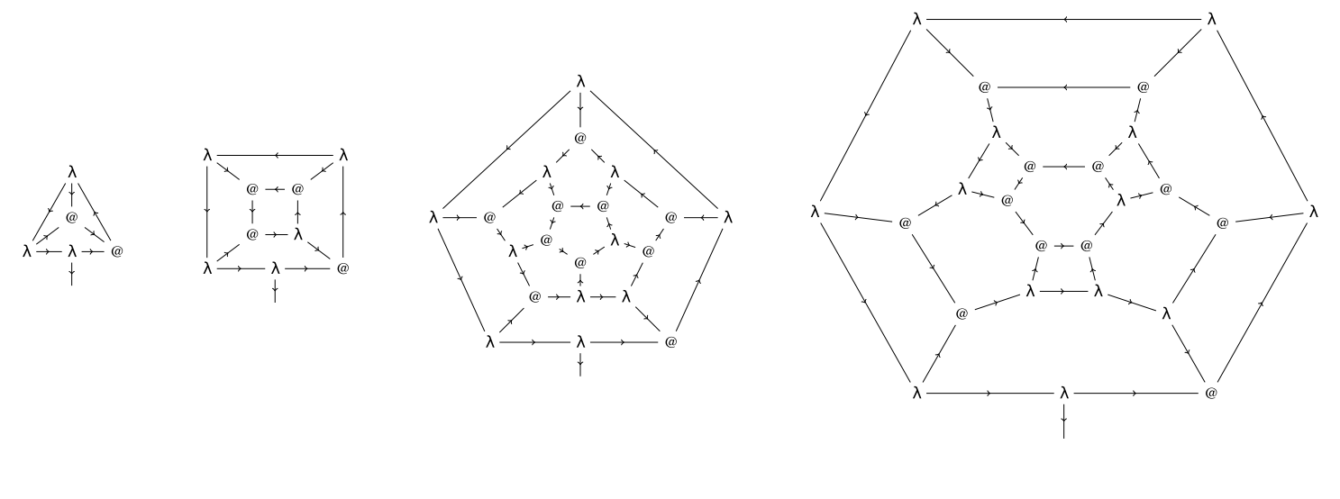
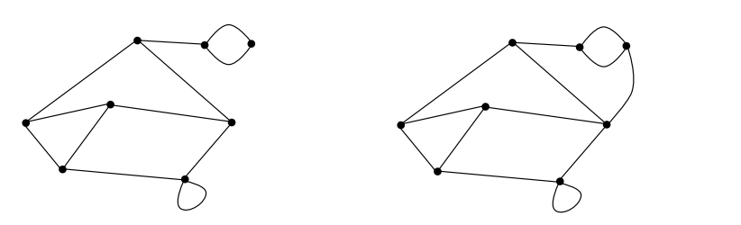
# Some enumerative, topological, and algebraic aspects of linear lambda calculus



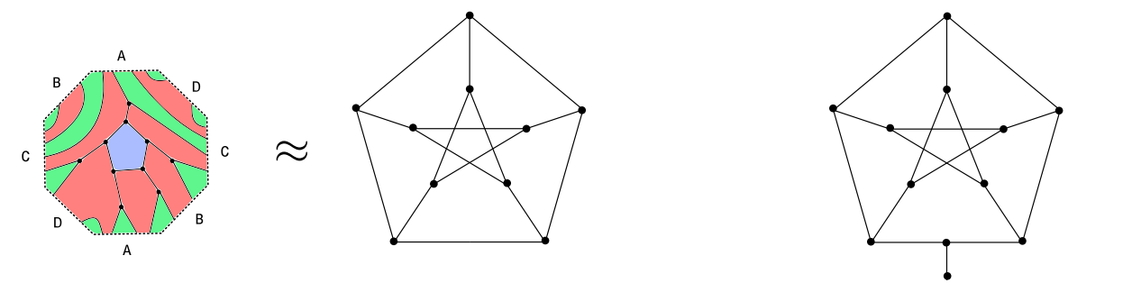
## Graphs on surfaces ("maps")



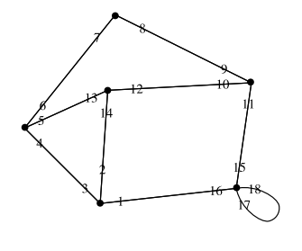
planar map    bridgeless (planar) map

Euler-Poincaré: #V - #E + #F = 2 - 2g

10 - 15 + 3 = -2



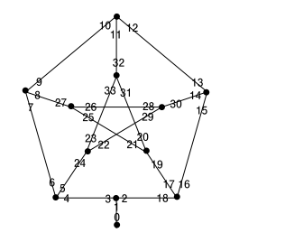
trivalent map (of genus 2)    rooted trivalent map



M = {1,…,18}
v = (1 2 3)(4 5 6)(7 8)(9 10 11)(12 13 14)(15 16 17 18)
e = (1 16)(2 14)(3 4)(5 13)(6 7)(8 9)(10 12)(11 15)(17 18)
f = (1 15 10 14)(2 13 4)(3 6 8 11 18 16)(5 12 9 7)(17)

combinatorial map

v = (0)(1 2 3)...(31 32 33)
e = (0 1)(2 18)(3 4)...



a trivalent (combinatorial) map is a permutation representation of

$$\mathcal{T} = \langle v, e \mid v^3 = e^2 = 1 \rangle \cong PSL(2,\mathbb{Z})$$

## Linear lambda calculus

$$\frac{\Gamma \vdash t \quad \Delta \vdash u}{\Gamma, \Delta \vdash t(u)} \qquad \frac{\Gamma, x \vdash t}{\Gamma \vdash \lambda x.t}$$

$$\frac{}{x \vdash x} \qquad \frac{\Theta \vdash u \quad \Gamma, x, \Delta \vdash t}{\Gamma, \Theta, \Delta \vdash t[u/x]} \qquad \boxed{\frac{\Gamma, y, x, \Delta \vdash t}{\Gamma, x, y, \Delta \vdash t}}$$

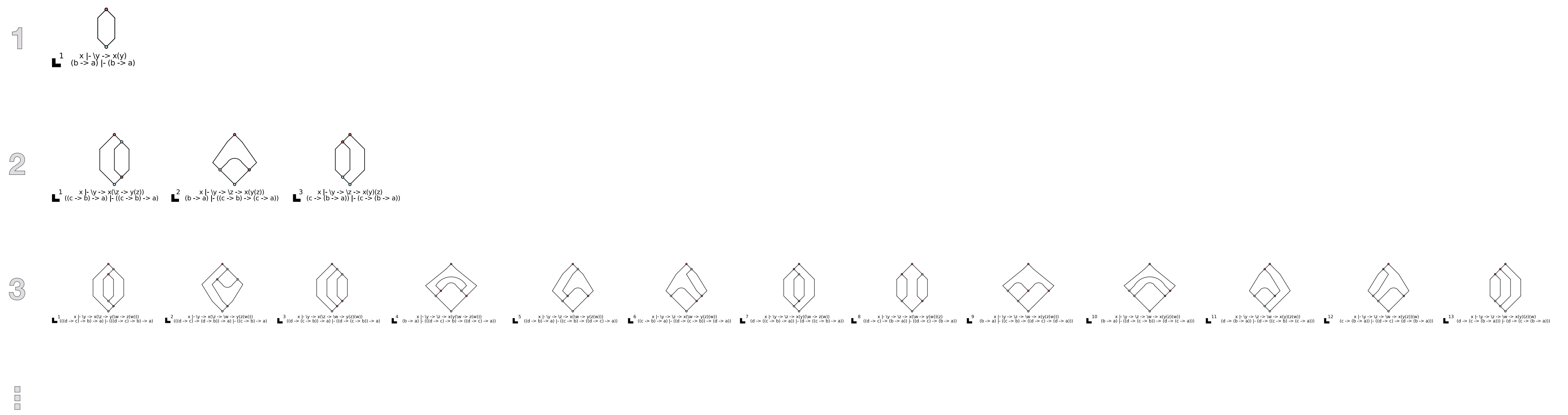
a (planar) operad of linear (planar) lambda terms

linearity constraint: # freevars - # applications + # abstractions = 1

| | | | |
|---|---|---|---|
| linear: | λx.λy.x(y) | non-linear: | λx.λy.x, λx.x(x) |
| planar: | λx.λy.λz.x(y(z))<br>λx.λy.λz.z(y(x)) | non-planar: | λx.λy.λz.x(z(y)) |
| normal: | λx.x(λy.y) | non-normal: | (λx.x)(λy.y) |
| decomposable: | λx.x(λy.y) | indecomposable: | λx.λy.x(y) |

## Enumerative connections

| family of (rooted) maps | family of lambda terms | sequence | OEIS entry |
|---|---|---|---|
| trivalent maps | linear terms[1,4] | 1,5,60,1105,27120,... | A062980 |
| planar trivalent maps | planar terms[4] | 1,4,32,336,4096,... | A002005 |
| bridgeless trivalent maps | indecomposable linear terms[4] | 1,2,20,352,8624,... | A267827 |
| bridgeless planar trivalent maps | indecomposable planar terms[4] | 1,1,4,24,176,1456,... | A000309 |
| maps (on oriented surfaces) | normal linear terms (mod ~)[3] | 1,2,10,74,706,8162,... | A000698 |
| planar maps | normal planar terms[2] | 1,2,9,54,378,2916,... | A000168 |
| bridgeless maps | normal indecomp. linear terms (mod ~)[5] | 1,1,4,27,248,2830,... | A000699 |
| bridgeless planar maps | normal indecomp. planar terms[6] | 1,1,3,13,68,399,... | A000260 |

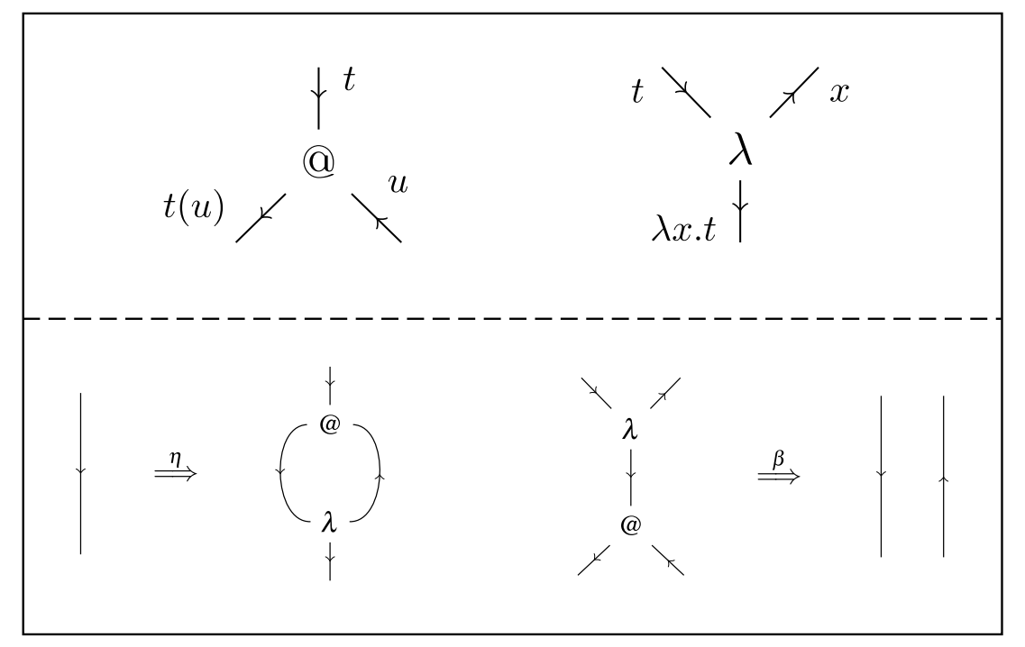


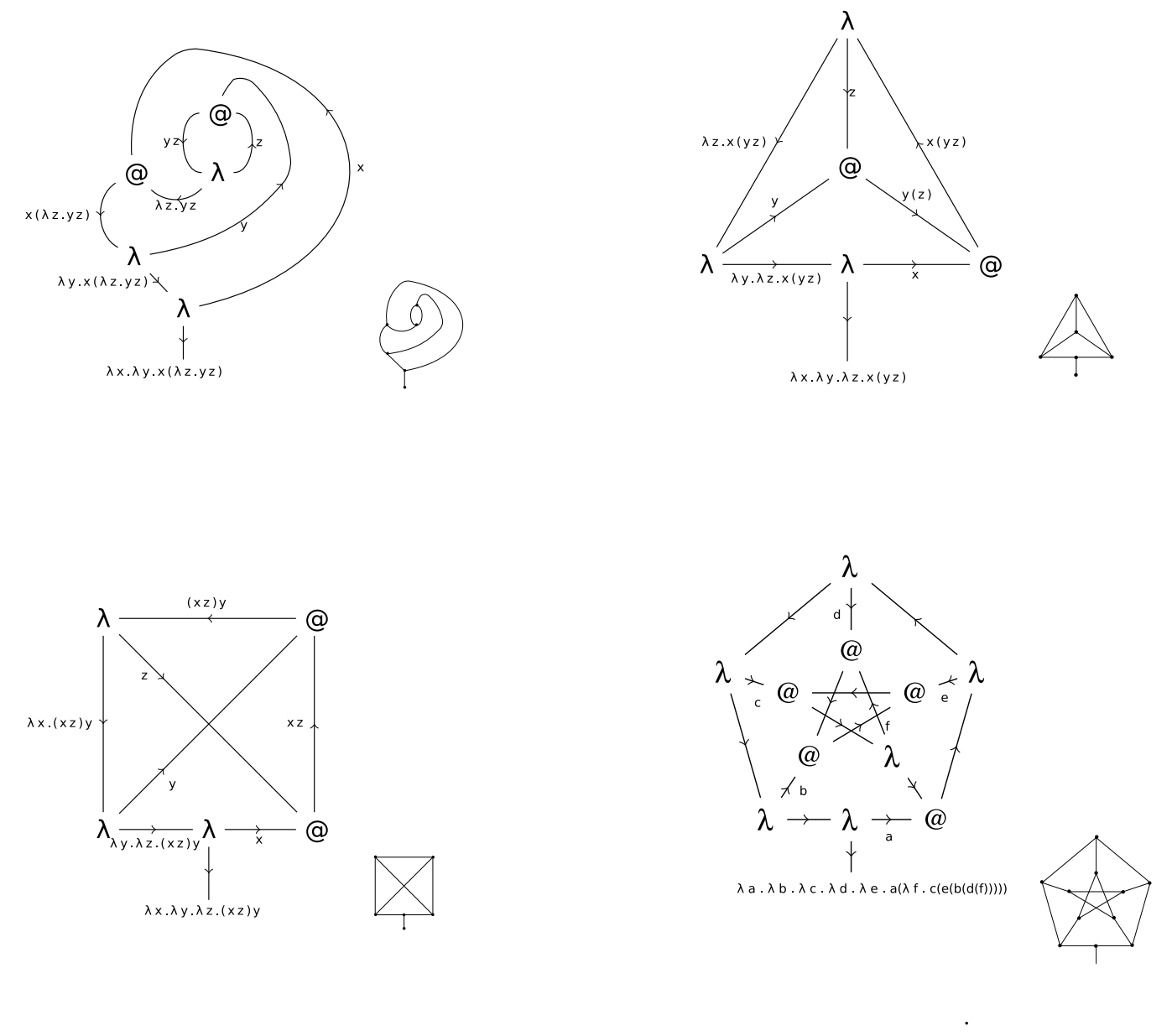
references

1. Bodini, Gardy, Jacquot (2013), Asymptotics and random sampling for BCI and BCK lambda terms, TCS
2. Z, Giorgetti (2015), A correspondence between rooted planar maps and normal planar lambda terms, LMCS
3. Z (2015), Counting isomorphism classes of beta-normal linear lambda terms, arXiv:1509.07596
4. Z (2016), Linear lambda terms as invariants of rooted trivalent maps, JFP
5. Courtiel, Yeats, Z (2016), Connected chord diagrams and bridgeless maps, arXiv:1611.04611
6. Z (2017), A sequent calculus for the Tamari order, arXiv:1701.02917

## String diagrams for linear lambda terms

(i.e., for a reflexive object in a smc bicategory)

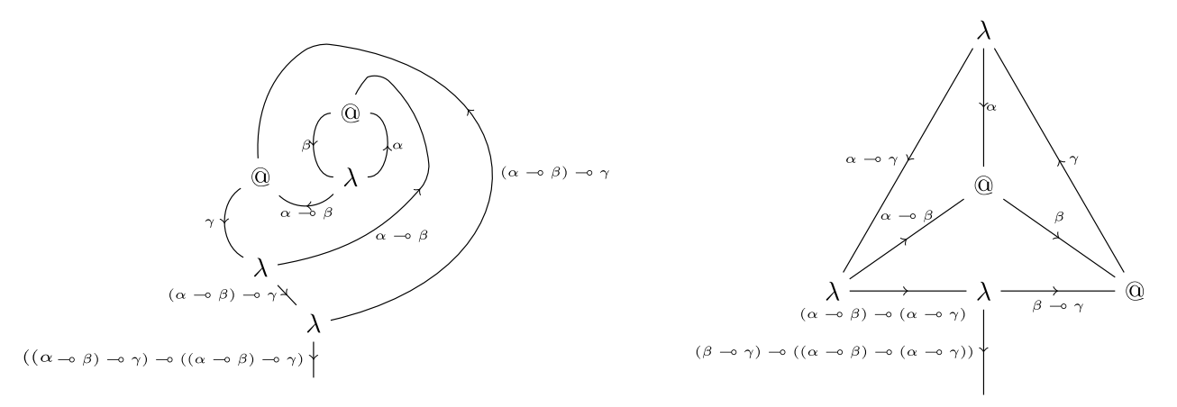




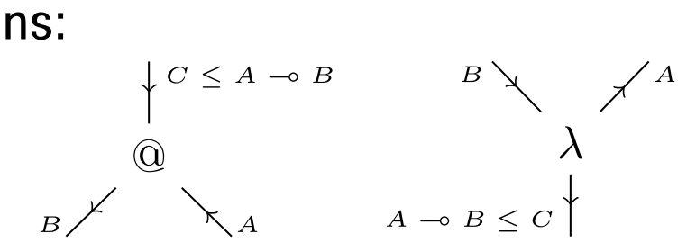
## Typing linear lambda terms

$$\frac{\Gamma \vdash t : A \multimap B \quad \Delta \vdash u : A}{\Gamma, \Delta \vdash t(u) : B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}$$

$$\frac{}{x : A \vdash x : A} \qquad \frac{\Theta \vdash u : A \quad \Gamma, x : A, \Delta \vdash t : B}{\Gamma, \Theta, \Delta \vdash t[u/x] : B} \qquad \frac{\Gamma, y : B, x : A, \Delta \vdash t : C}{\Gamma, x : A, y : B, \Delta \vdash t : C}$$



abstract typing conditions:



where A,B,C range over elements of an *imploid*...

An **imploid** is a preordered set equipped with an "implication" operation

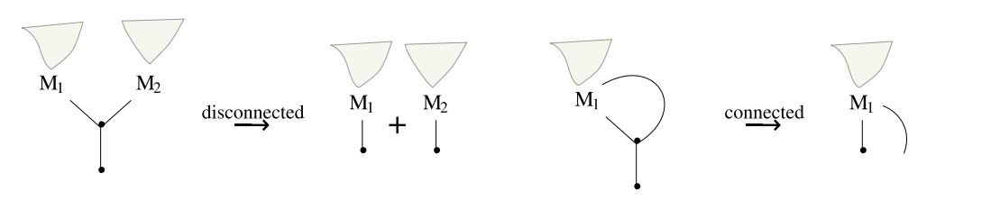
$$\frac{a_2 \le a_1 \quad b_1 \le b_2}{a_1 \multimap b_1 \le a_2 \multimap b_2}$$

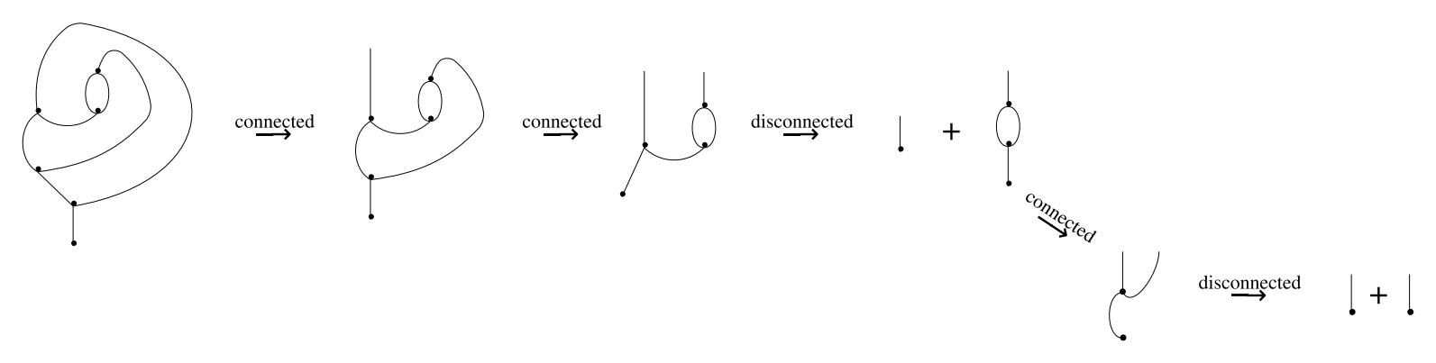
satisfying a ***composition law***: $b \multimap c \le (a \multimap b) \multimap (a \multimap c)$

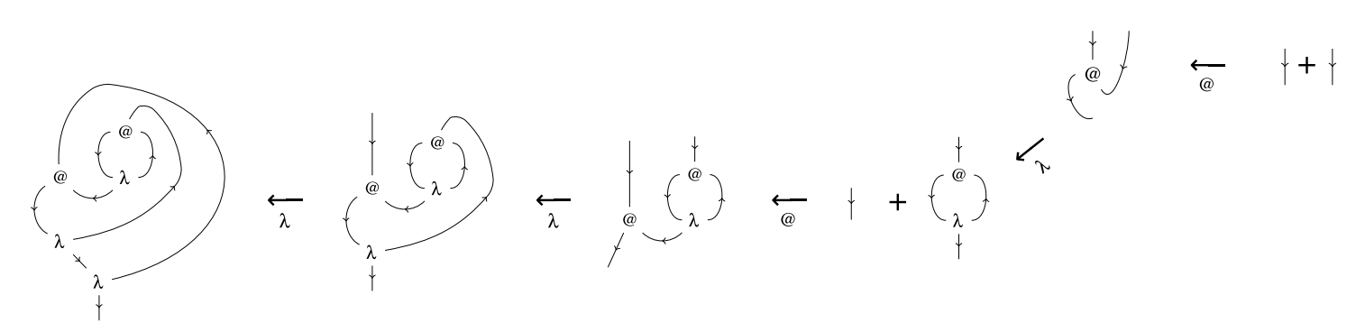
A **unital** imploid moreover has an element $I$
satisfying an ***identity law*** $I \le a \multimap a$
and a ***unit law***: $I \multimap a \le a$

(cf. Ross Street (2013), "Skew-closed categories")

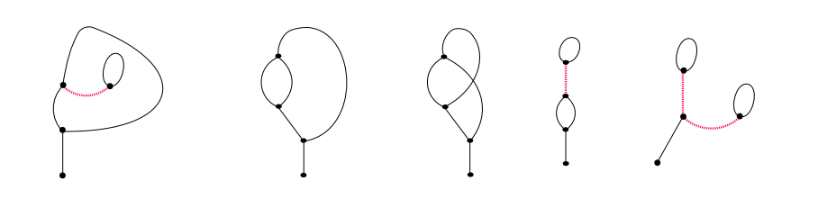
## Linear lambda terms as invariants of rooted trivalent maps

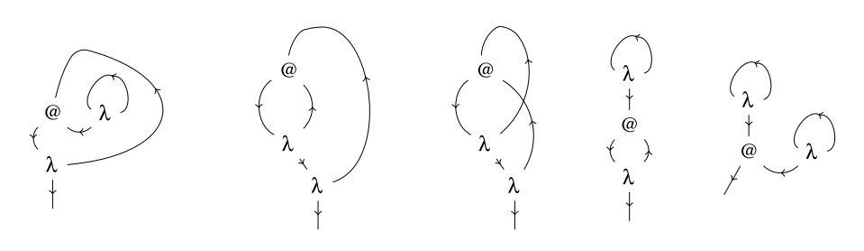






corollary: bridges = closed subterms

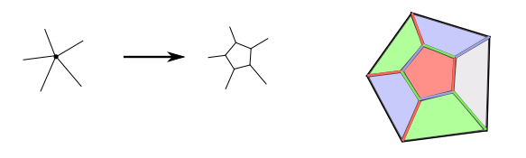




## A reformulation of the 4CT

**Tait (1880)**

every planar map has a proper face 4-coloring iff
every trivalent planar map has a proper edge 3-coloring



any group induces an imploid by: $a \multimap b \overset{\text{def}}{=} b \cdot a^{-1}$

consider the Klein Four Group...

$$\begin{array}{c|ccc} & R & G & B \\ \hline R & I & B & G \\ G & B & I & R \\ B & G & R & I \end{array}$$

Theorem: *every planar term has a proper Klein-typing...*

(proper = only closed subterms have type I)