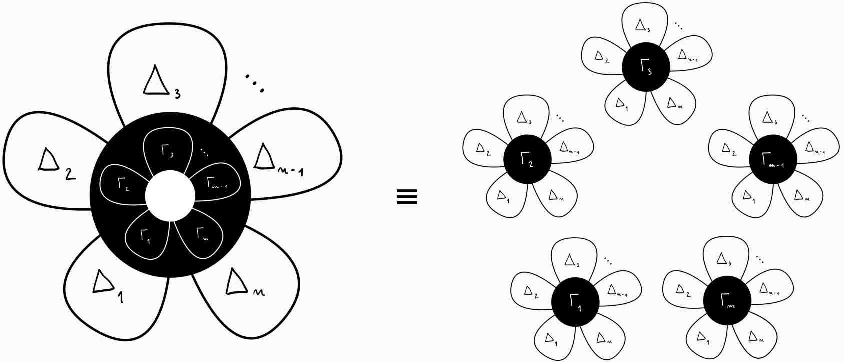
Deletion (and dually, Insertion) splits in two:

$\phi \triangleright$



# Disjunction and Falsity

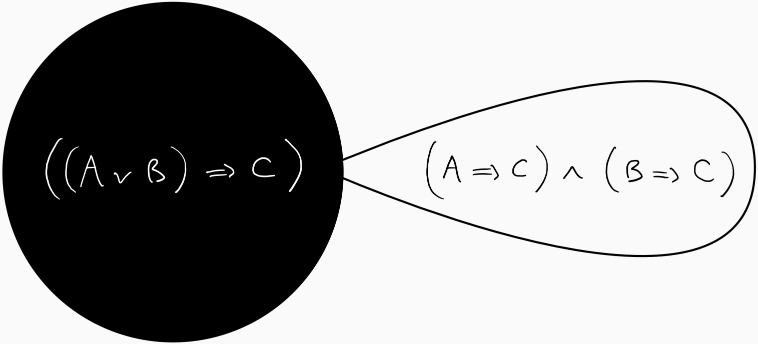
Reproduction rule for *case reasoning*:



# Example: disjunction elimination

$$((A \vee B) \Rightarrow C) \Rightarrow (A \Rightarrow C) \wedge (B \Rightarrow C)$$

# Example: disjunction elimination



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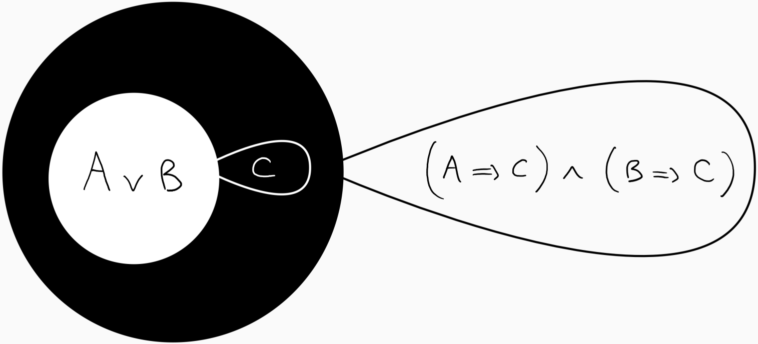

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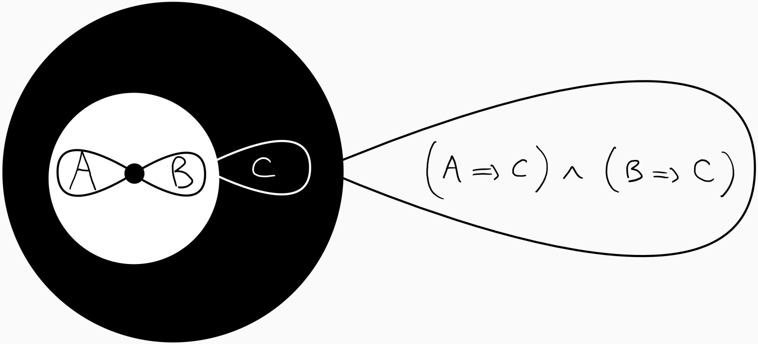

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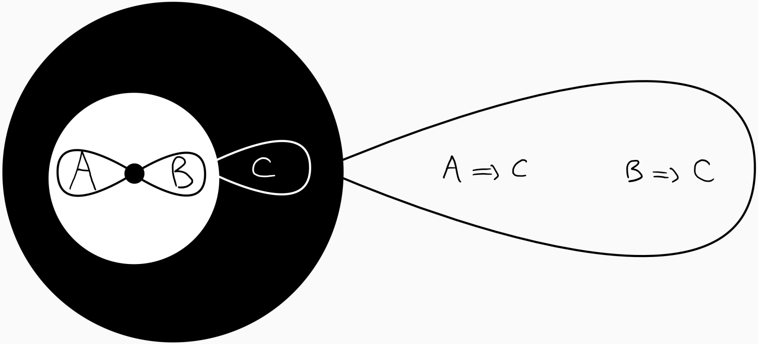

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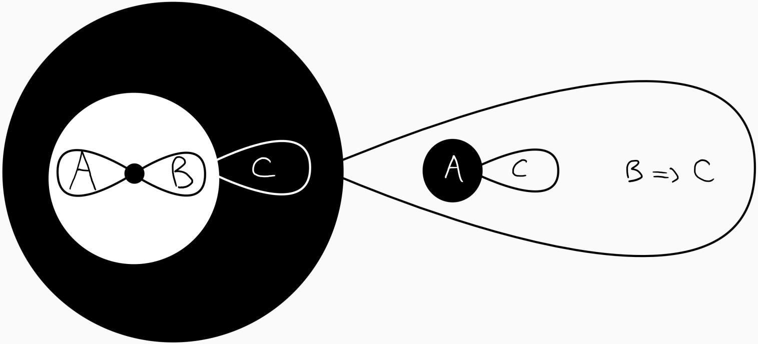

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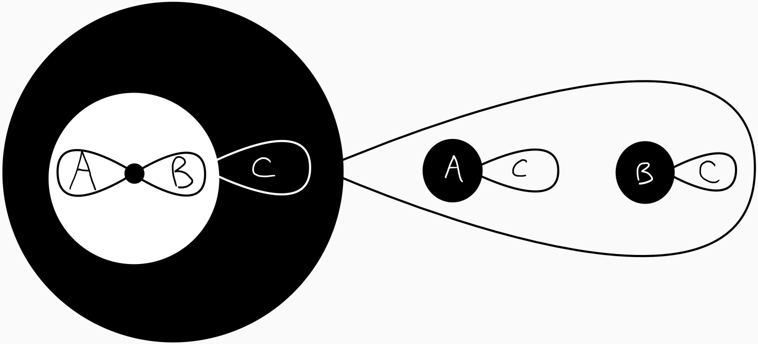

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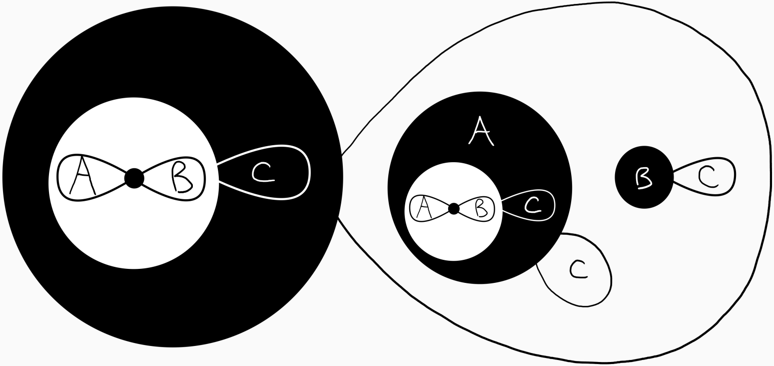

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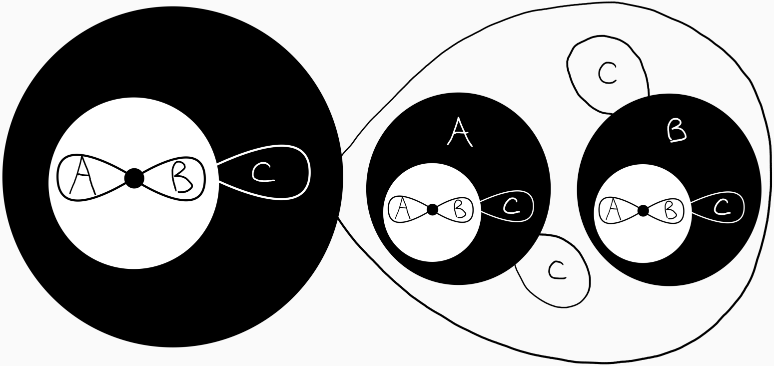

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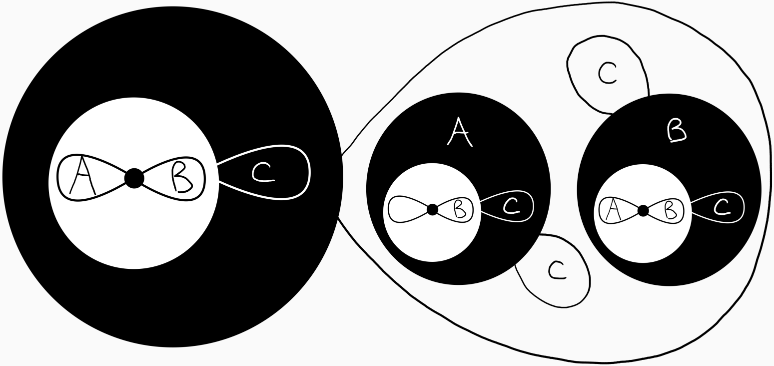
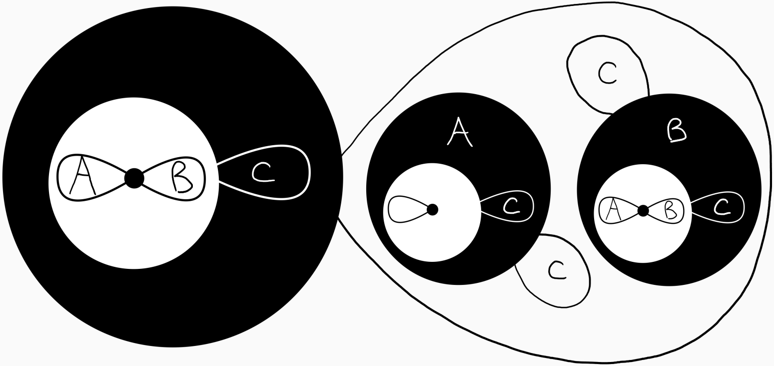
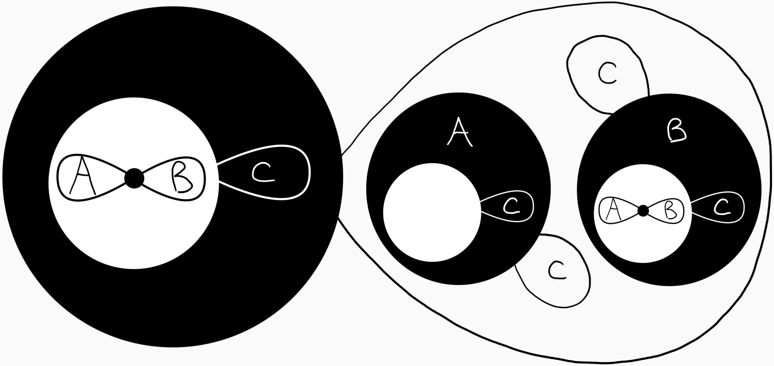


# Example: disjunction elimination



# Example: disjunction elimination



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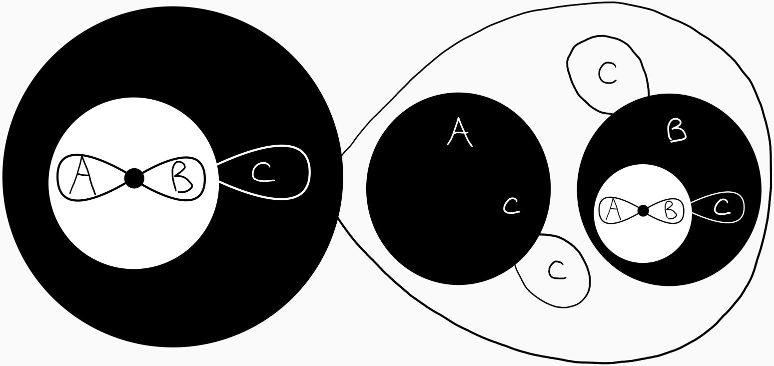
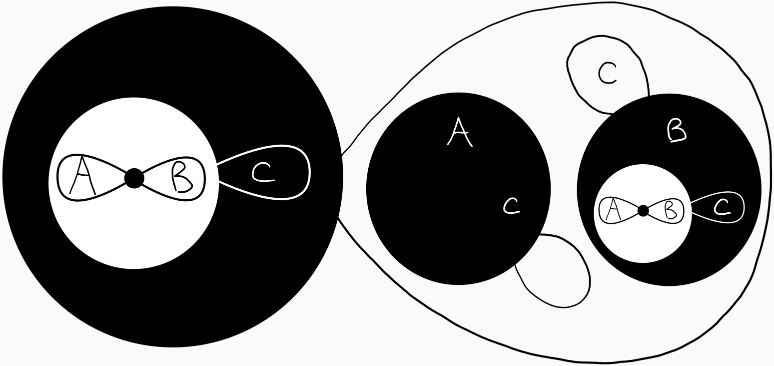


# Example: disjunction elimination



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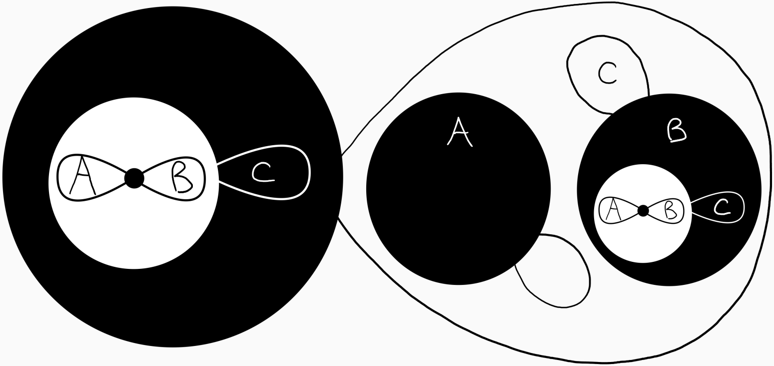

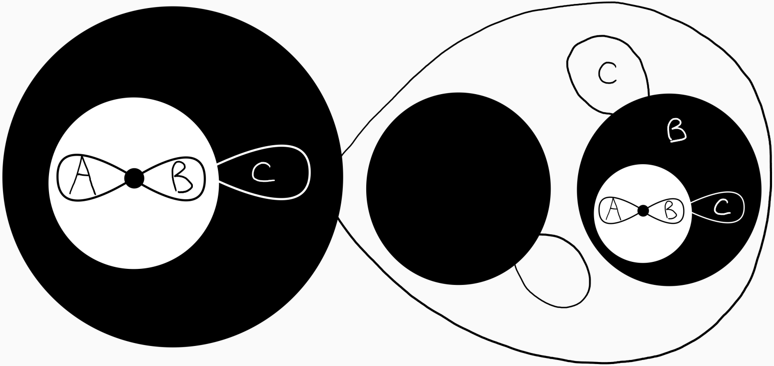
A
B
C
C
B
A
B
C

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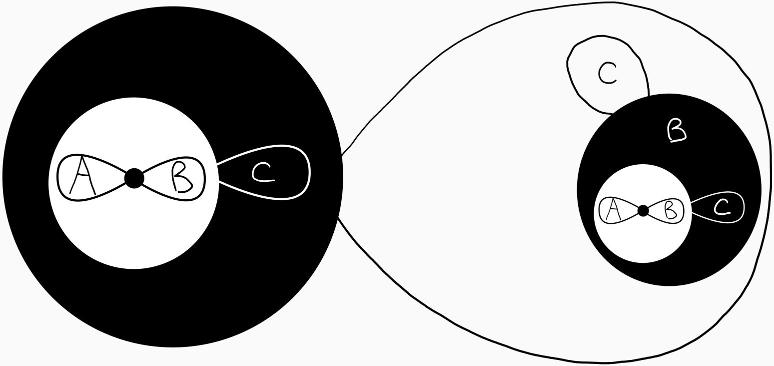

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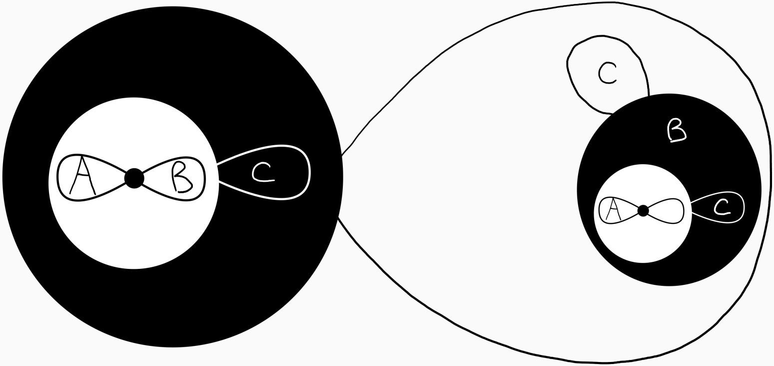

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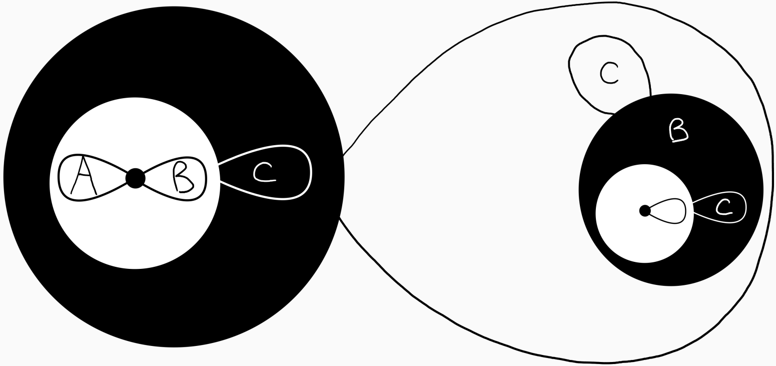

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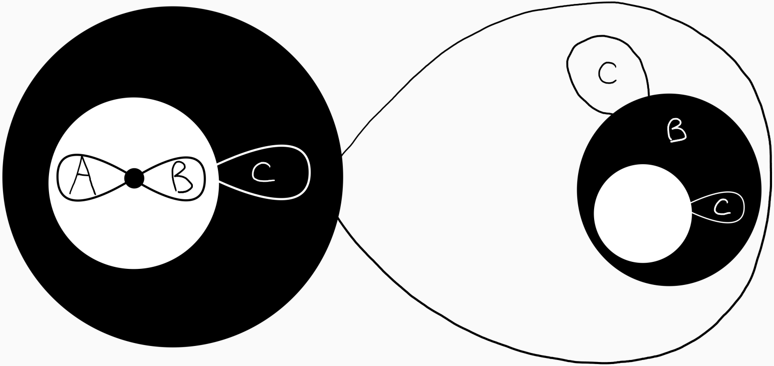

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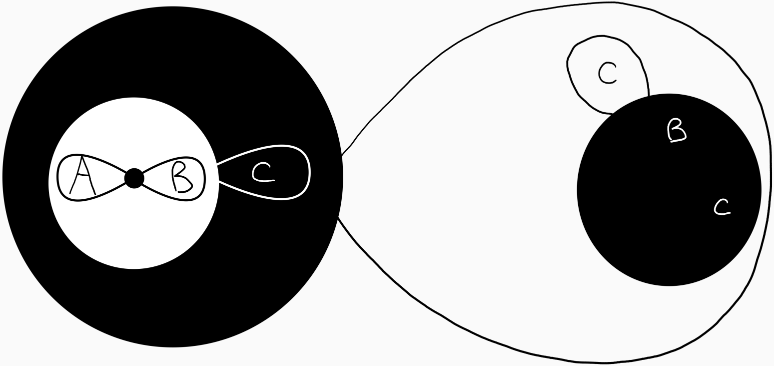

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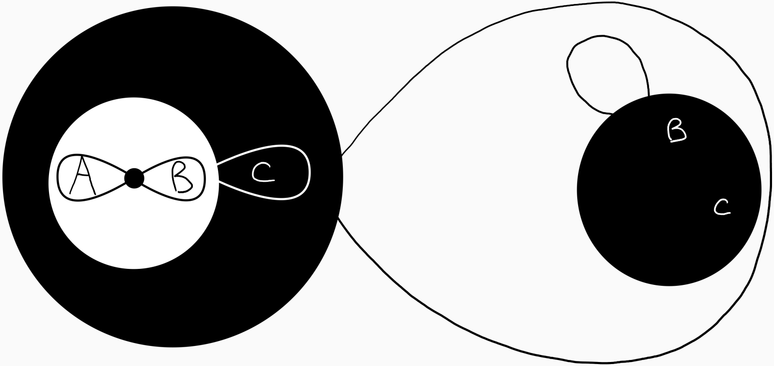

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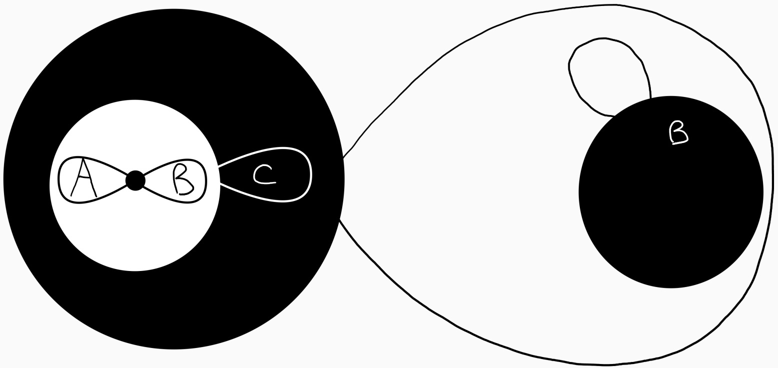

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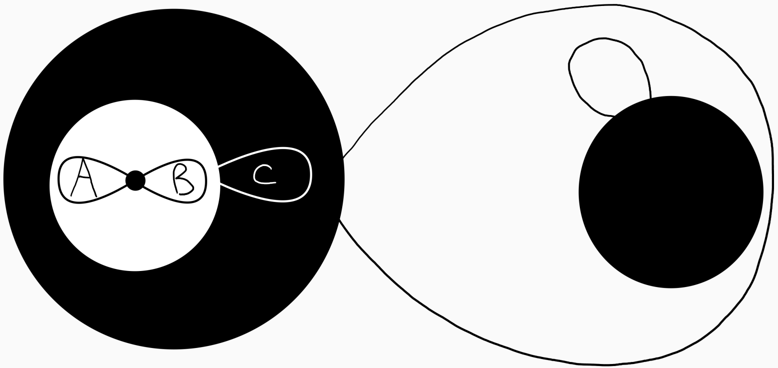

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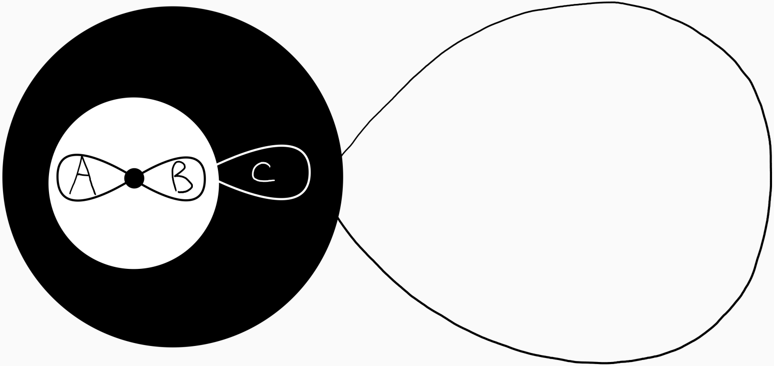

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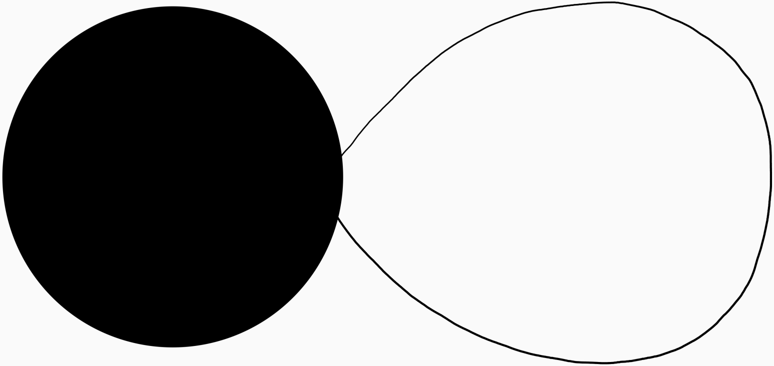

# Example: disjunction elimination

- As in EGs, **full formula decomposition** is *trivial*
- We want the butter *and* the money: what about reversibility?

What we have so far...

$$\phi, \Delta \boxed{\phi} \equiv \phi, \Delta \boxed{\phantom{\phi}} \qquad \text{(Wind Pollination)}$$

$$\Gamma, \phi \mathbin{\supset} \Delta \boxed{\phi}; \mathcal{C} \equiv \Gamma, \phi \mathbin{\supset} \Delta \boxed{\phantom{\phi}}; \mathcal{C} \qquad \text{(Self Pollination)}$$

$$\mathbin{\supset} \Delta \equiv \Delta \qquad \text{(Decomposition)}$$

$$\Gamma, (\mathbin{\supset} \{\Gamma_i\}_i^n) \mathbin{\supset} \Delta \equiv \Gamma \mathbin{\supset} \{\Gamma_i \mathbin{\supset} \Delta\}_i^n \qquad \text{(Reproduction)}$$

# Culture ✂

Other rules (Insertion and Deletion) are *oriented* and irreversible, thus **polarized**:

$$\triangleright \ \phi \qquad \text{(Grow)}$$

$$\Gamma \sqsupset \Delta ; \mathcal{C} \ \triangleright \ \Gamma \sqsupset \mathcal{C} \qquad \text{(Love)}$$

$$\phi \ \triangleright \qquad \text{(Fall)}$$

$$\Gamma \sqsupset \mathcal{C} \ \triangleright \ \Gamma \sqsupset \Delta ; \mathcal{C} \qquad \text{(Hate)}$$

## Cult elimination

New rule to handle *solved goals* (no computational content):

$$\Gamma \supset \varnothing ; \mathcal{C} \equiv \varnothing \qquad \text{(Empty Petal)}$$

**Theorem (Soundness)**

*If* $\Gamma \ \triangleright \ \Delta$*, then* $[\![\Delta]\!] \vdash [\![\Gamma]\!]$ *is provable.*

**Theorem (Completeness of Nature)**

*If* $[\![\phi]\!]$ *is true in every Kripke structure, then* $\phi \equiv \varnothing$.

**Corollary (Admissibility of Culture)**

*If* $\phi \ \triangleright^* \ \varnothing$*, then* $\phi \equiv \varnothing$.

# Quantifiers

- Add **variable binders** to *gardens*:

$$\Gamma, \Delta ::= \mathcal{X} \cdot \Phi \qquad \text{(Gardens)}$$
$$\Phi, \Psi ::= \phi_1, \ldots, \phi_n \qquad \text{(Bouquets)}$$
$$\mathcal{X}, \mathcal{Y} ::= x_1, \ldots, x_n \qquad \text{(Sprinklers)}$$

- And two **reversible** instantiation rules:

$$\mathcal{X}, x \cdot \Phi \rhd \mathcal{C} \equiv (\mathcal{X} \cdot \Phi[t/x] \rhd \mathcal{C}[t/x]), (\mathcal{X}, x \cdot \Phi \rhd \mathcal{C}) \qquad \text{(Pistil Sprinkle)}$$
$$\Gamma \rhd \mathcal{X}, x \cdot \Phi; \mathcal{C} \equiv \Gamma \rhd \mathcal{X} \cdot \Phi[t/x]; \mathcal{X}, x \cdot \Phi; \mathcal{C} \qquad \text{(Petal Sprinkle)}$$

- Now rules apply to **bouquets** instead of gardens
- **Flowers** $\Longleftrightarrow$ (arbitrary depth) intuitionistic **geometric formulas**:

$$[\![\mathcal{X} \cdot \Phi \rhd \{\mathcal{Y}_i \cdot \Psi_i\}_i^n]\!] = \forall \mathcal{X}.(\bigwedge [\![\Phi]\!] \Rightarrow \bigvee_i \exists \mathcal{Y}_i. \bigwedge [\![\Psi_i]\!])$$

# The Flower Prover

*“A demo is worth a thousand pictures…”*

# Paradigm

Another instance of **Proof-by-Action**:

- **Direct manipulation** of the *goals* themselves
- **Formulas** still supported, but superfluous
- **Modal** interface to interpret click and DnD:

| | | |
|---|---|---|
| Proof mode | $\Longleftrightarrow$ | Natural (reversible) rules |
| Edit mode | $\Longleftrightarrow$ | Cultural (non-reversible) rules |
| Navigation mode | $\Longleftrightarrow$ | Contextual closure (functoriality) |

- All possible actions are immediately visible/accessible:
  $\Longrightarrow$ **discoverable** and **touch-friendly**

# Towards Curry-Howard

Proof-by-Action is inherently dynamic:

- Rules/actions **erase** proved goals/flowers

$$\textbf{proof} \quad = \quad \textbf{reduction} \text{ steps towards } \varnothing$$

- Could we **annotate** flowers with proof terms instead?

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$$t : \phi$$

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*Blurring the frontier between proofs and types*

— Miquel (2020)

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