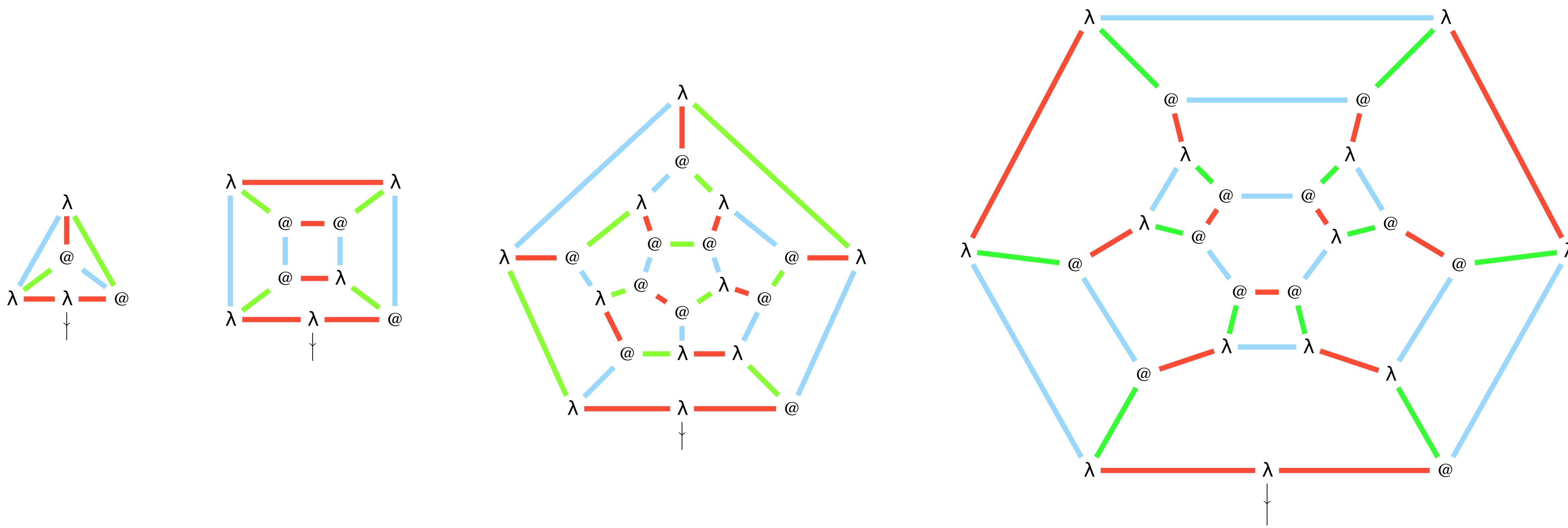
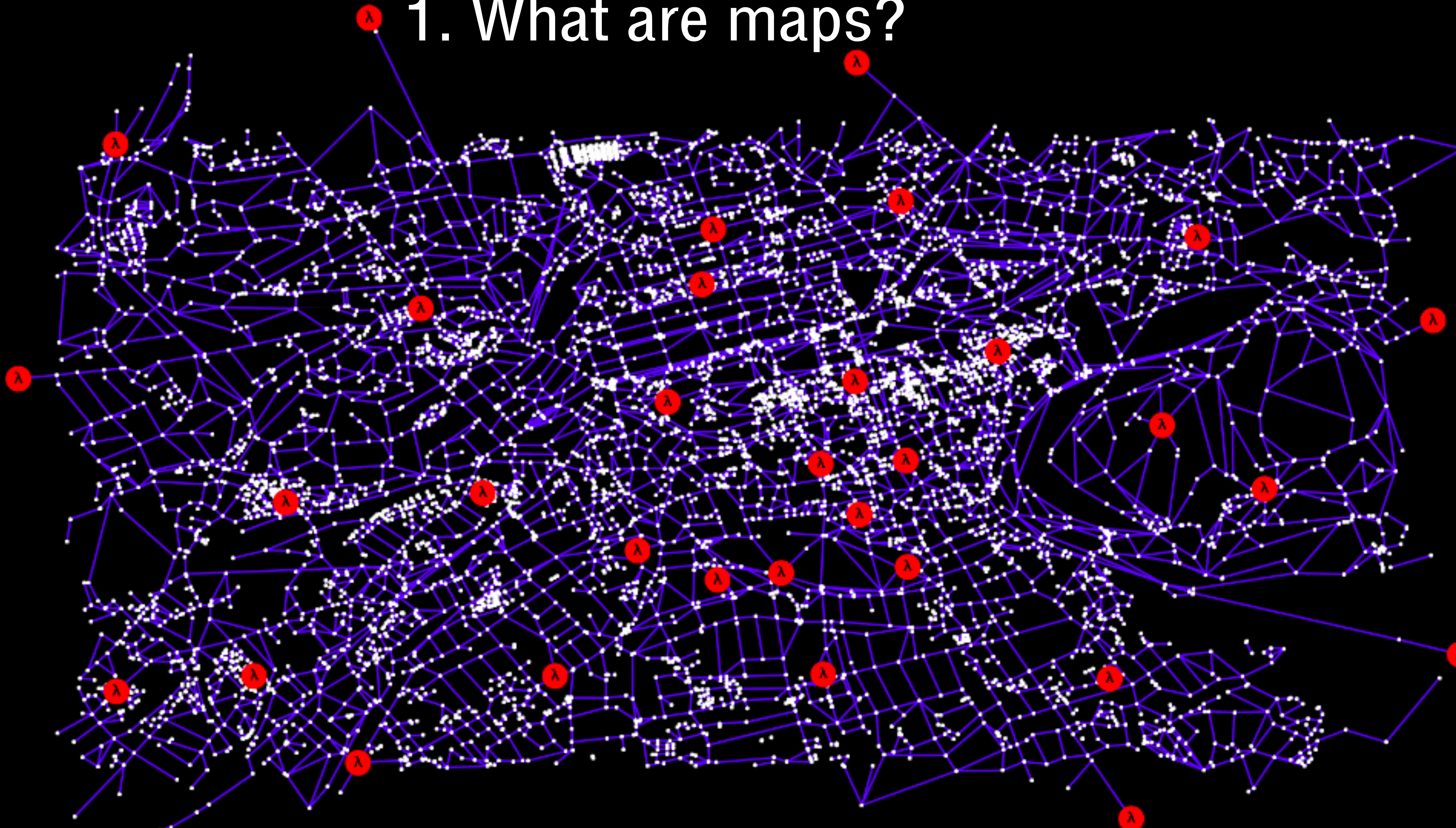
Lambda Calculus and the Four Colour Theorem



LFCS Seminar
21 November 2017
University of Edinburgh

Noam Zeilberger
School of Computer Science
University of Birmingham

1. What are maps?



(Reproduced with permission of PuntTV.)

Definition

a ***map*** is:

a *2-cell embedding* of a graph
into a (connected, oriented) surface

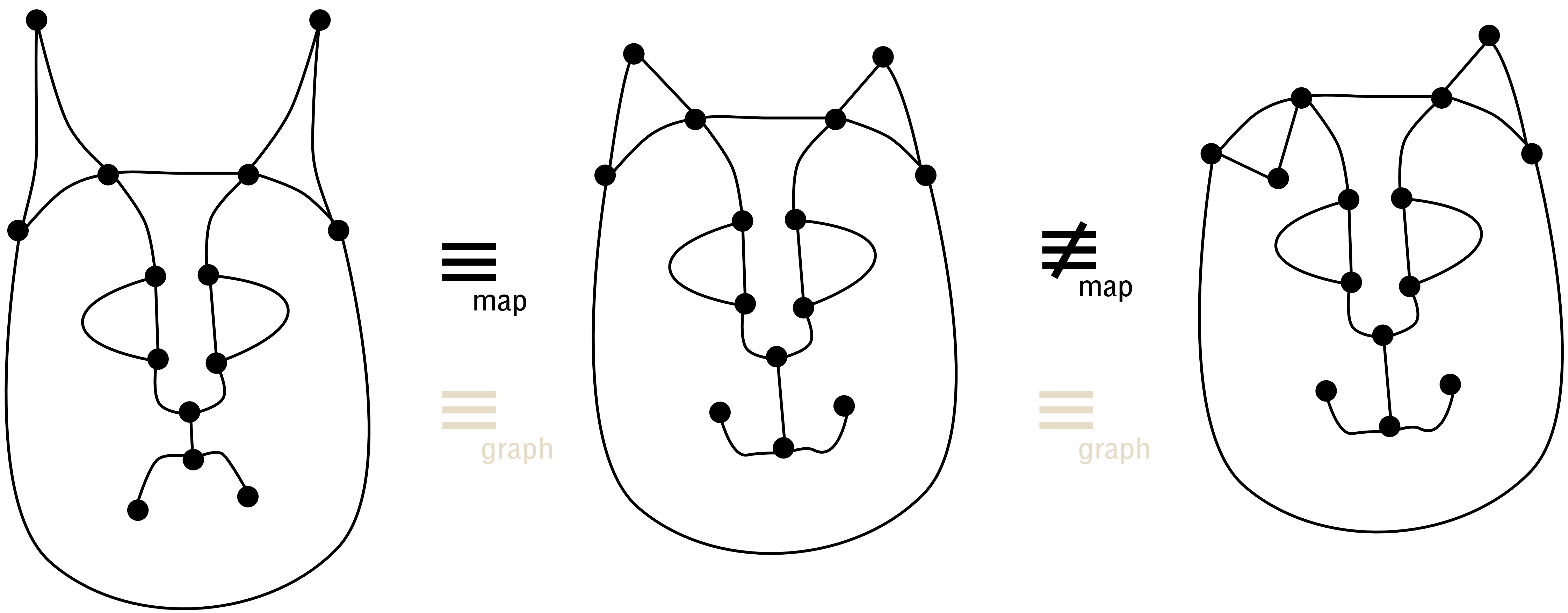
or equivalently

a (connected) graph equipped w/a cyclic ordering
of the half-edges around each vertex

or equivalently

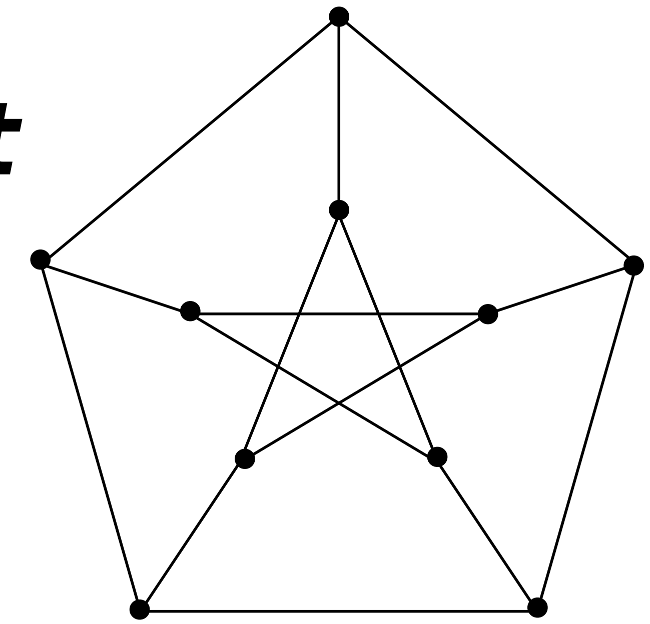
a (transitive) permutation representation
of the group $\langle v, e, f \mid e^2 = vef = 1 \rangle$

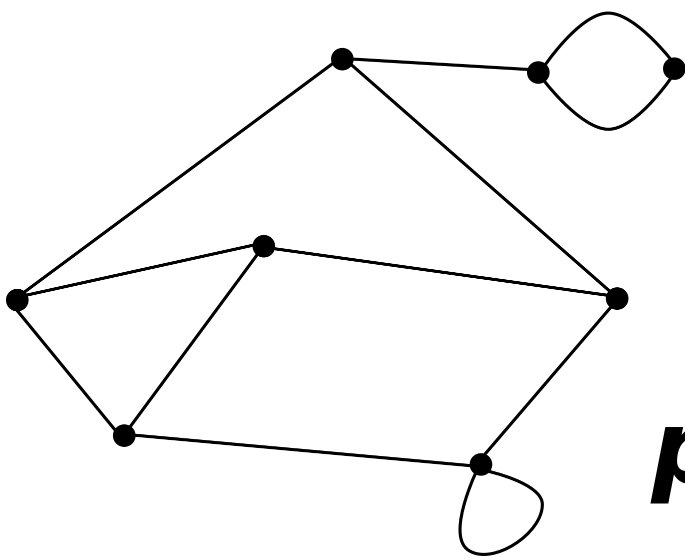
"graph" vs "map"



Special kinds of maps

3-valent

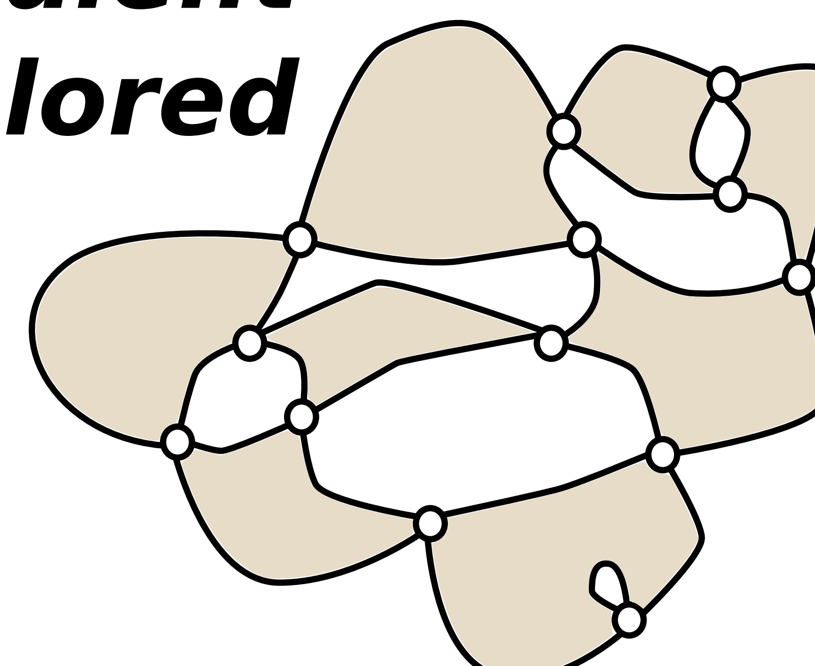


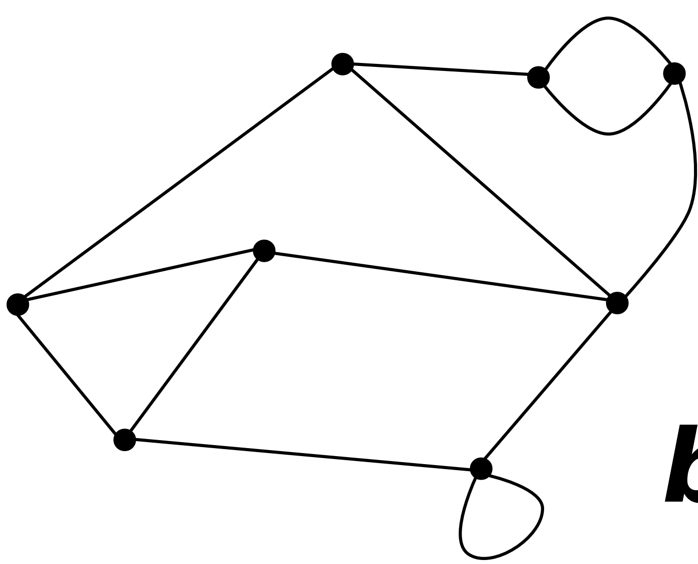


v - e + f = 8 - 12 + 6
= 2

planar

4-valent
face 2-colored

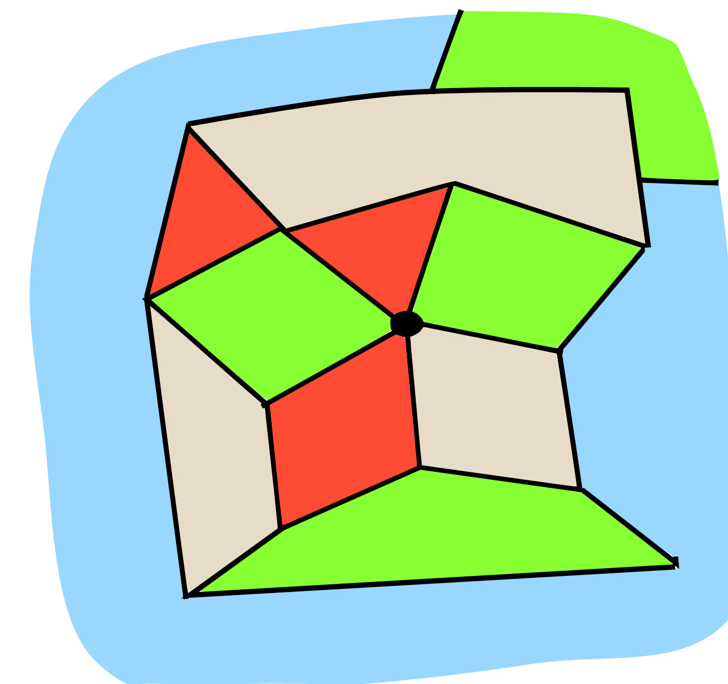




bridgeless

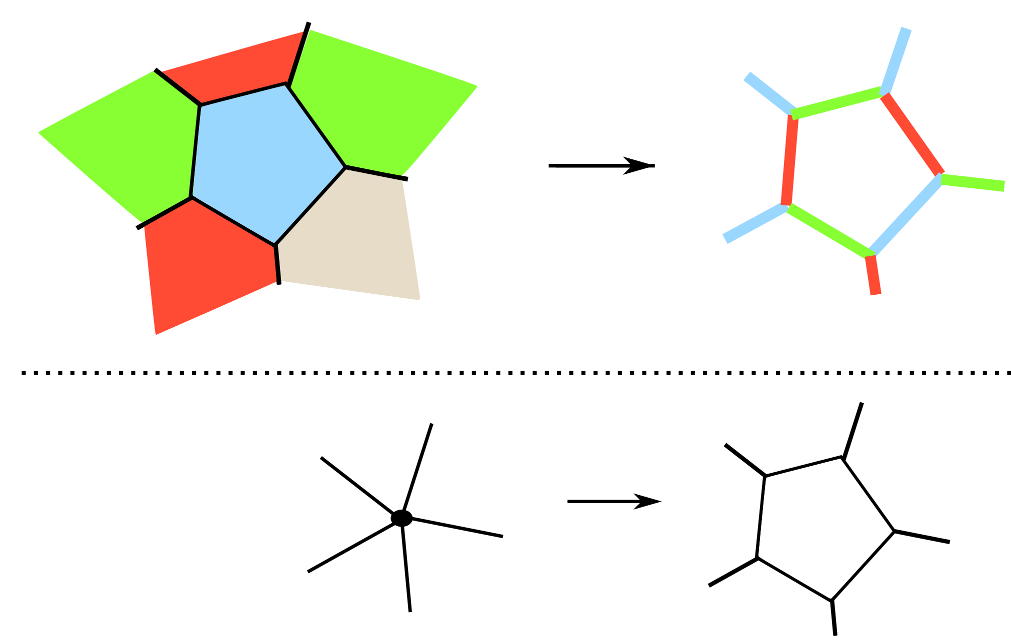
Four Colour Theorem

The 4CT is a statement about maps.



every bridgeless planar map has a proper face 4-coloring

By a well-known reduction (Tait 1880), 4CT is equivalent to a statement about 3-valent maps



every bridgeless planar 3-valent map has a proper edge 3-coloring

Map enumeration

From time to time in a graph-theoretical career one's thoughts turn to the Four Colour Problem. It occurred to me once that it might be possible to get results of interest in the theory of map-colourings without actually solving the Problem. For example, it might be possible to find the average number of colourings on vertices, for planar triangulations of a given size.

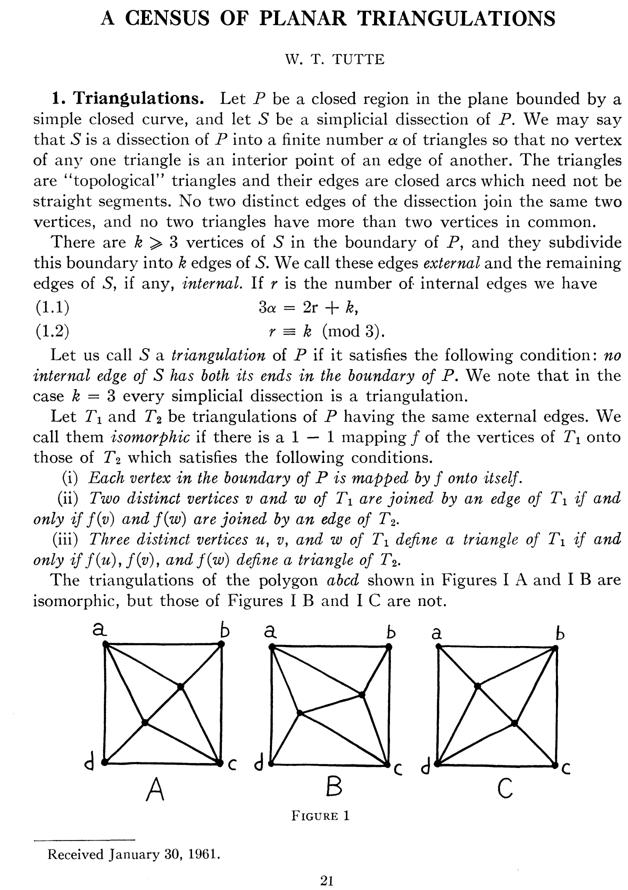
One would determine the number of triangulations of 2n faces, and then the number of 4-coloured triangulations of 2n faces. Then one would divide the second number by the first to get the required average. I gathered that this sort of retreat from a difficult problem to a related average was not unknown in other branches of Mathematics, and that it was particularly common in Number Theory.

W. T. Tutte, Graph Theory as I Have Known It

Map enumeration

Tutte wrote a seminal series of papers:

W. T. Tutte (1962), A census of planar triangulations. Canadian Journal of Mathematics 14:21–38
W. T. Tutte (1962), A census of Hamiltonian polygons. Can. J. Math. 14:402–417
W. T. Tutte (1962), A census of slicings. Can. J. Math. 14:708–722
W. T. Tutte (1963), A census of planar maps. Can. J. Math. 15:249–271
W. T. Tutte (1968), On the enumeration of planar maps. Bulletin of the American Mathematical Society 74:64
W. T. Tutte (1969), On the enumeration of four-colored maps. SIAM Journal on Applied Mathematics 17:454-



A CENSUS OF PLANAR TRIANGULATIONS

W. T. TUTTE

1. Triangulations. Let P be a closed region in the plane bounded by a simple closed curve, and let S be a simplicial dissection of P. We may say that S is a dissection of P into a finite number α of triangles so that no vertex of any one triangle is an interior point of an edge of another. The triangles are "topological" triangles and their edges are closed arcs which need not be straight segments. No two distinct edges of the dissection join the same two vertices, and no two triangles have more than two vertices in common.

There are $k \geqslant 3$ vertices of S in the boundary of P, and they subdivide this boundary into k edges of S. We call these edges *external* and the remaining edges of S, if any, *internal*. If r is the number of internal edges we have

(1.1) $$3\alpha = 2r + k,$$

(1.2) $$r \equiv k \pmod 3.$$

Let us call S a *triangulation* of P if it satisfies the following condition: *no internal edge of S has both its ends in the boundary of P.* We note that in the case $k = 3$ every simplicial dissection is a triangulation.

Let T_1 and T_2 be triangulations of P having the same external edges. We call them *isomorphic* if there is a $1 - 1$ mapping f of the vertices of T_1 onto those of T_2 which satisfies the following conditions.

(i) *Each vertex in the boundary of P is mapped by f onto itself.*

(ii) *Two distinct vertices v and w of T_1 are joined by an edge of T_1 if and only if $f(v)$ and $f(w)$ are joined by an edge of T_2.*

(iii) *Three distinct vertices u, v, and w of T_1 define a triangle of T_1 if and only if $f(u)$, $f(v)$, and $f(w)$ define a triangle of T_2.*

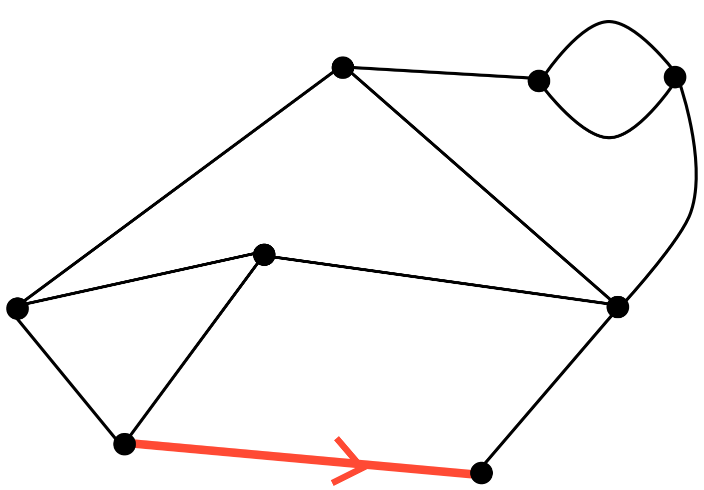
The triangulations of the polygon *abcd* shown in Figures I A and I B are isomorphic, but those of Figures I B and I C are not.

a b a b a b
d c d c d c
A B C

FIGURE 1

Received January 30, 1961.

One of his insights was to consider ***rooted*** maps



Key property: rooted maps have no non-trivial automorphisms

2. What on earth does this have to do with lambda calculus?

Some enumerative connections

family of rooted maps	family of lambda terms	sequence	OEIS
trivalent maps	linear terms[1,4]	1,5,60,1105,27120,...	A062980
planar trivalent maps	planar terms[4]	1,4,32,336,4096,...	A002005
bridgeless trivalent maps	unit-free linear terms[4]	1,2,20,352,8624,...	A267827
bridgeless planar trivalent maps	unit-free planar terms[4]	1,1,4,24,176,1456,...	A000309
maps	normal linear terms (mod ~)[3]	1,2,10,74,706,8162,...	A000698
planar maps	normal planar terms[2]	1,2,9,54,378,2916,...	A000168
bridgeless maps	normal unit-free linear terms (mod ~)[5]	1,1,4,27,248,2830,...	A000699
bridgeless planar maps	normal unit-free planar terms[6]	1,1,3,13,68,399,...	A000260

references

1. O. Bodini, D. Gardy, A. Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS 502: 227-238
2. Z, A. Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS 11(3:22): 1-39
3. Z (2015), Counting isomorphism classes of beta-normal linear lambda terms, arXiv:1509.07596
4. Z (2016), Linear lambda terms as invariants of rooted trivalent maps, J. Functional Programming 26(e21)
5. J. Courtiel, K. Yeats, Z (2016), Connected chord diagrams and bridgeless maps, arXiv:1611.04611
6. Z (2017), A sequent calculus for a semi-associative law, FSCD 2017

Families of lambda terms

a term is ***linear*** if every (free or bound) var is used exactly once

$\lambda x.x$ ✓ $\lambda x.\lambda y.x$ ✗

$\lambda x.\lambda y.xy$ ✓ $\lambda x.\lambda y.x(xy)$ ✗

a term is ***planar*** if vars are used in the order they're bound

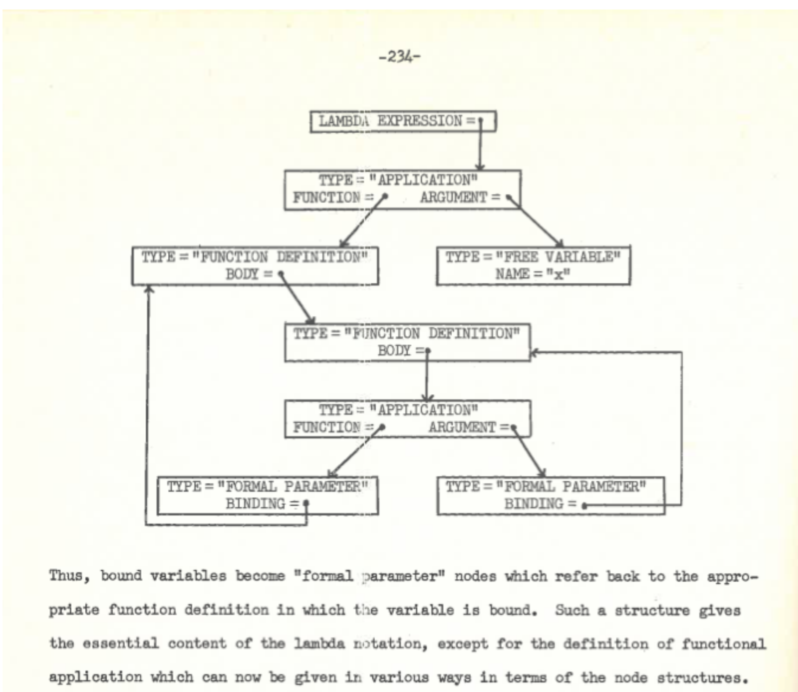
$\lambda x.\lambda y.\lambda z.x(yz)$ ✓ $\lambda x.\lambda y.\lambda z.(xz)y$ ✗

a term is ***unit-free*** if it has no closed subterms

$x \vdash \lambda y.yx$ ✓ $x \vdash x(\lambda y.y)$ ✗

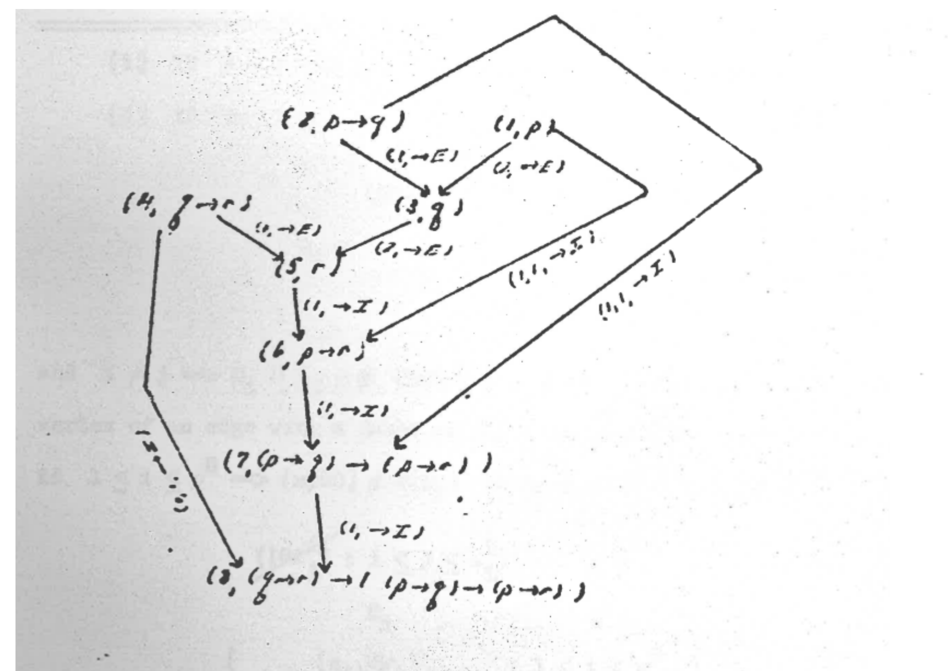
A graphical representation of lambda terms

1970



D. E. Knuth, "Examples of formal semantics",
in Symposium on Semantics of Algorithmic Languages.

1974



A derivation in the System I (Section 3)

R. Statman, Structural Complexity of Proofs,
PhD Thesis, Stanford University

1987

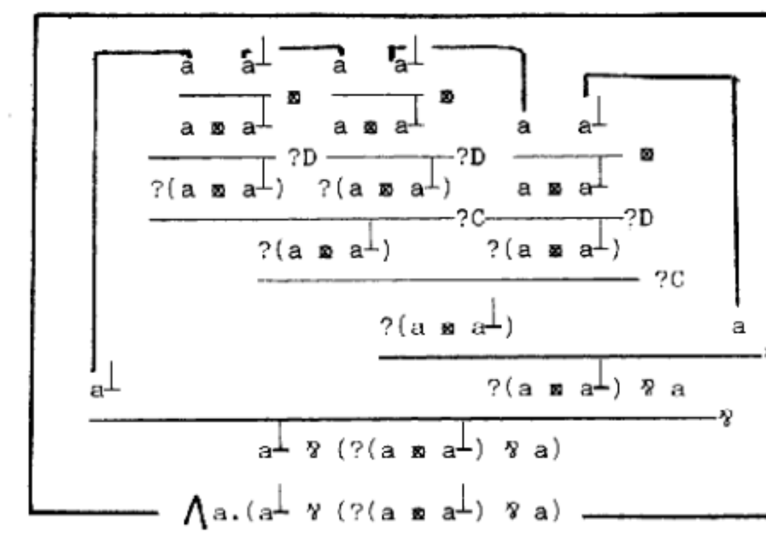
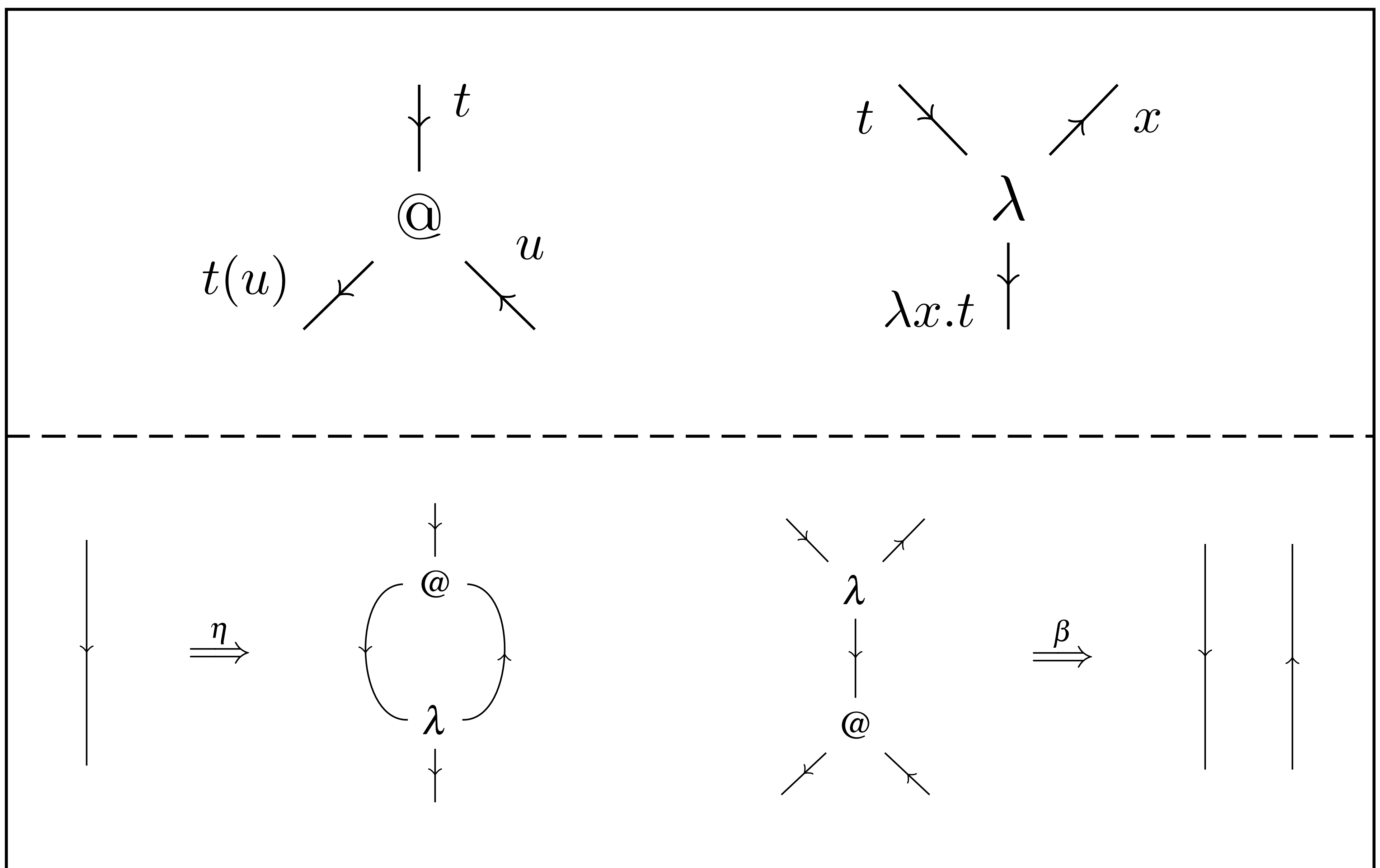


Fig. 57(b). The number "3" represented in PN2.

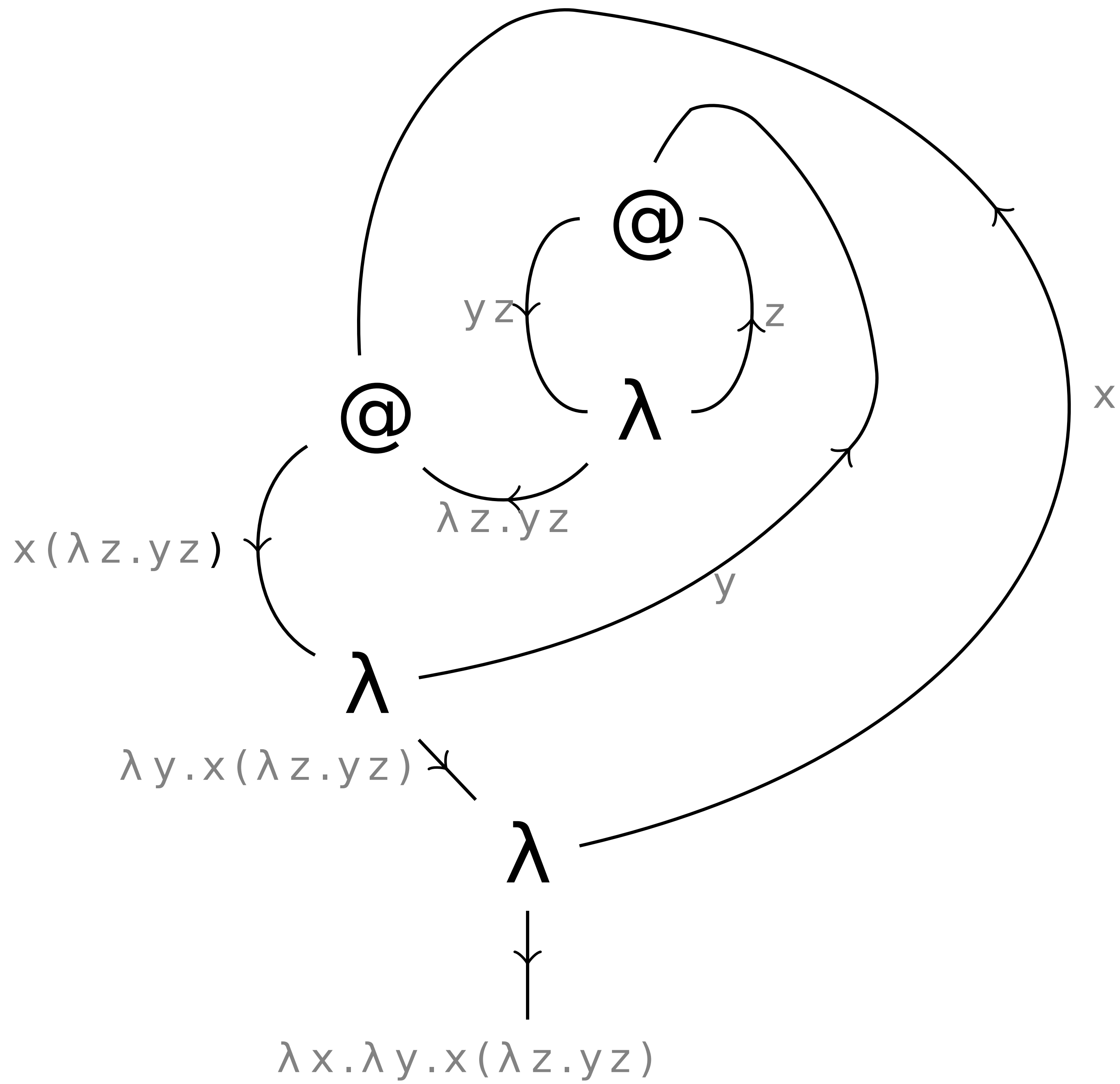
J.-Y. Girard, Linear Logic, TCS

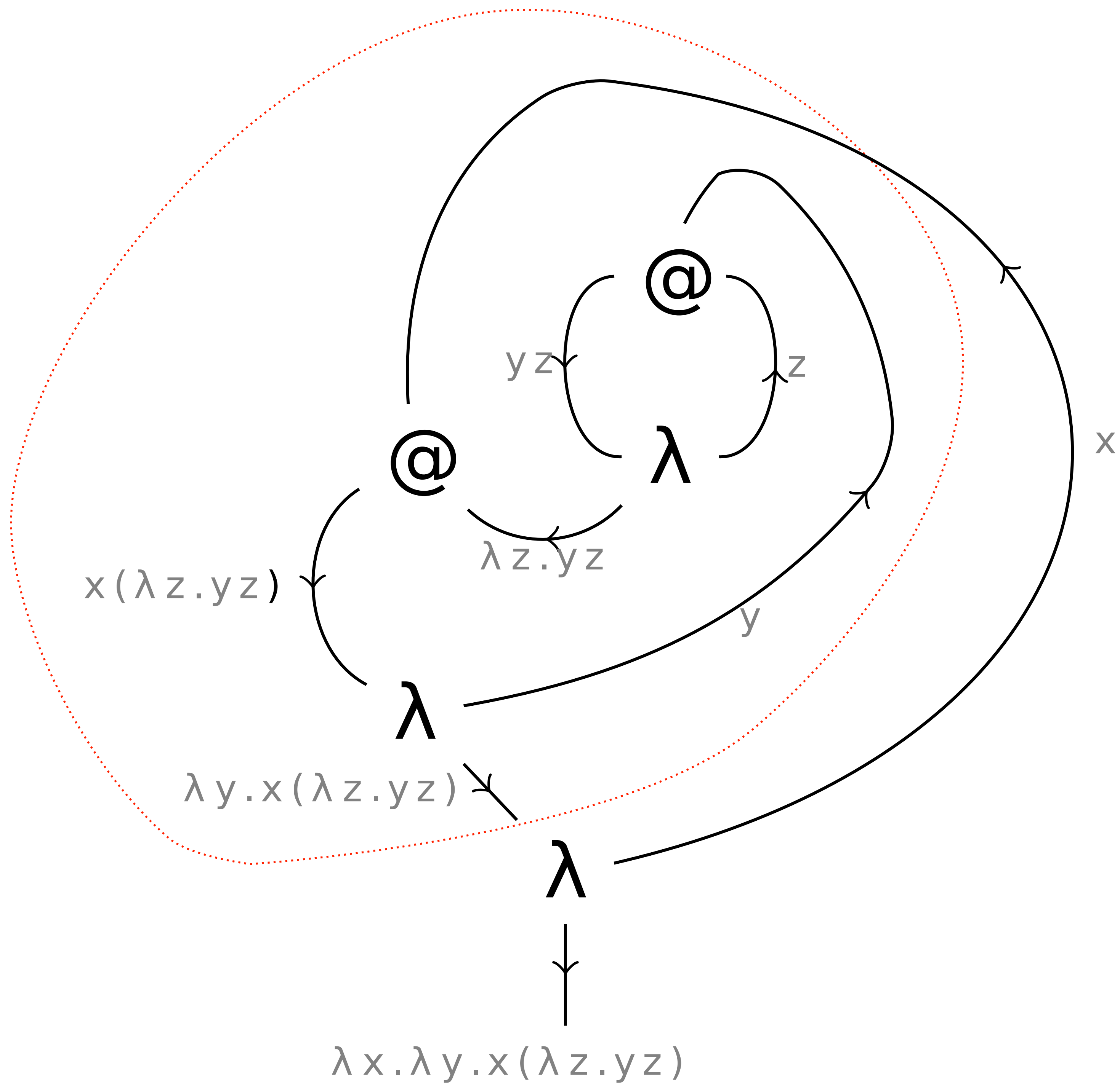
String diagrams

Formally, linear lambda terms may be interpreted as endomorphisms of a **reflexive object** in a *symmetric monoidal closed bicategory*

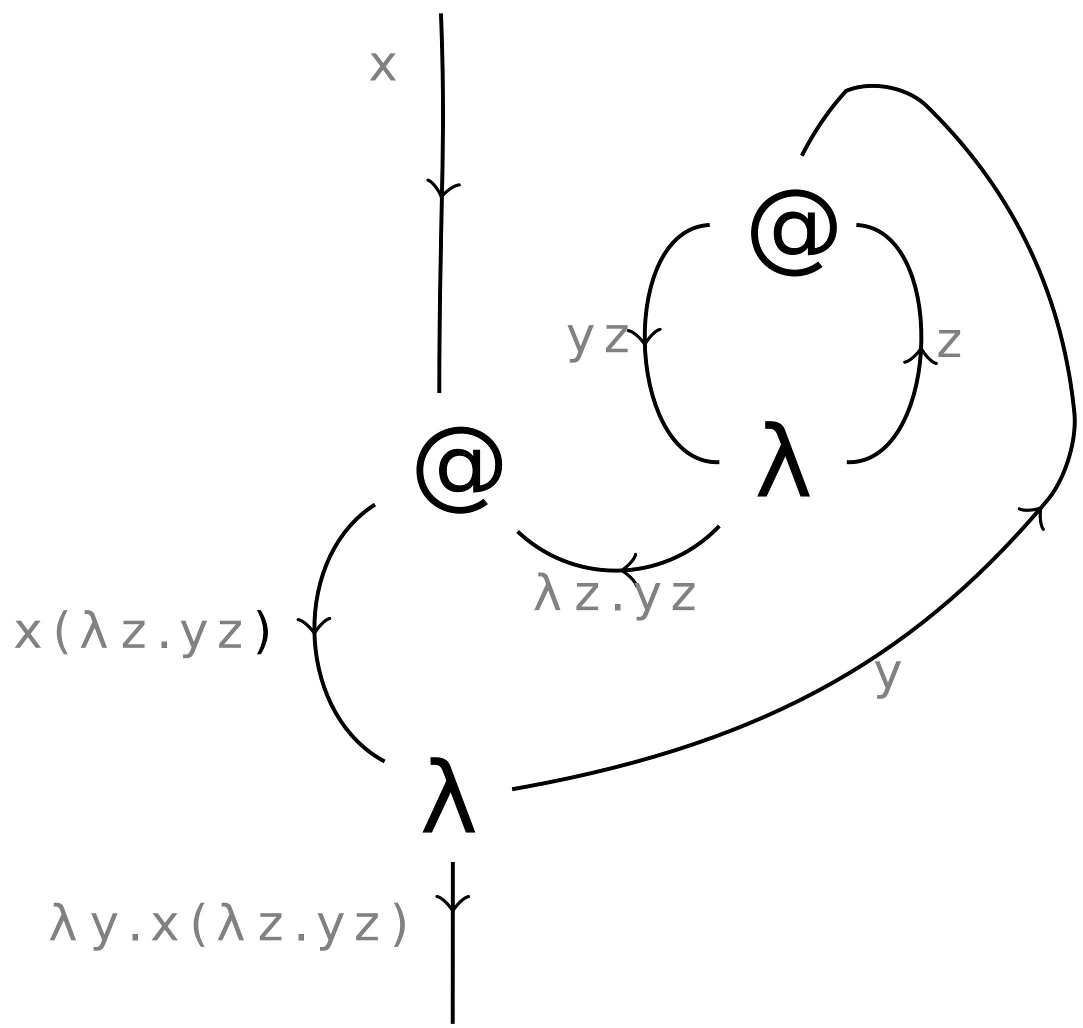


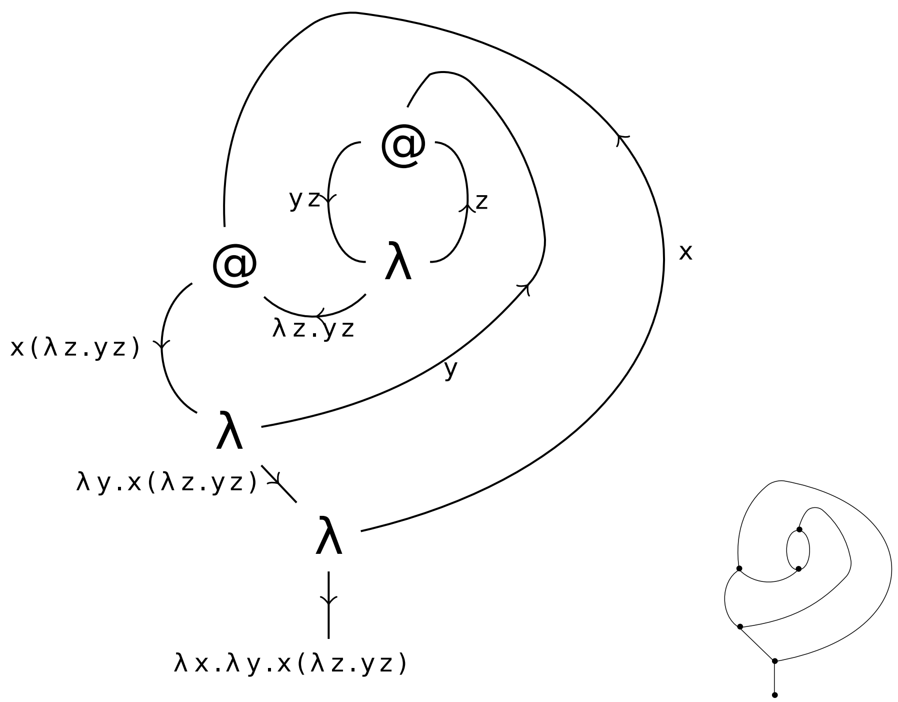
$$U \underset{\lambda}{\overset{@}{\rightleftarrows}} U \multimap U \qquad (@ \dashv \lambda)$$

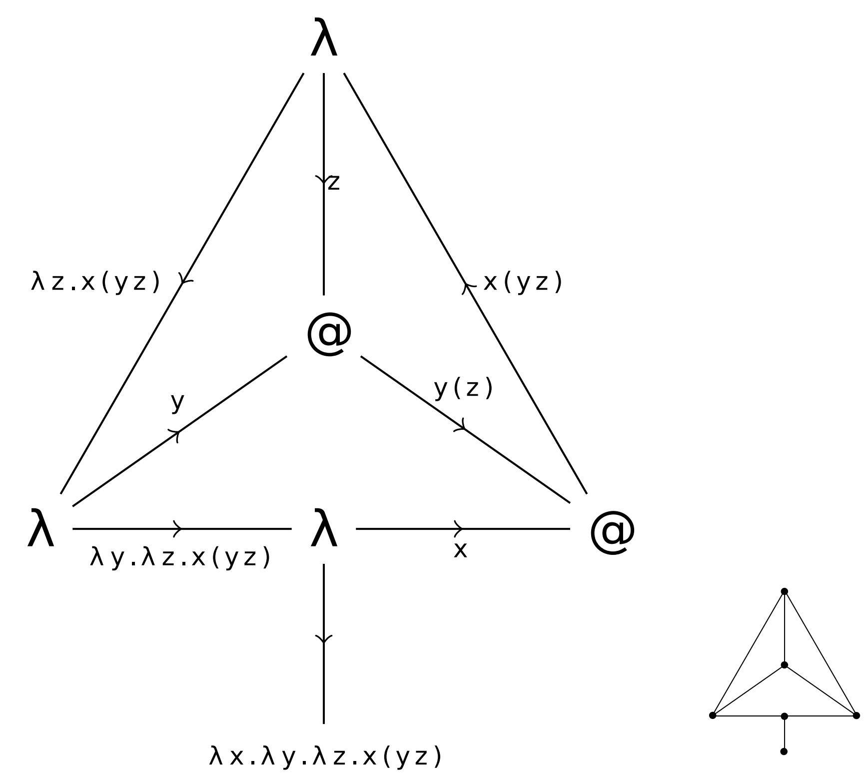

@
@
λ
λ
λ
yz
z
x
λz.yz
x(λz.yz)
y
λy.x(λz.yz)
λx.λy.x(λz.yz)

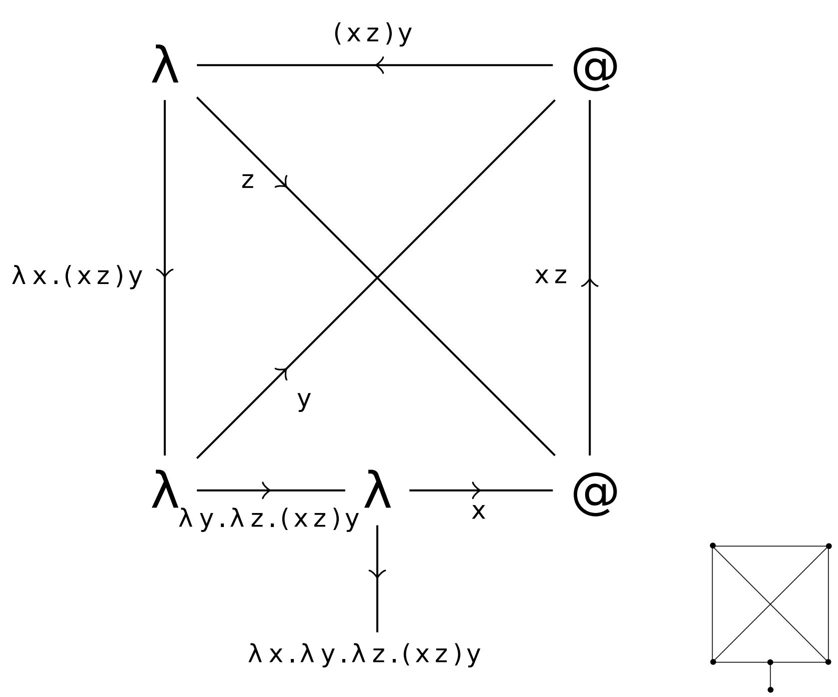


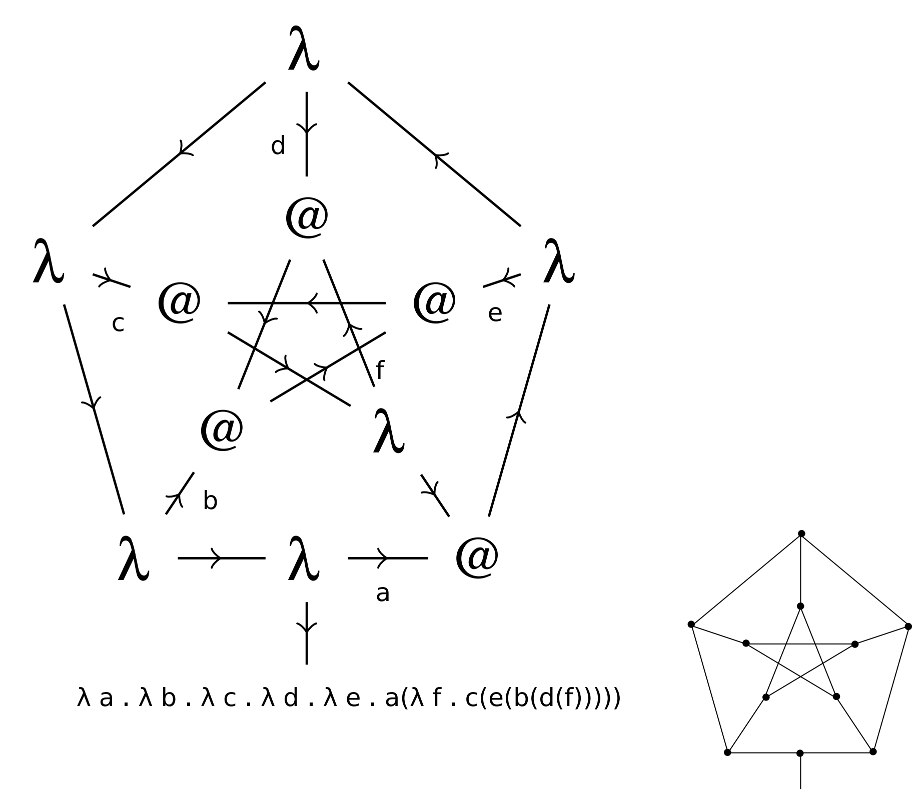
@
@
λ
λ
λ
yz
z
x
λz.yz
x(λz.yz)
y
λy.x(λz.yz)
λx.λy.x(λz.yz)


x
@
yz
z
λ
@
λz.yz
x(λz.yz)
y
λ
λy.x(λz.yz)


x
yz
z
λz.yz
x(λz.yz)
y
λy.x(λz.yz)
λx.λy.x(λz.yz)

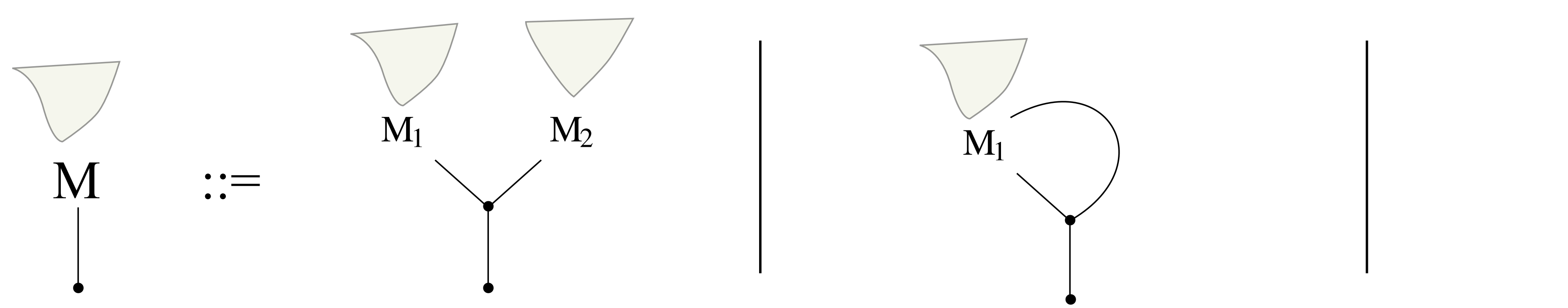

z
λz.x(yz)
x(yz)
y
y(z)
λy.λz.x(yz)
x
λx.λy.λz.x(yz)


(xz)y
z
λx.(xz)y
xz
y
λy.λz.(xz)y
x
λx.λy.λz.(xz)y


d
c
e
f
b
a
λa.λb.λc.λd.λe.a(λf.c(e(b(d(f)))))

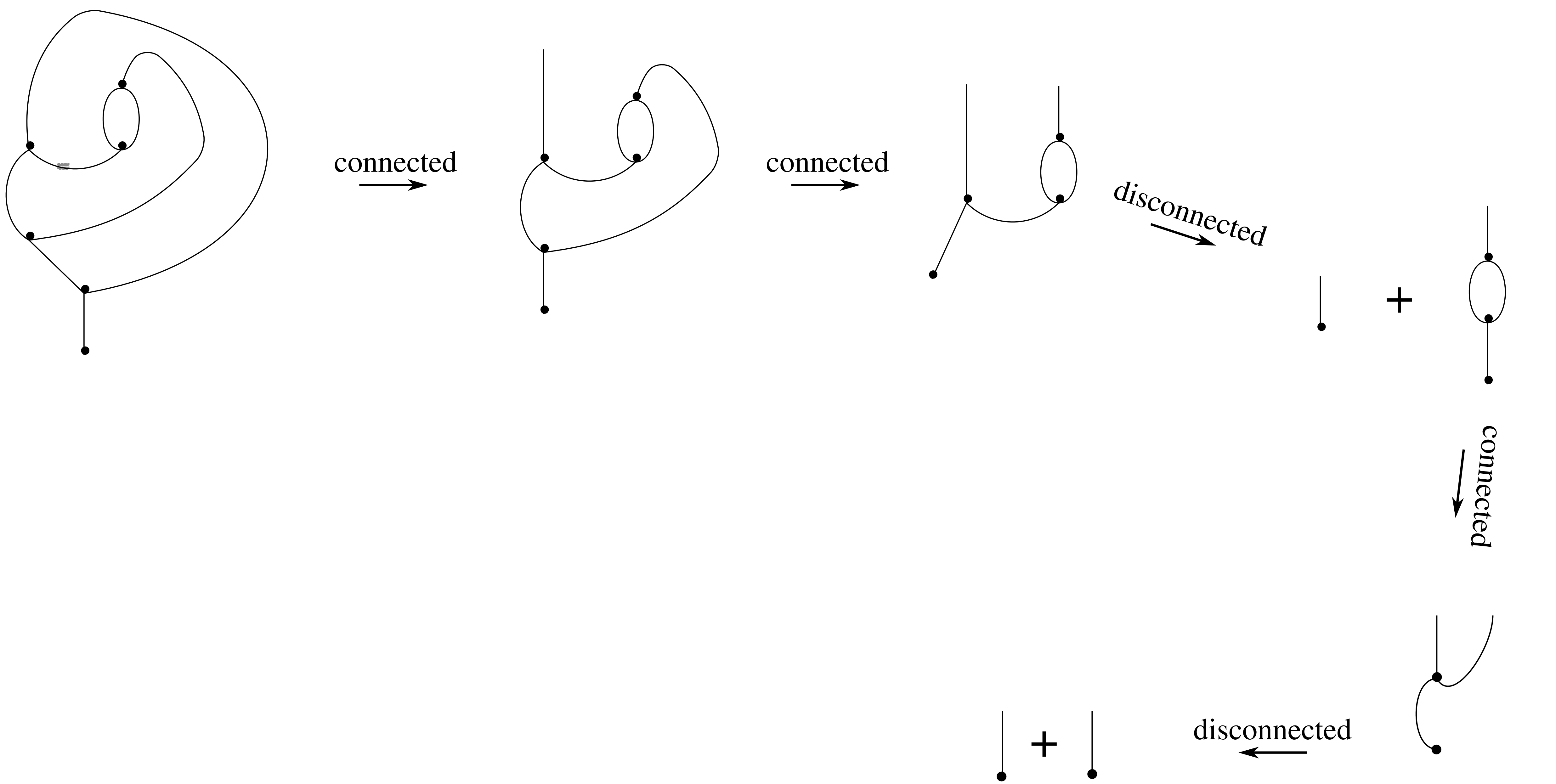
Linear λ-terms as invariants of 3-valent maps

Consider an arbitrary 3-valent map M, possibly with a boundary of "dangling" edges. What are its potential shapes?

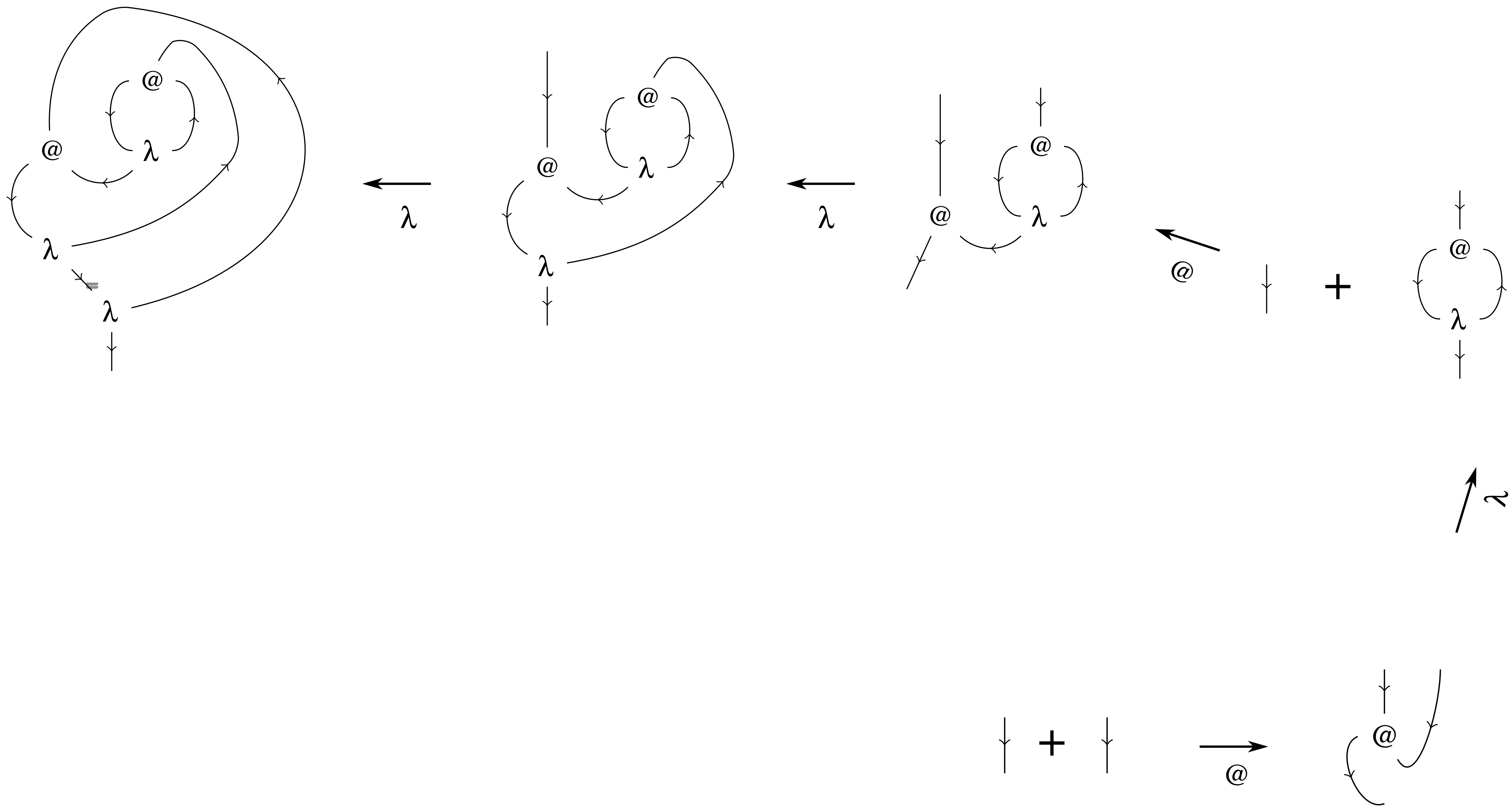


This exactly mirrors the inductive structure of linear lambda terms!

Linear λ-terms as invariants of 3-valent maps

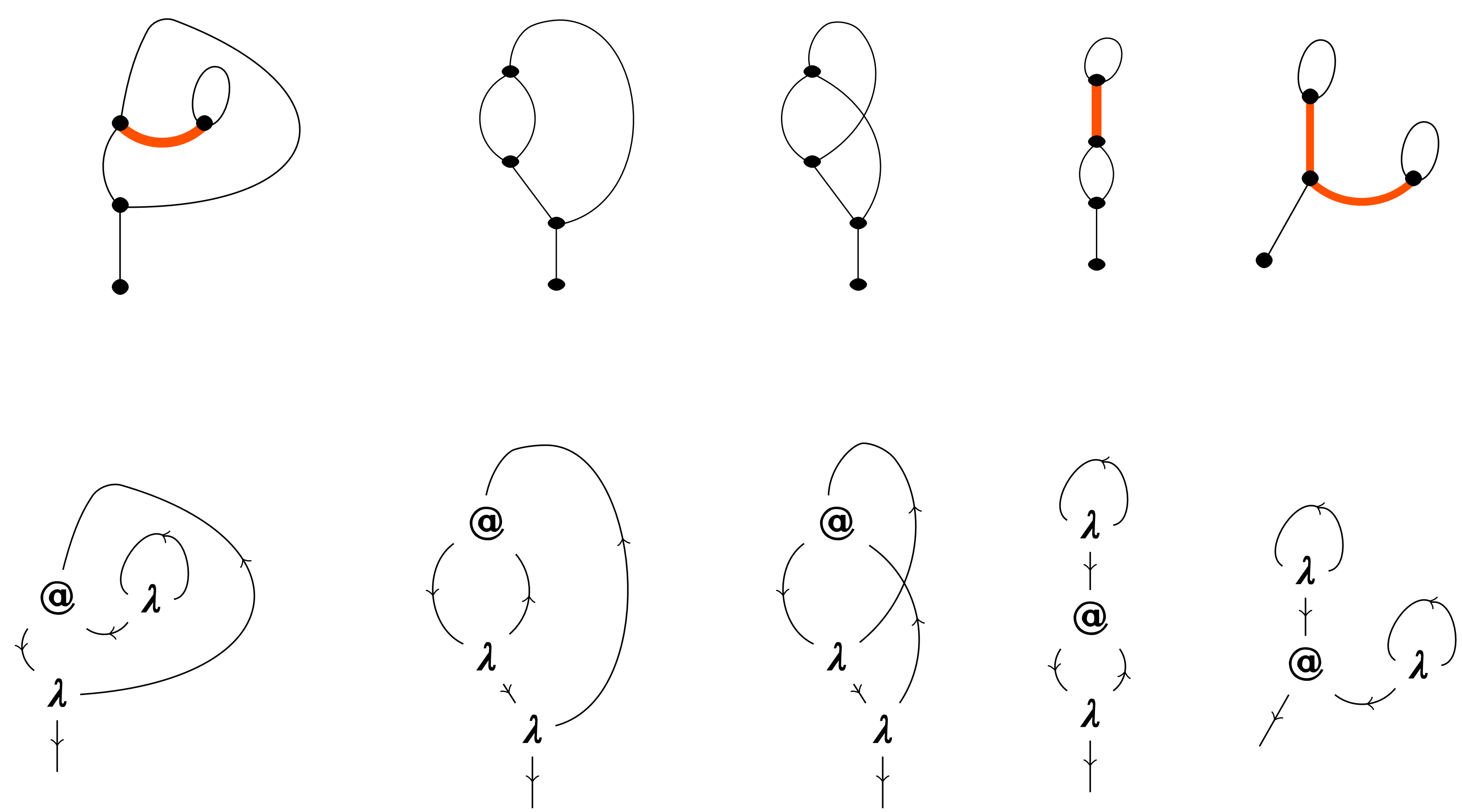


Linear λ-terms as invariants of 3-valent maps

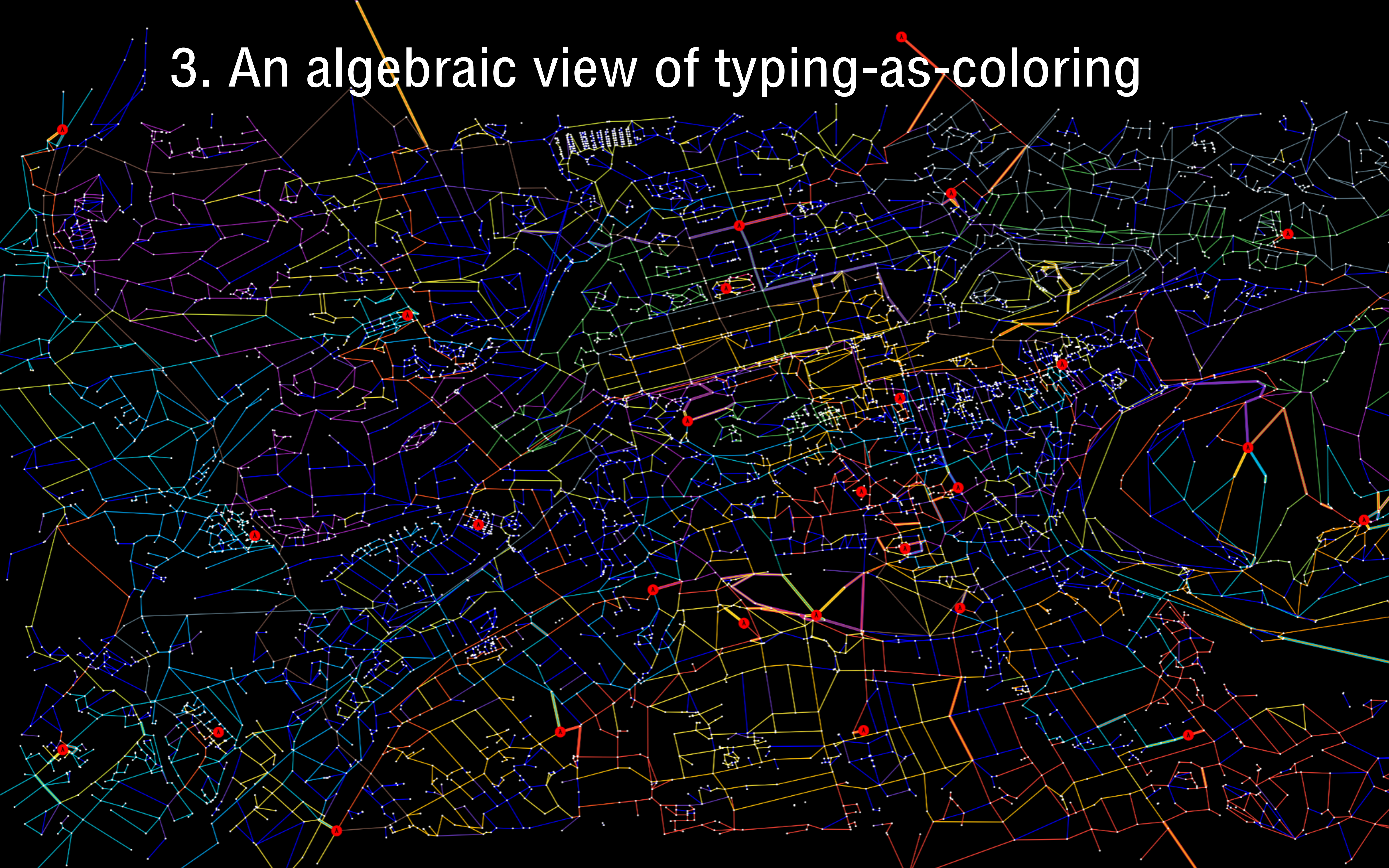


Linear λ-terms as invariants of 3-valent maps

∴ planar iff no var exchange, bridgeless iff no closed subterms

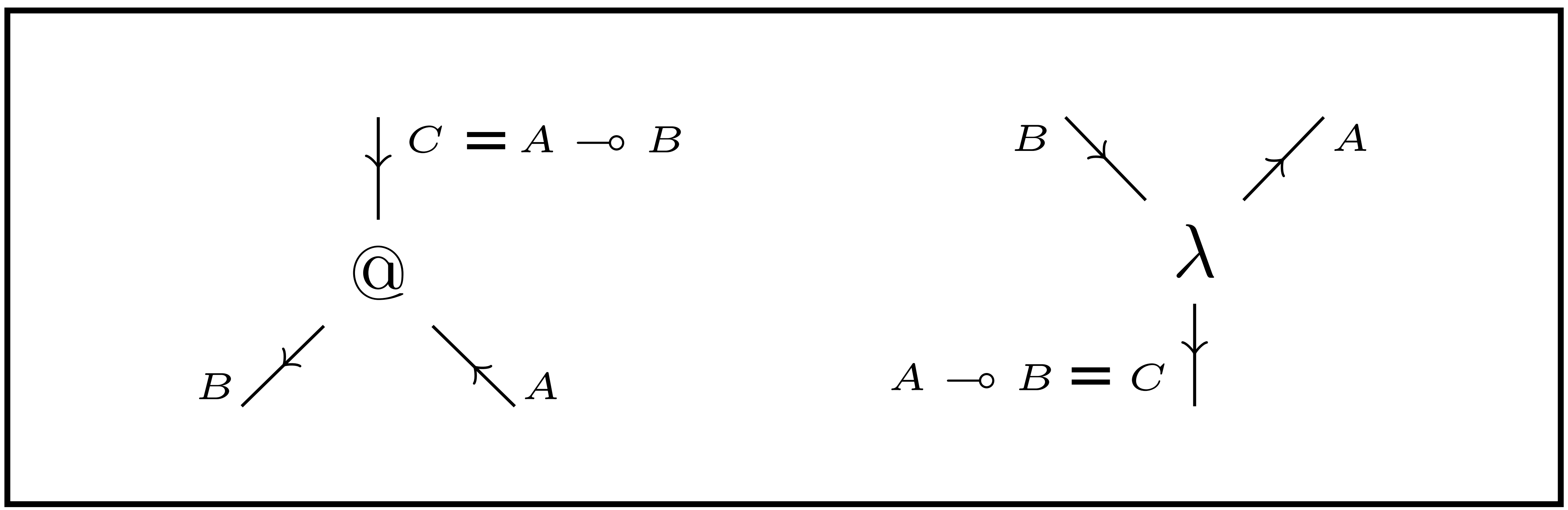


3. An algebraic view of typing-as-coloring



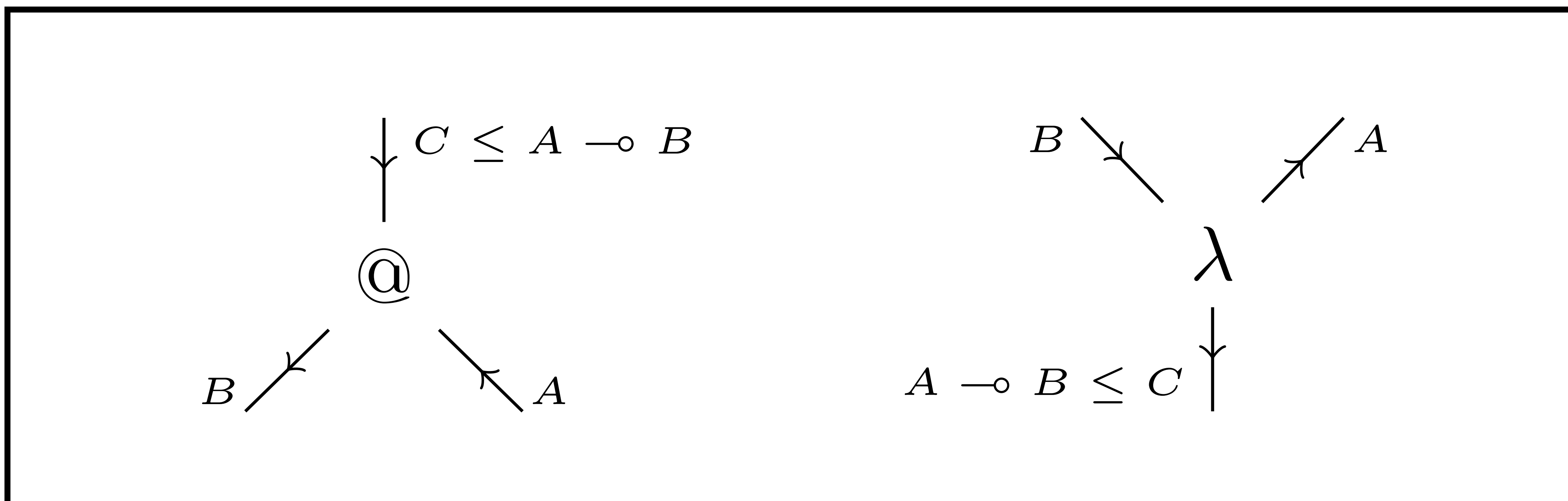
Definition (provisional)

a **typing** is an assignment of "types" to subterms satisfying the following equations:



Definition (better)

a **typing** is an assignment of "types" to subterms satisfying the following *inequations*:



Linear type algebras

An **imploid** is a preorder (P,≤) equipped with a binary operation

$$\frac{a_2 \leq a_1 \quad b_1 \leq b_2}{a_1 \multimap b_1 \leq a_2 \multimap b_2}$$

satisfying a *composition law*

$$b \multimap c \leq (a \multimap b) \multimap (a \multimap c)$$

It is **unital** if it moreover has an element I satisfying

$$I \leq a \multimap a$$

$$I \multimap a \leq a$$

It is **commutative** if it moreover satisfies

$$a \leq (a \multimap b) \multimap b$$

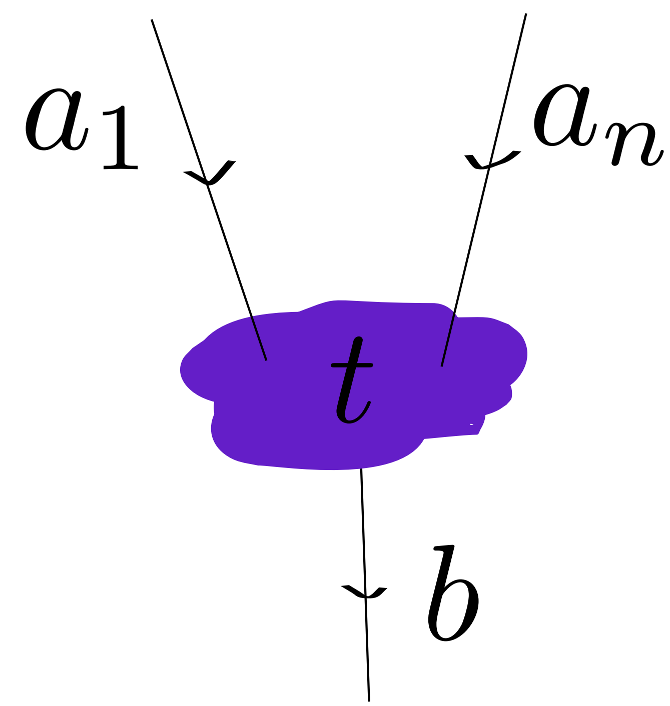
For example: every abelian group yields a commutative, unital imploid

$$a \multimap b \stackrel{\text{def}}{=} b - a$$

Linear type algebras

Proposition.

Let t be a linear term with n free variables, and suppose



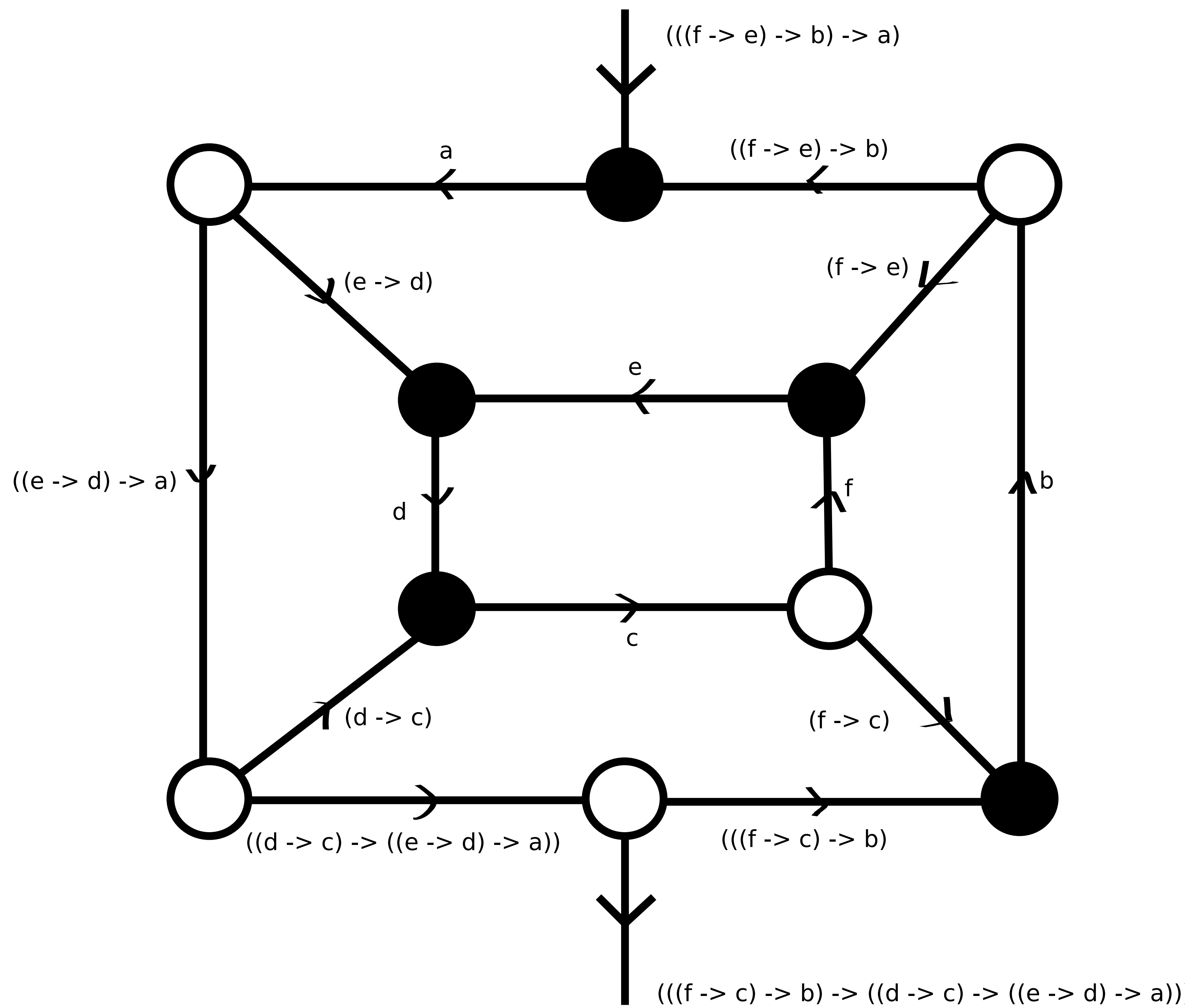
is valid typing of *t* in a commutative, unital imploid.

Then $I \leq a_1 \multimap \cdots \multimap a_n \multimap b$.

(Note: necessary but not sufficient!)

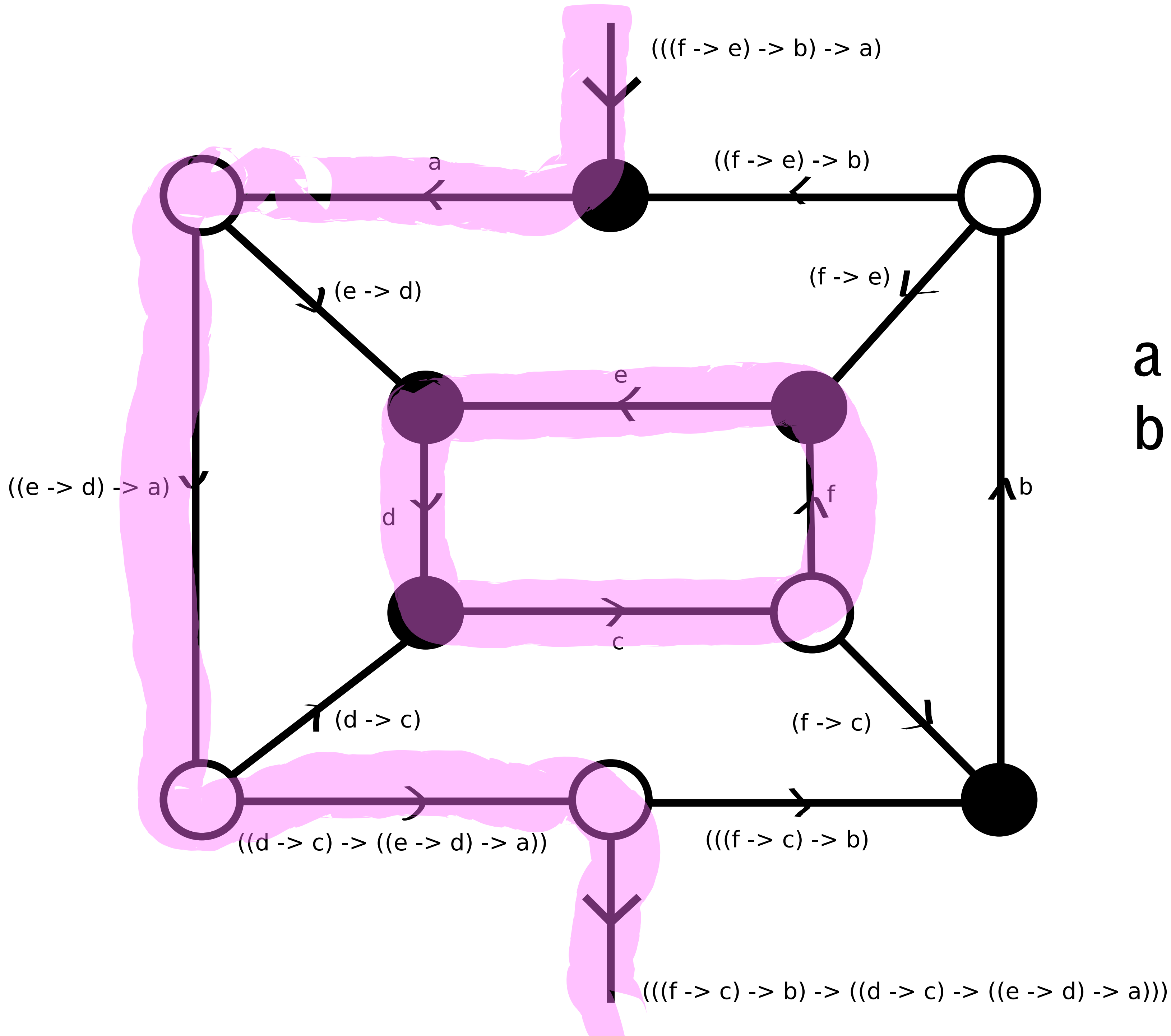
An abstract typing

$x \vdash \lambda y.\lambda z.\lambda w.x(\lambda u.y(\lambda v.z(w(uv))))$



A Z2-typing

x ⊢ λy.λz.λw.x(λu.y(λv.z(w(uv))))

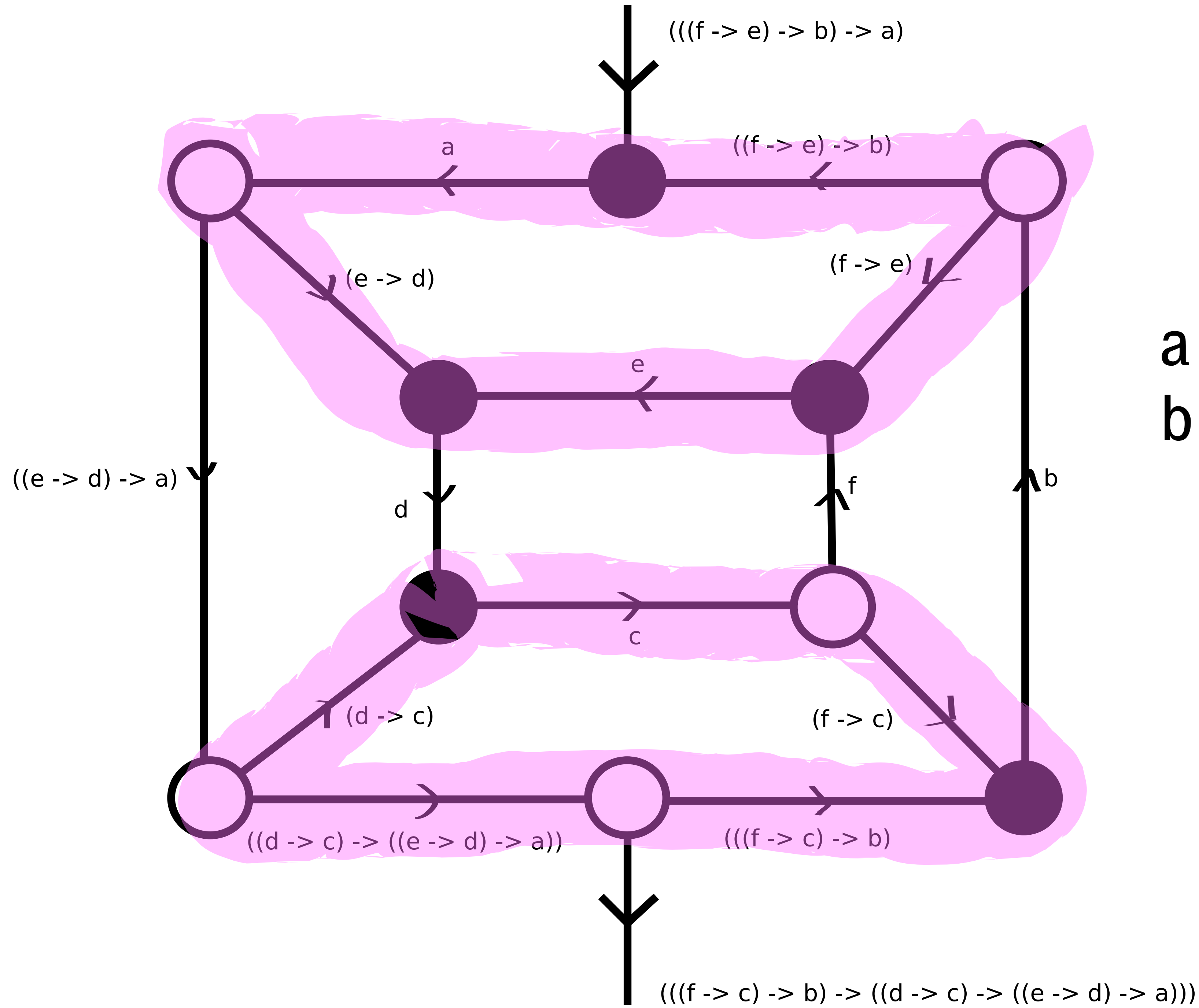


a = c = d = e = f = 1

b = 0

A Z2-typing

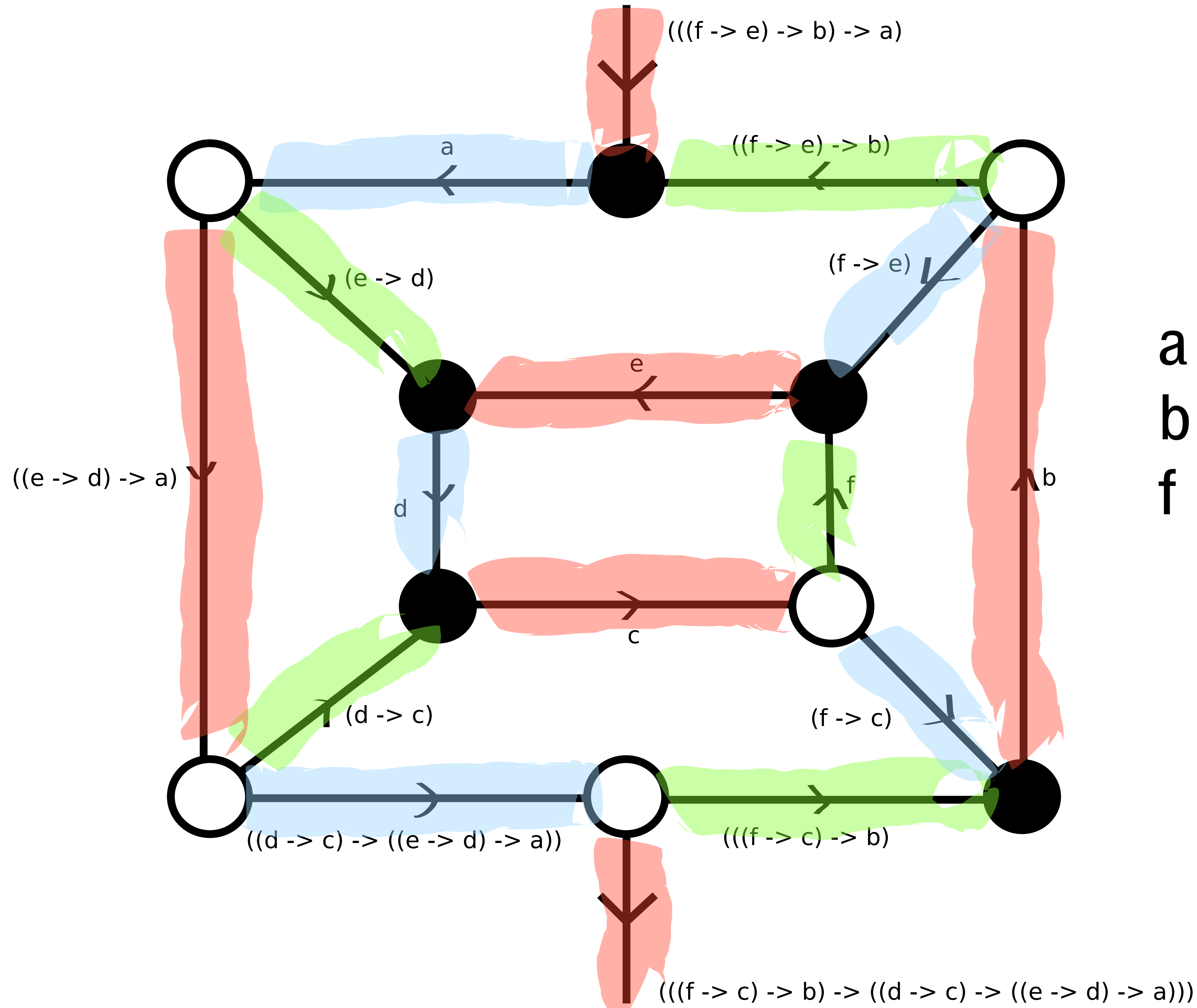
$x \vdash \lambda y.\lambda z.\lambda w.x(\lambda u.y(\lambda v.z(w(uv))))$



a = c = e = 1
b = d = f = 0

A (Z2 × Z2)-typing

x ⊢ λy.λz.λw.x(λu.y(λv.z(w(uv))))



a = d = (1,0)
b = c = e = (0,1)
f = (1,1)

Some typing exercises

We say that a typing is *proper* if no subterm is assigned a type ≥ *I*.

1. Prove that every unit-free linear term has a proper $(\mathbb{Z}_2)^n$-typing, for some n.

2. Prove that every unit-free planar lambda term has a proper $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-typing.