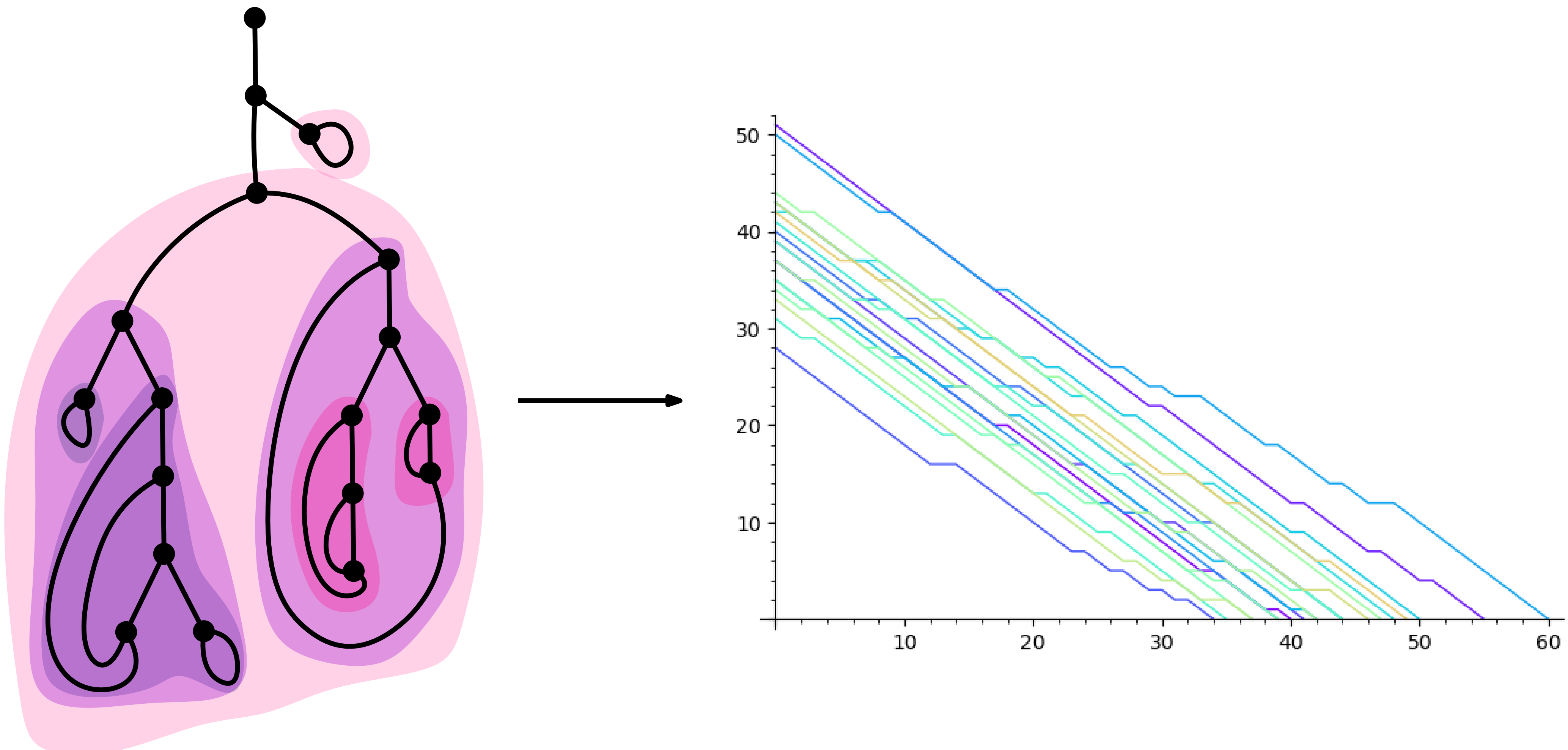
A lower bound on reduction legnth for random closed linear λ-terms



LambdaComb KIckoff Meeting, 11 April 2022

Olivier Bodini (LIPN, Paris 13)

Michael Wallner (TU Wien)

Bernhard Gittenberger (TU WIen)

Noam Zeilberger (LIX, Polytechnique)

Alexandros Singh (LIPN, Paris 13)

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- We're interested in terms up to α-equivalence:

$$(\lambda x.xx)(\lambda x.xx) \overset{\alpha}{=} (\lambda y.yy)(\lambda x.xx) \overset{\alpha}{\neq} (\lambda y.ya)(\lambda x.xx)$$

Computing with the λ-calculus

- Substitution rule:

$$T_1[v := T_2]$$

"replace free occurences of v in T_1 with T_2"

(α-converting T if necessary, to avoid capturing variables of T_2)

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- Dynamics of the λ-calculus: β-reductions
(λ-terms together with β-reduction are enough to encode any computation!)

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  - Order in which redices are reduced matters!

$$(\lambda x.z)((\lambda x.(x\ x))(\lambda x.(x\ x))) \begin{cases} \to (\lambda x.z)((x\ x)[x := (\lambda x.(x\ x))]) = \dots \\ \to z[x := (\lambda x.x\ x)(\lambda x.x\ x)] = z \end{cases}$$

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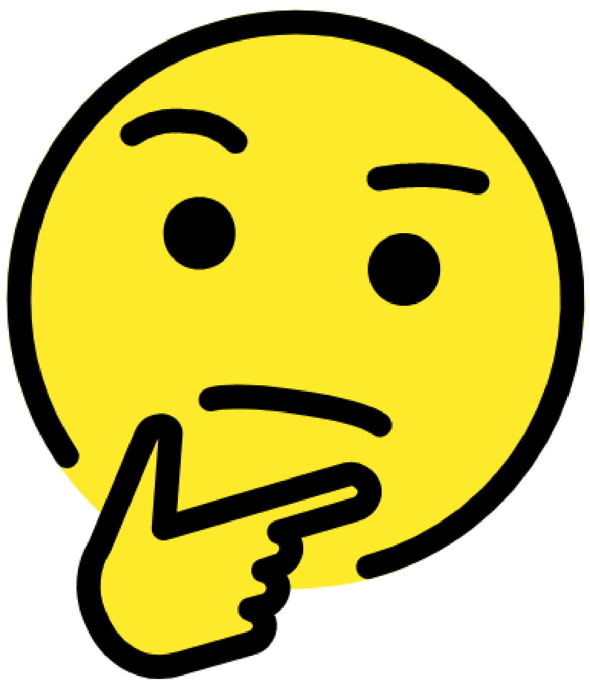

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  [DGKRTZ13]
  For terms expressed in the previously-presented syntax and size defined recursively as:

$$|x| = 0, |(a\ b)| = 1 + |a| + |b|, |\lambda x.t| = 1 + |t|$$

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  [DGKRTZ13,BGLZ16]
  For terms expressed using de Bruijn indices or combinators (together with appropriate size functions)

Parameter sensitive to the definition of the syntax and the size of terms!

- Almost every simply-typed λ-term has a long β-reduction sequence
  [SAKT17]

Subfamilies of λ-terms

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Linear Terms: bound variables occur **exactly** once

$\lambda x.\lambda y.(y\ x)a$

$\lambda x.\lambda y.(y\ a)(b\ x)$

$\lambda x.a(\lambda z.(\lambda y.y\ (x\ z)))$

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asymptotically!

seq. of random variables!

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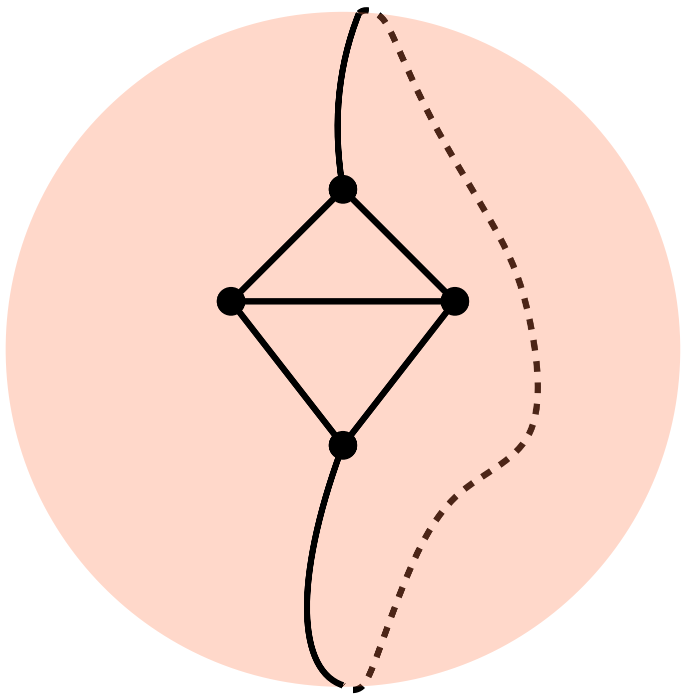
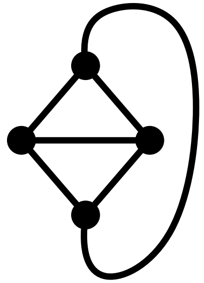
What are maps?





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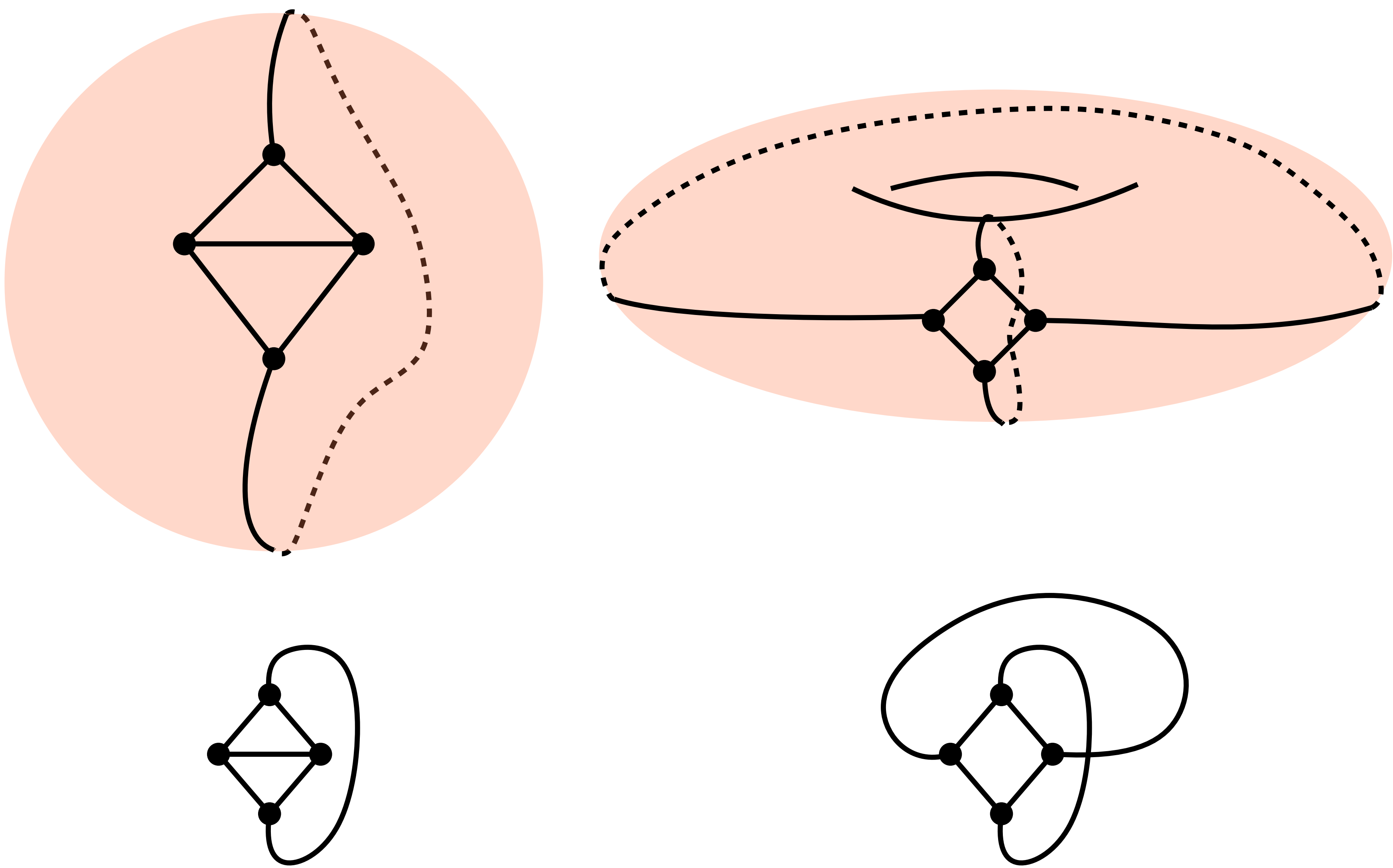

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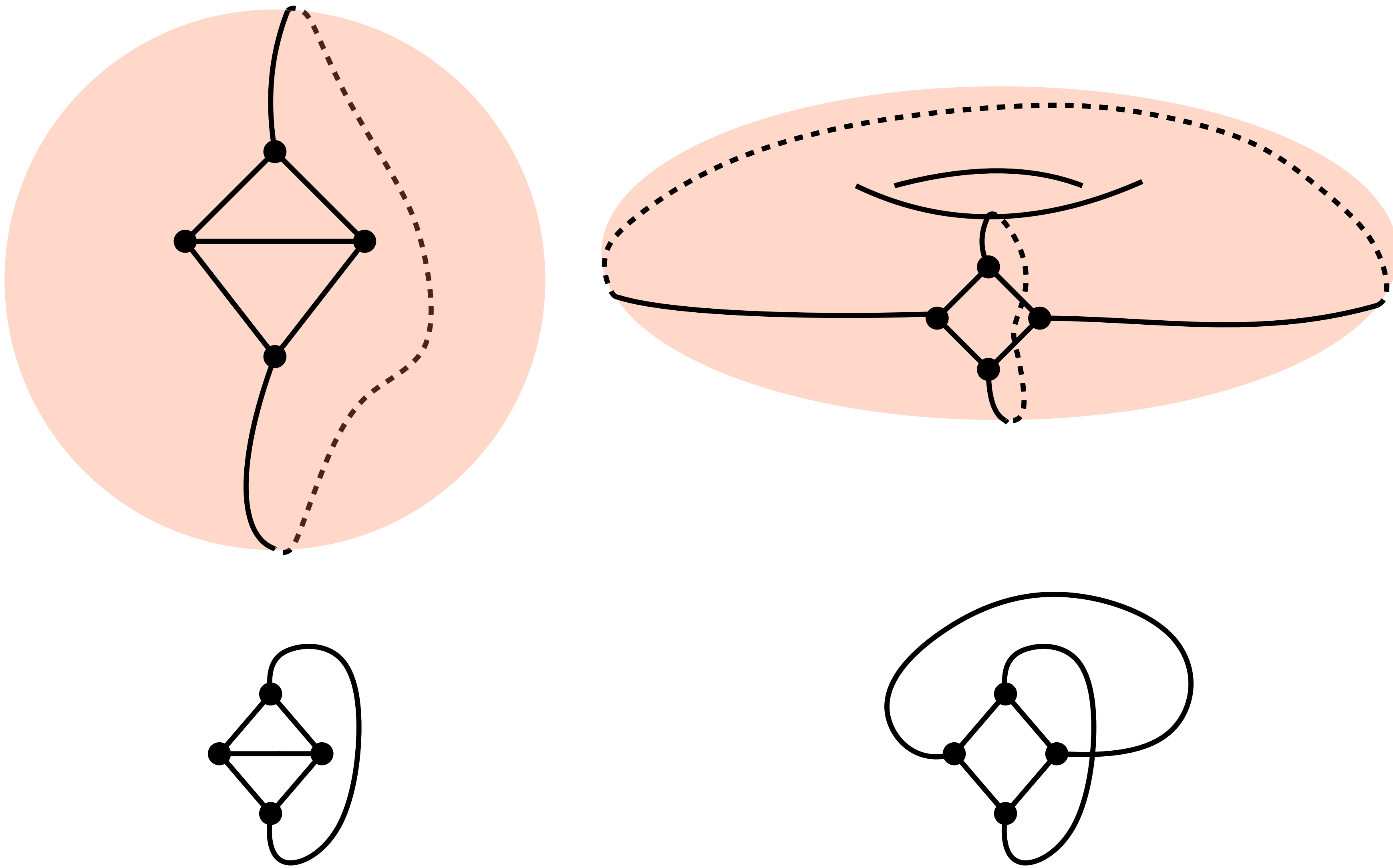
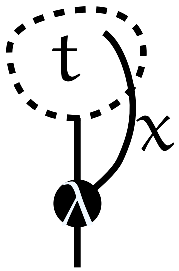


We’re interested in unrestricted genus, restricted vertex degrees

Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16]

$\bullet = x$



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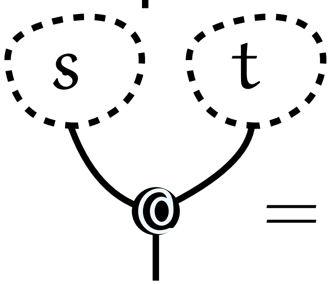

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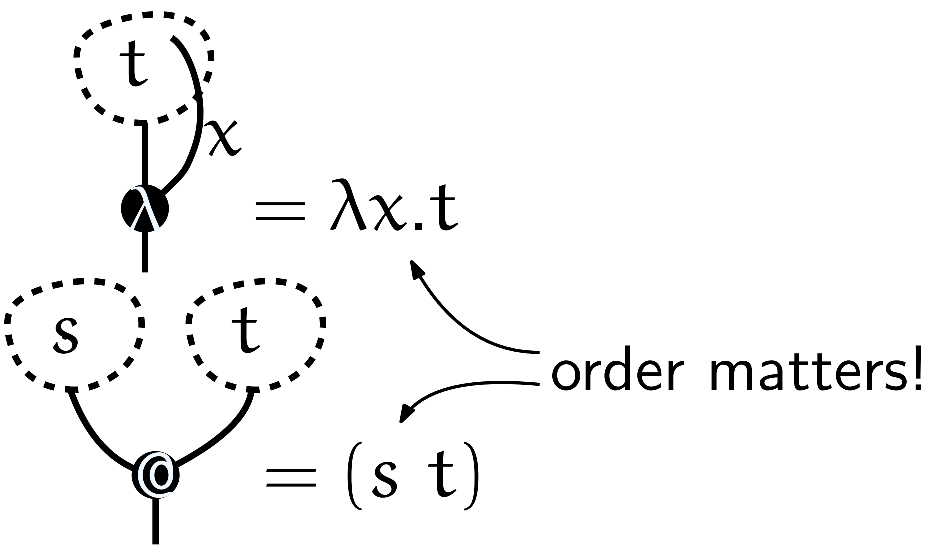

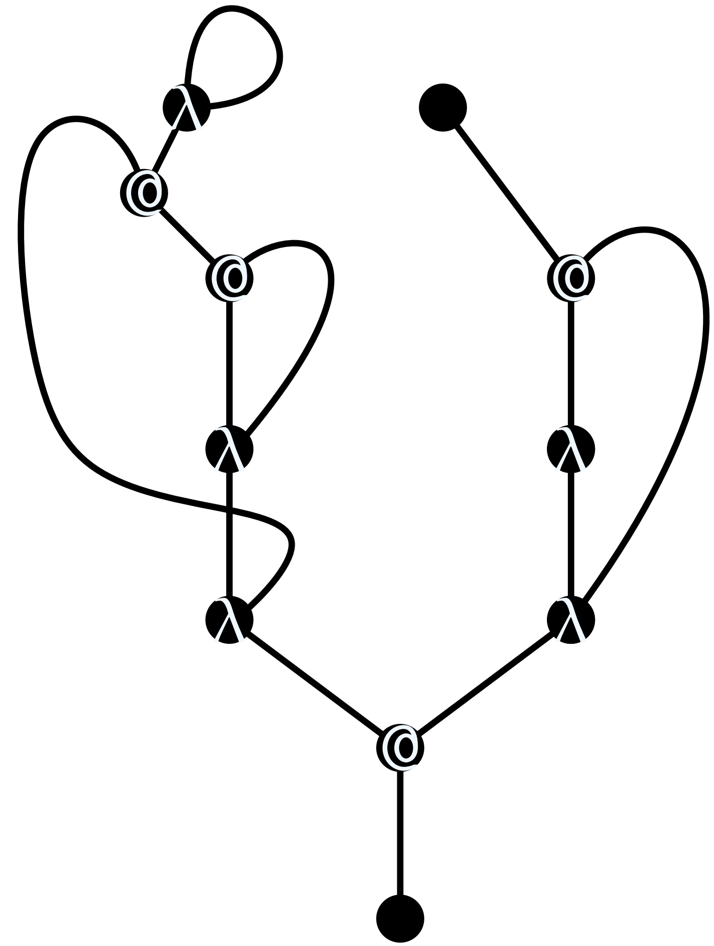

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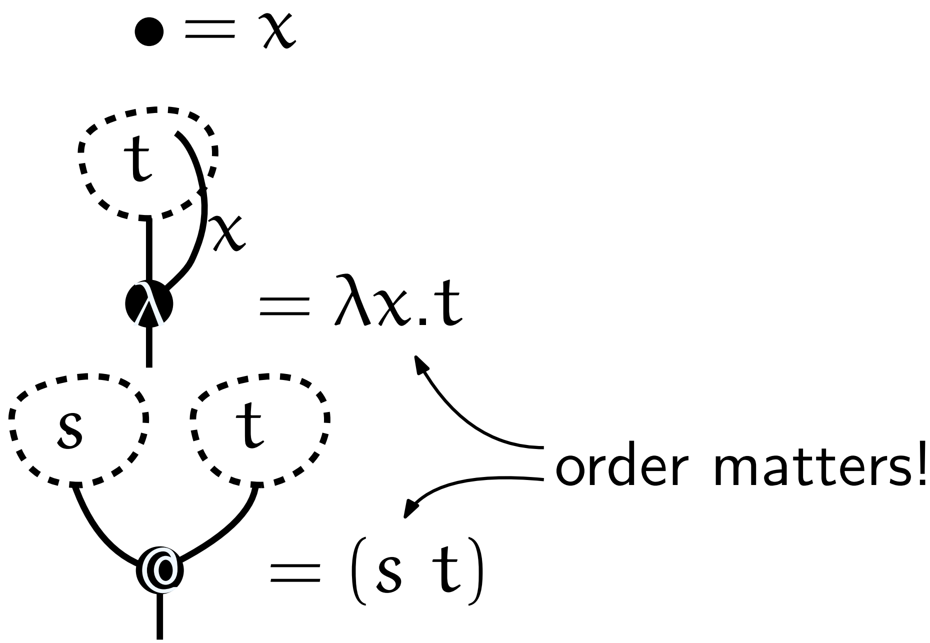

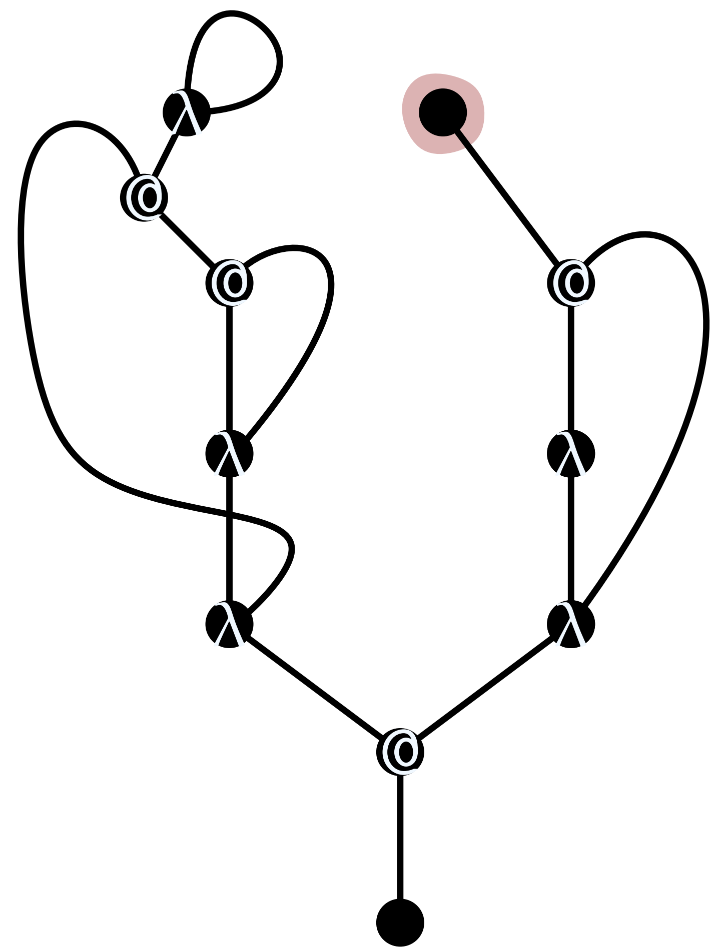

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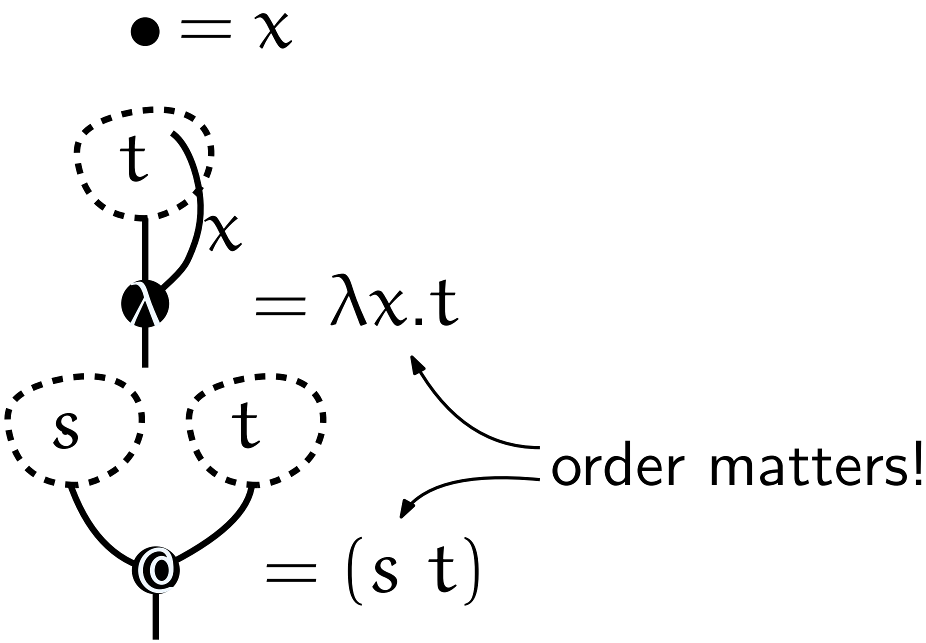

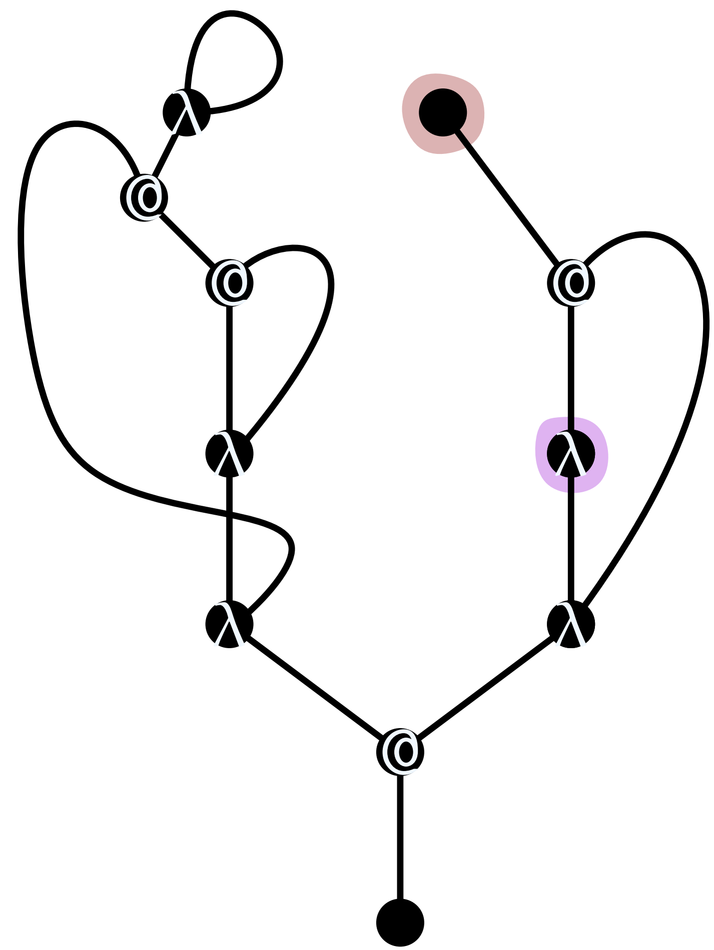

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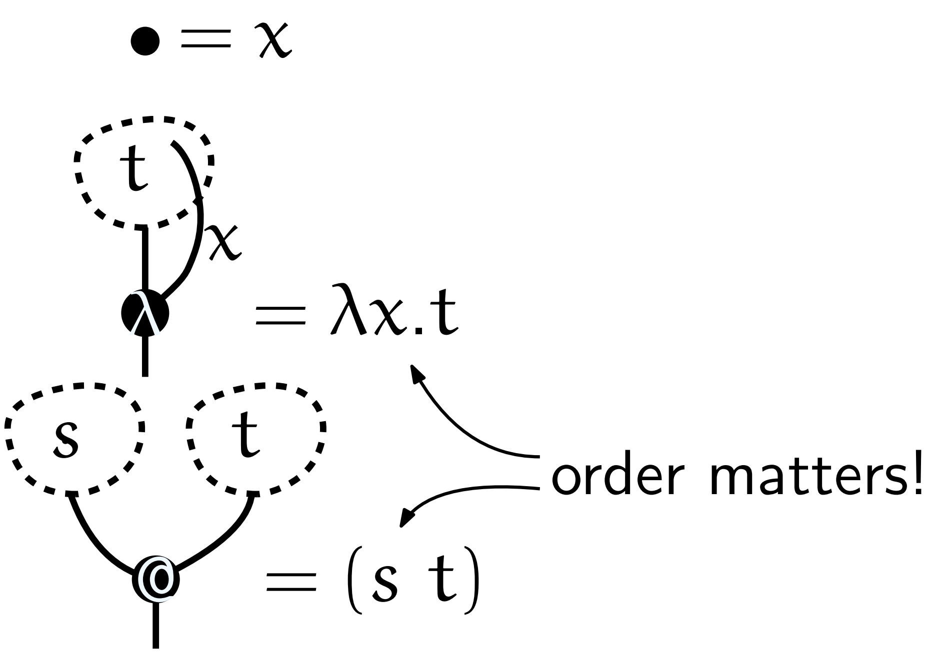

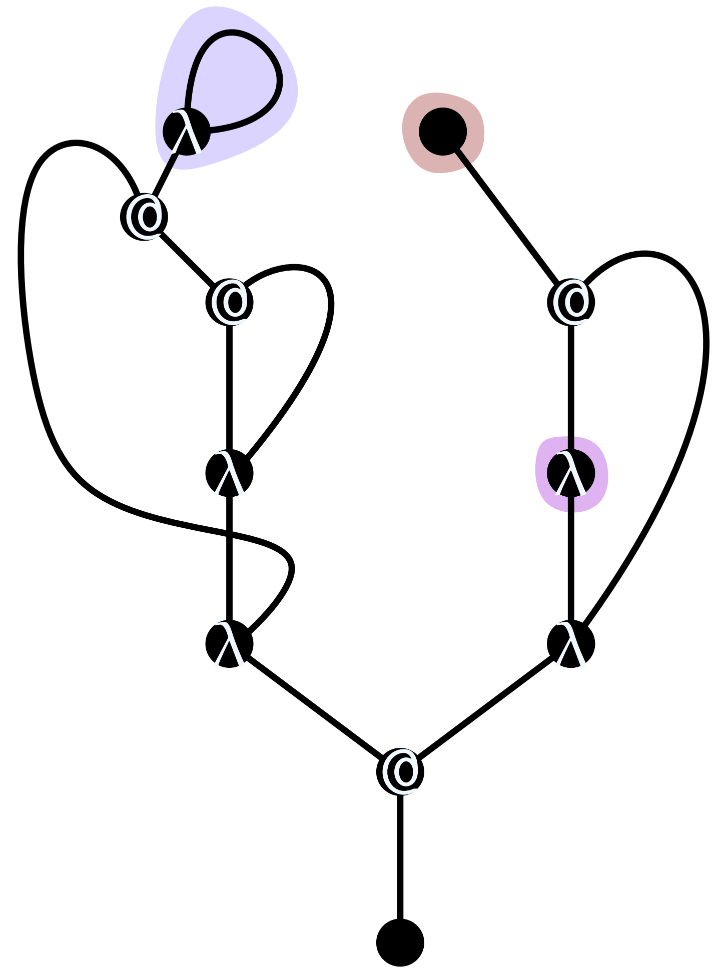

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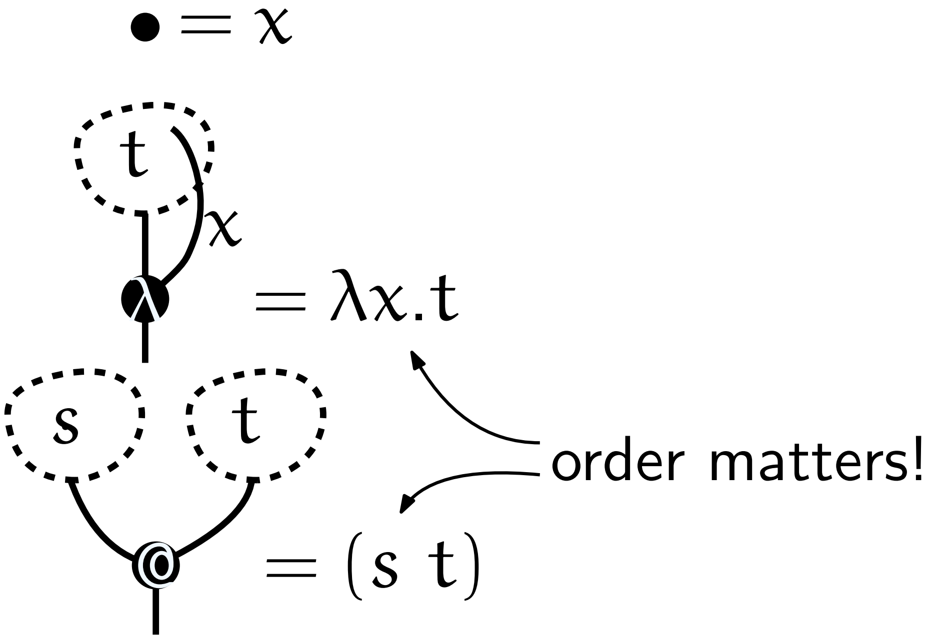

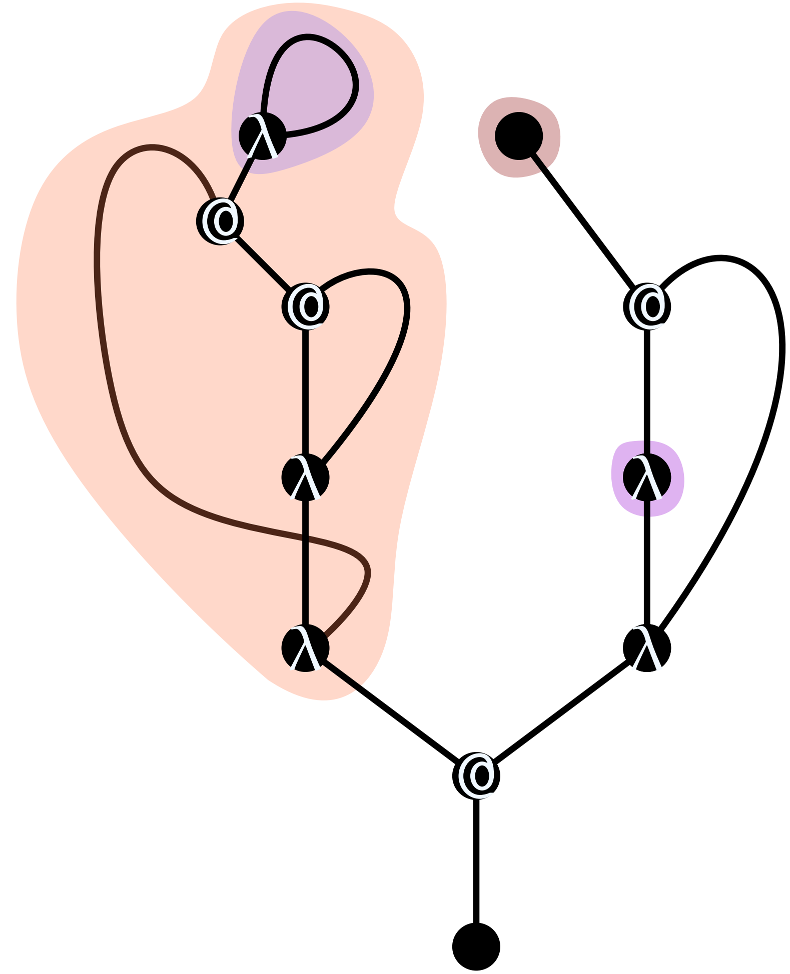

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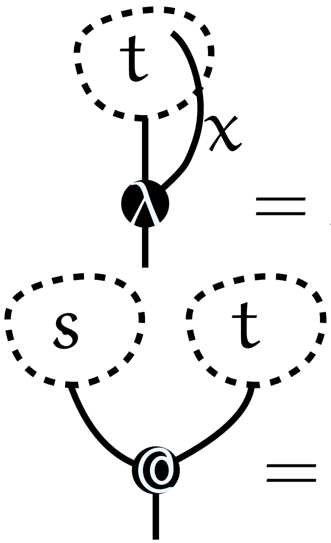

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order matters!

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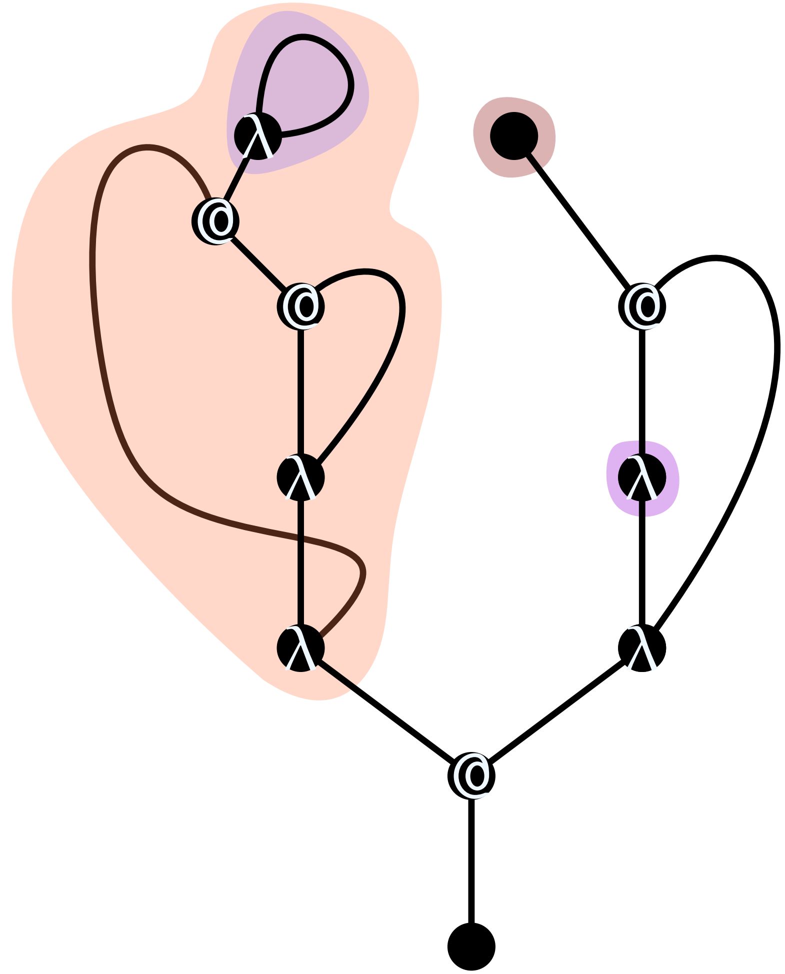

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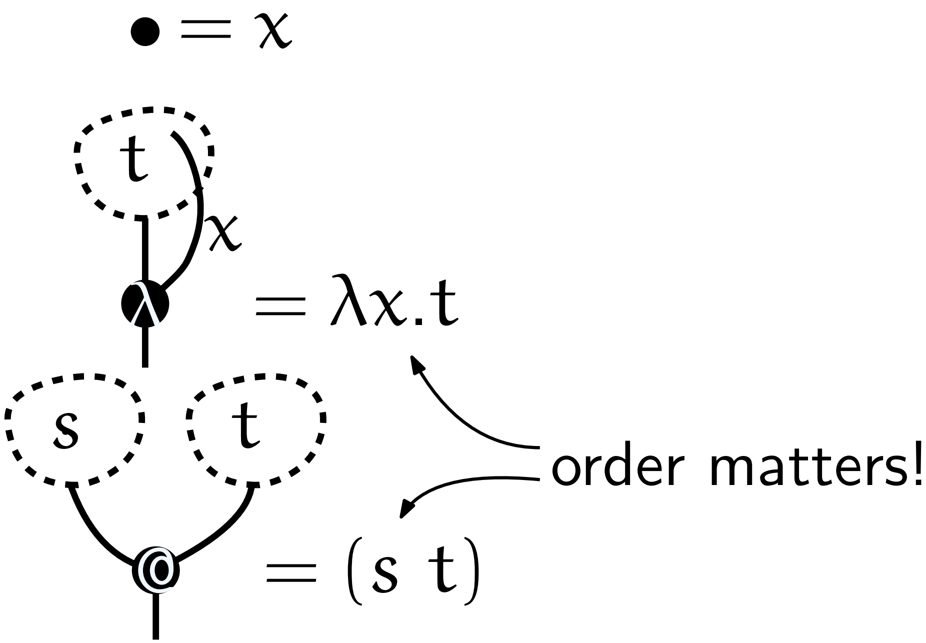

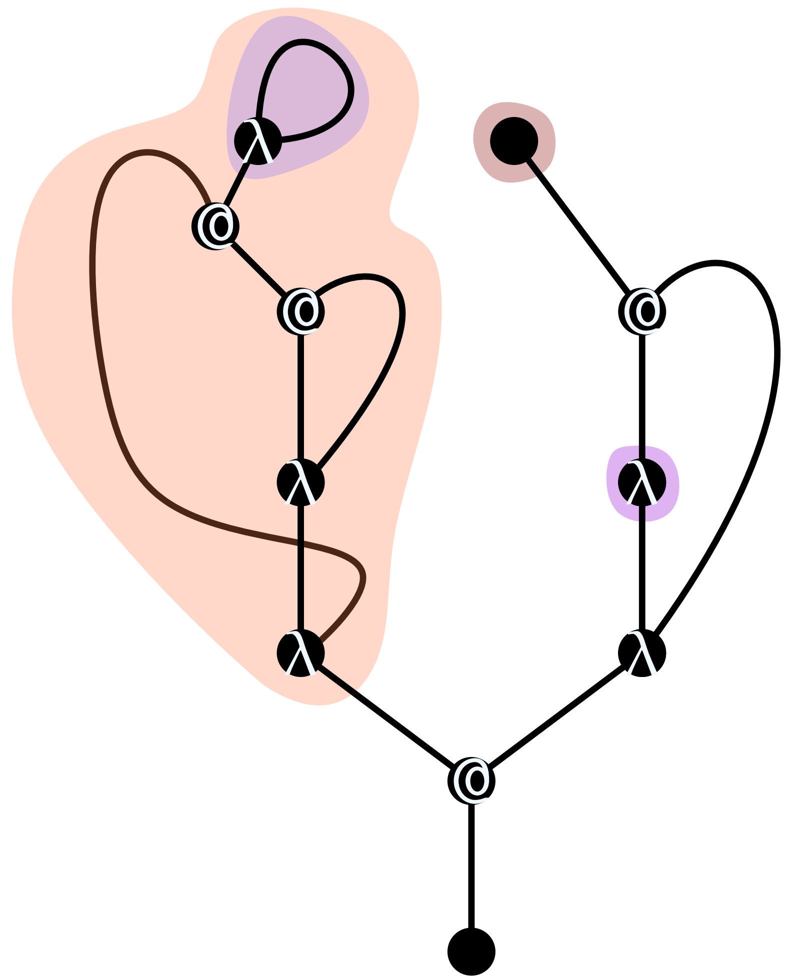

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Closed linear terms $\leftrightarrow$ trivalent maps
Closed affine terms $\leftrightarrow$ (2,3)-valent maps
Established in [BGJ13, BGGJ13]

Why should you, a combinatorialist, be interested in λ-terms?

Decomposing (closed) rooted trivalent maps [BGJ13]

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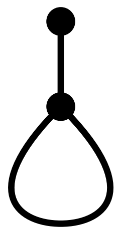

edges

$$\mathsf{T}(z) = z^2$$

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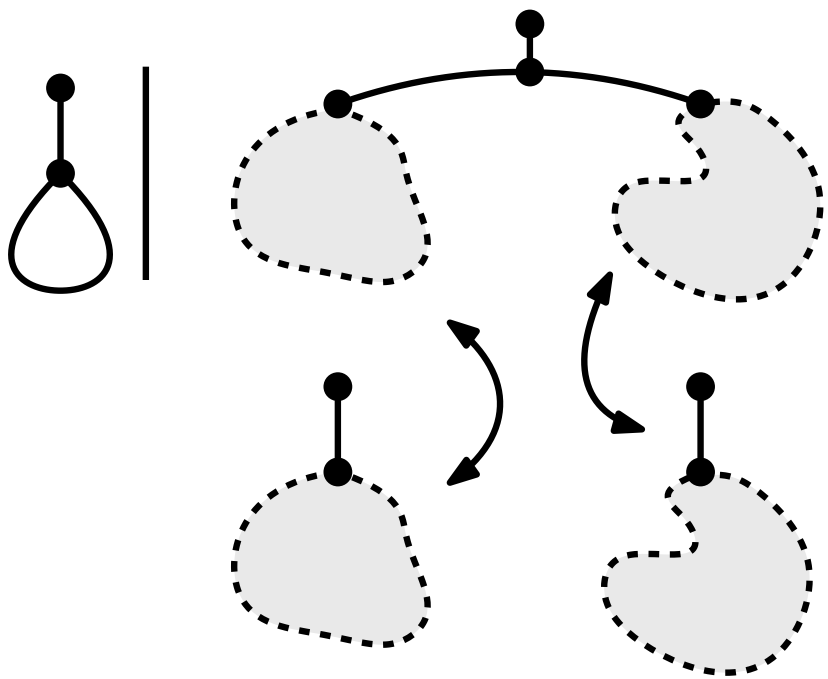

edges

$$\mathsf{T}(z) = z^2 \ + \ z\mathsf{T}(z)^2$$

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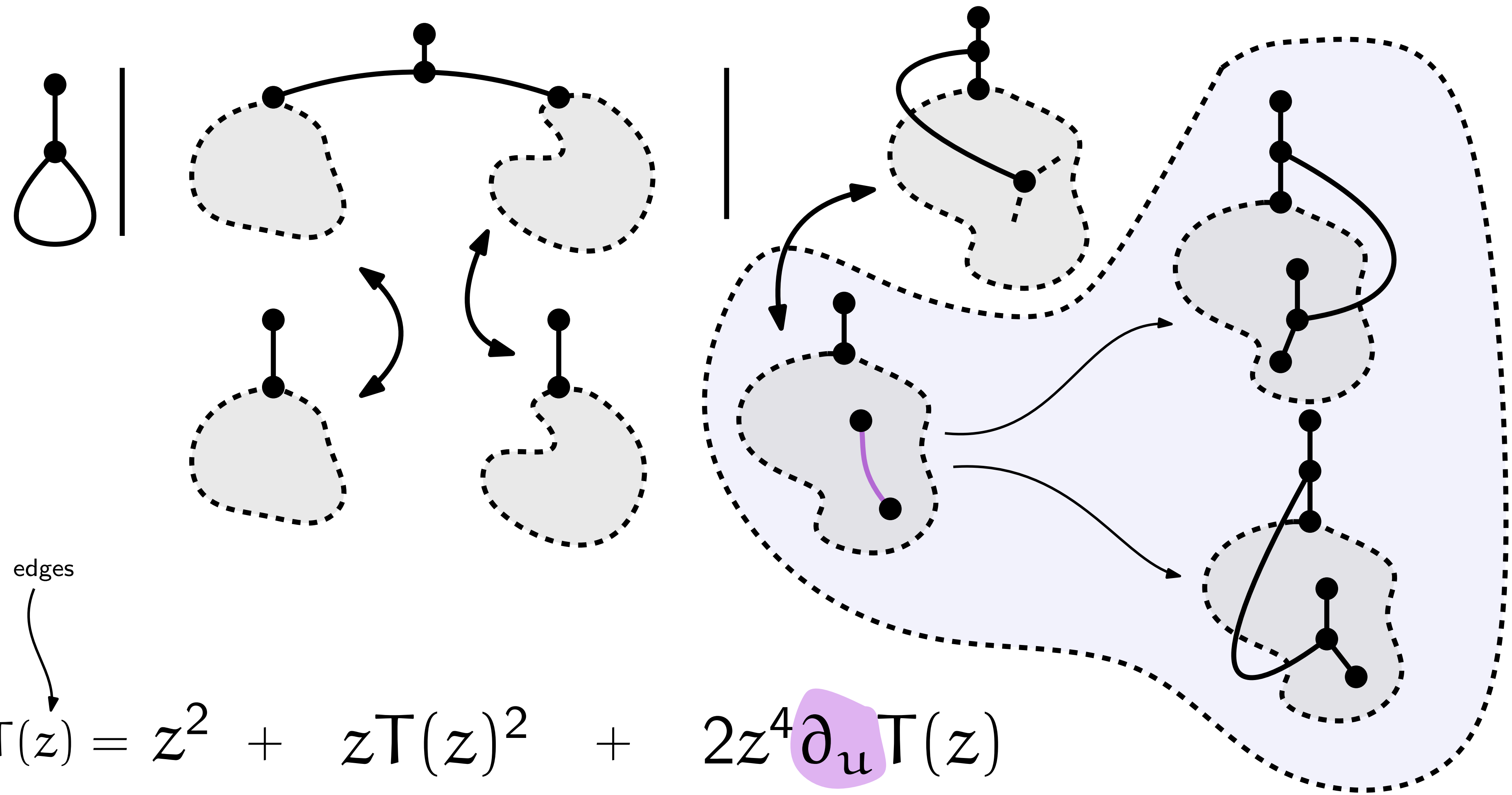
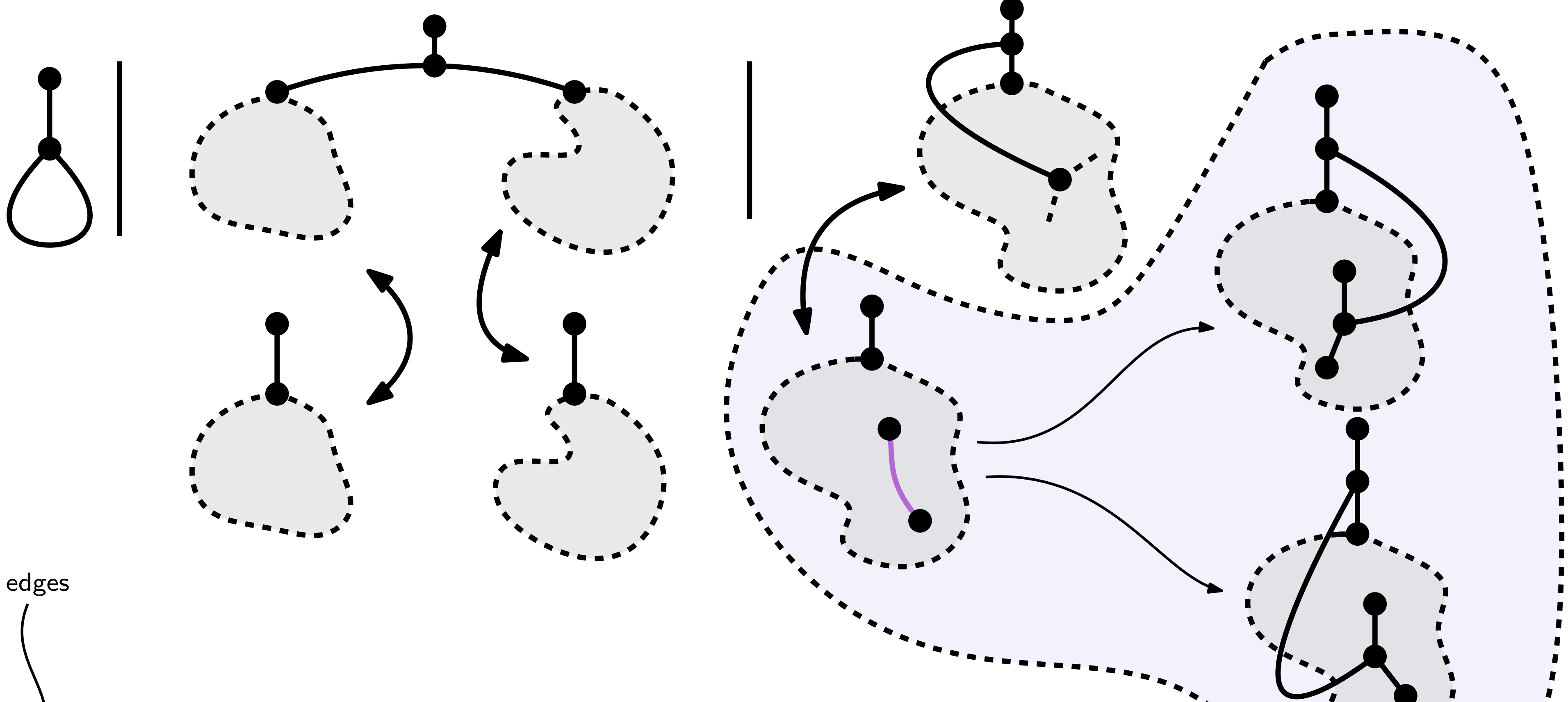


edges

$$\mathsf{T}(z) = z^2 \;+\; z\mathsf{T}(z)^2 \;+\; 2z^4\partial_u\mathsf{T}(z)$$

Why should you, a combinatorialist, be interested in λ-terms?

Decomposing (closed) rooted trivalent maps [BGJ13]
and open linear terms!



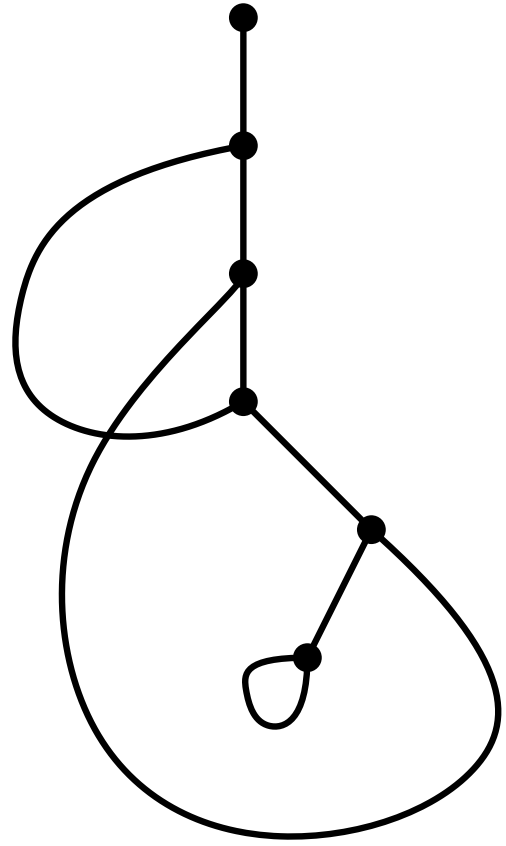
edges

$$T(z) = z^2 \quad + \quad zT(z)^2 \quad + \quad 2z^4 \partial_u T(z)$$

subterms

$$\mathrm{lin.term} = \lambda x.x \,\Big|\, (s\ t) \,\Big|\, \begin{array}{l} \lambda x.t[u := (x\ u)] \text{ or} \\ \lambda x.t[u := (u\ x)] \end{array}$$

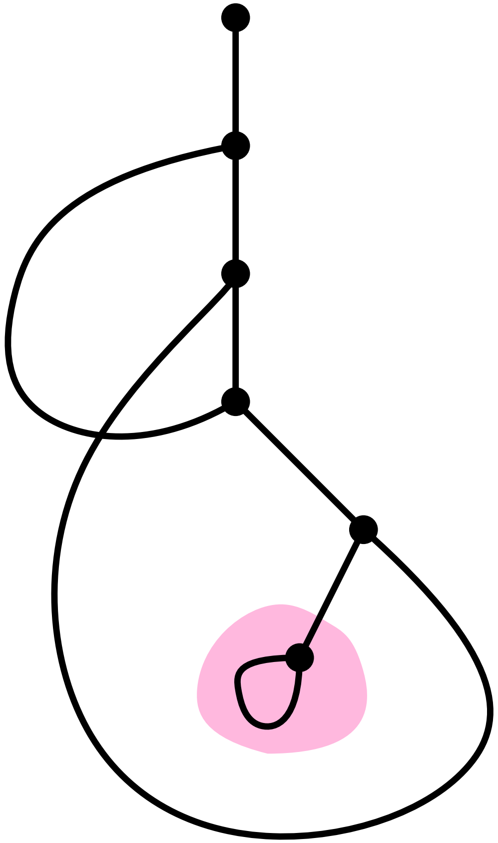
Some of our previous results: limit distributions



$$\lambda x.\lambda y.(y\ \lambda w.w)x$$

Some of our previous results: limit distributions

loops = # id-subterms



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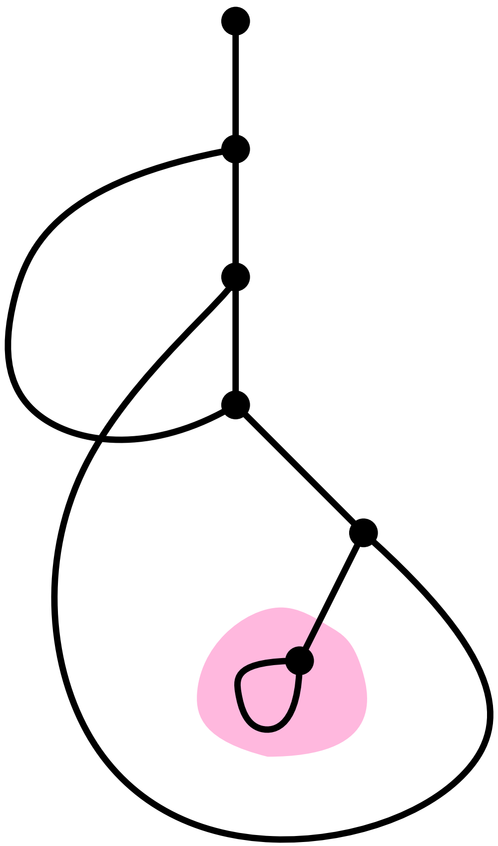

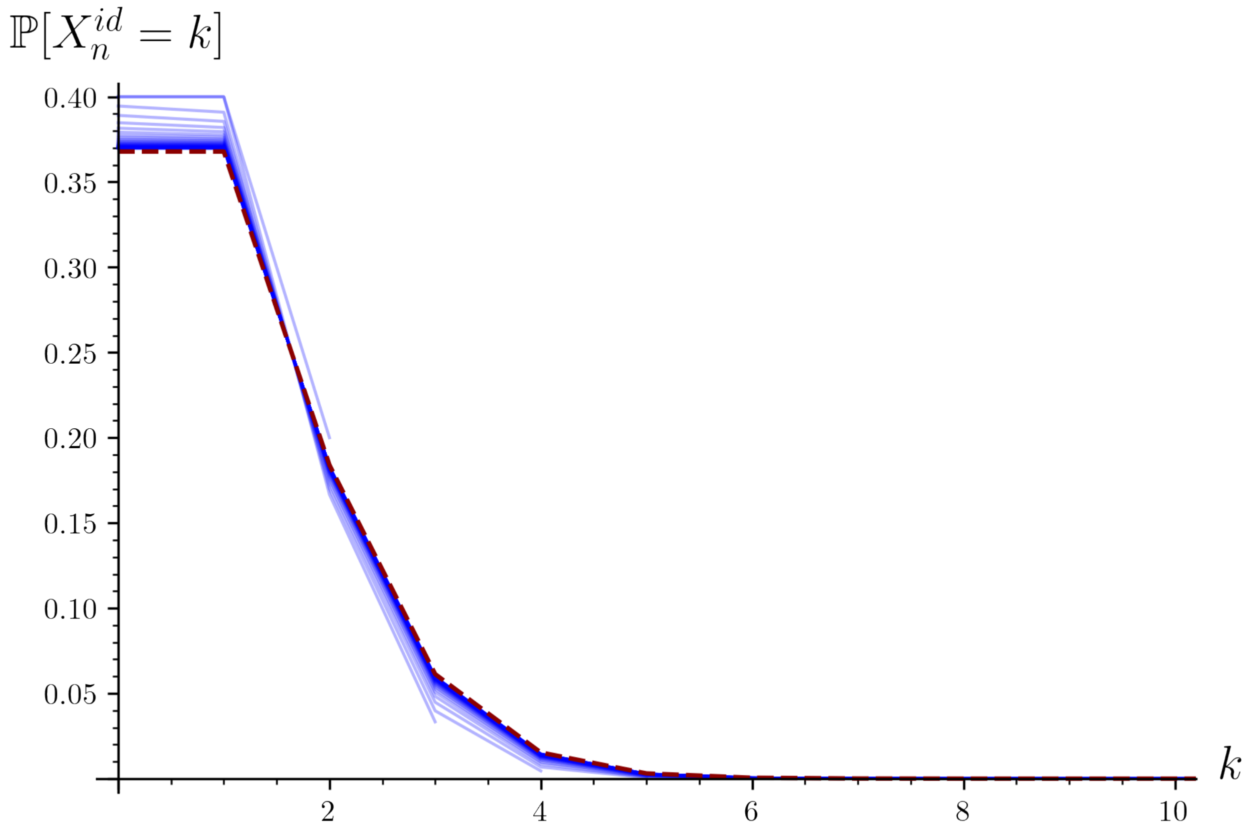

$\lambda x.\lambda y.(y\ \lambda w.w)x$

$X_n^{id} \xrightarrow{D} \text{Poisson}(1)$

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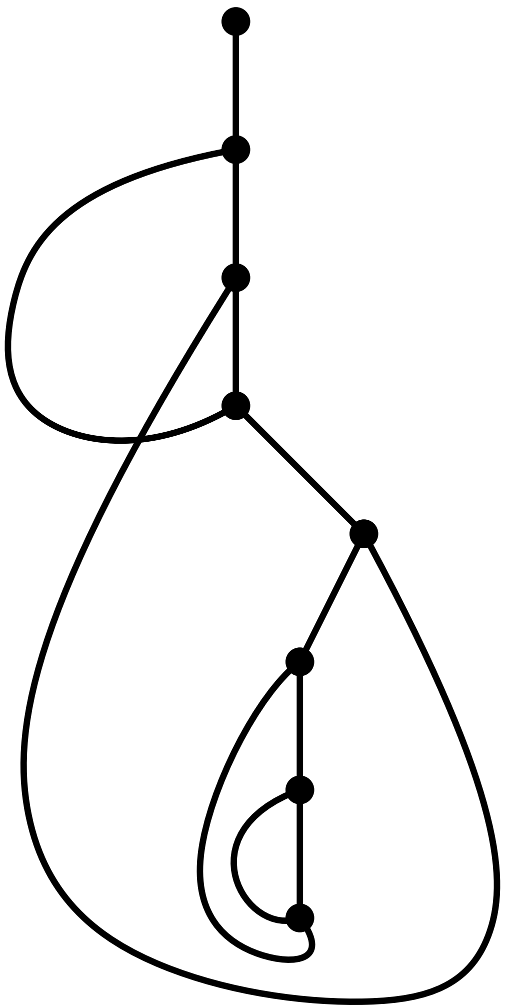

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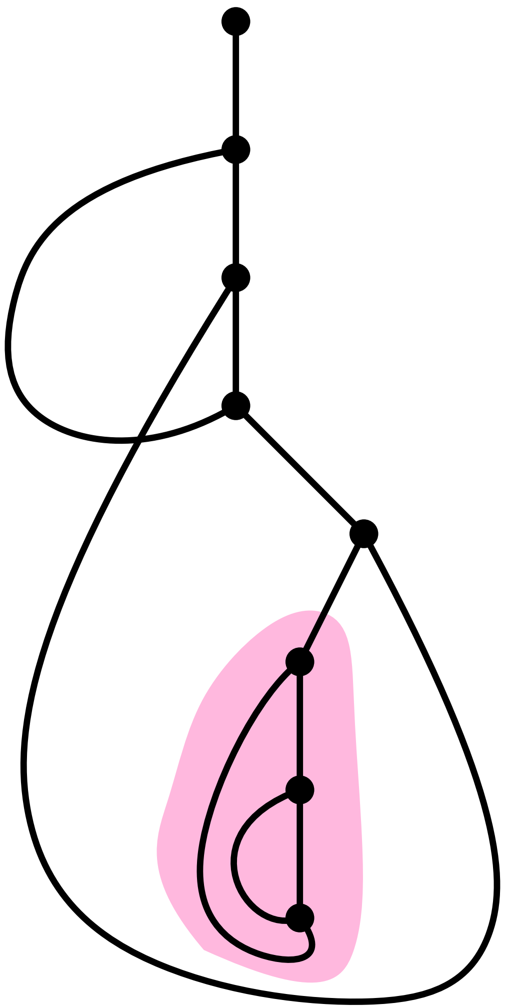
Some of our previous results: limit distributions

\# bridges = # closed subterms



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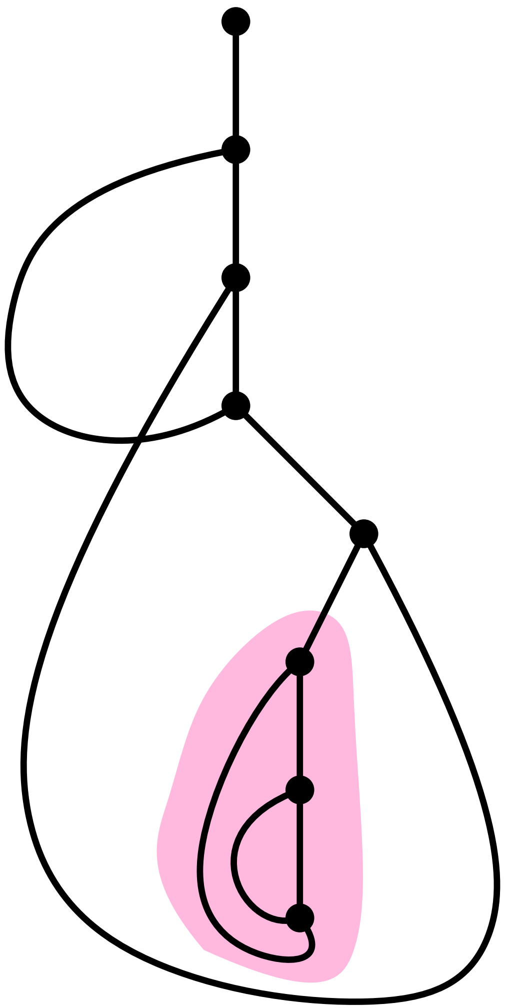

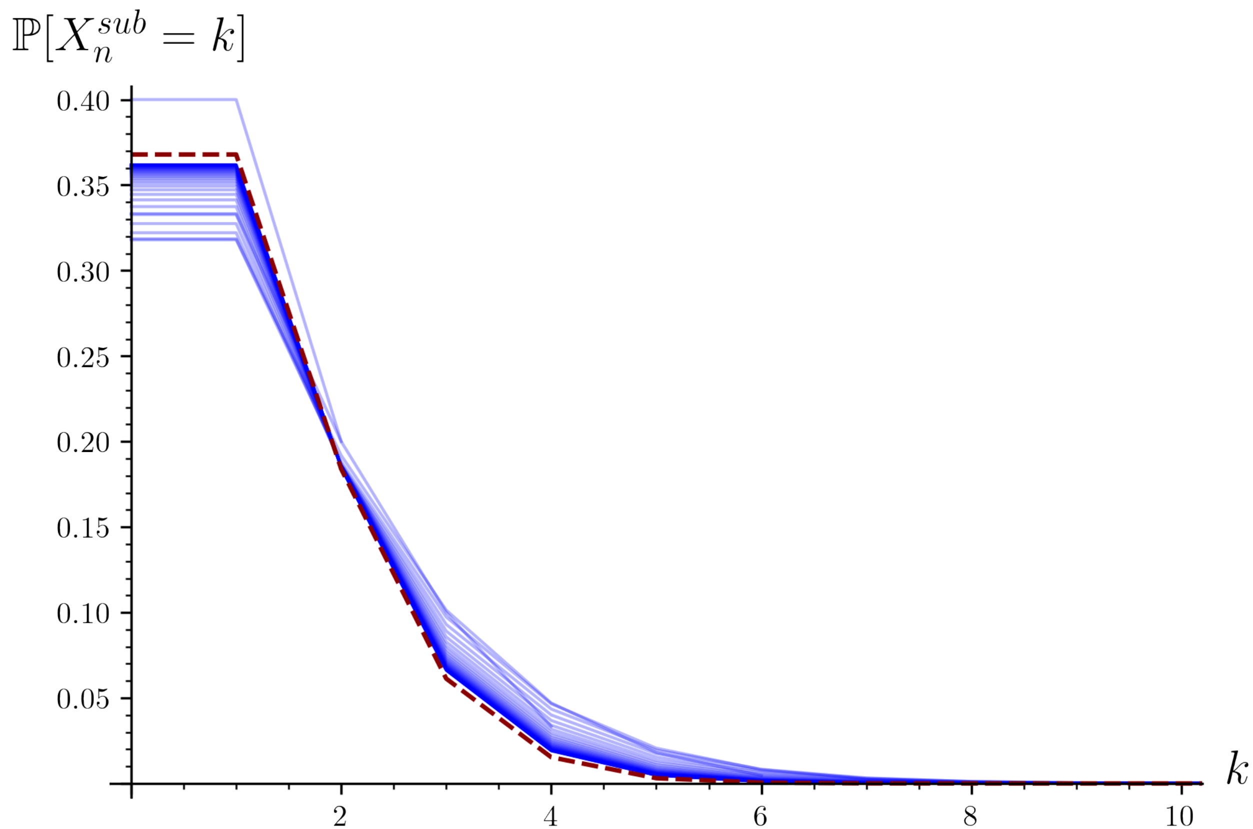

$\lambda x.\lambda y.(y\ \lambda z.\lambda w.zw)x$ $\qquad$ $X_n^{sub} \xrightarrow{D} \mathrm{Poisson}(1)$

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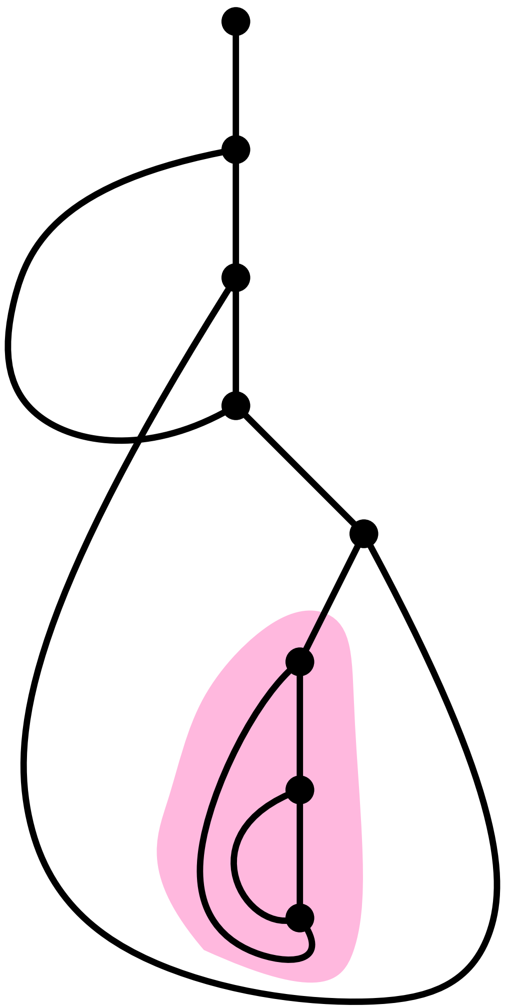

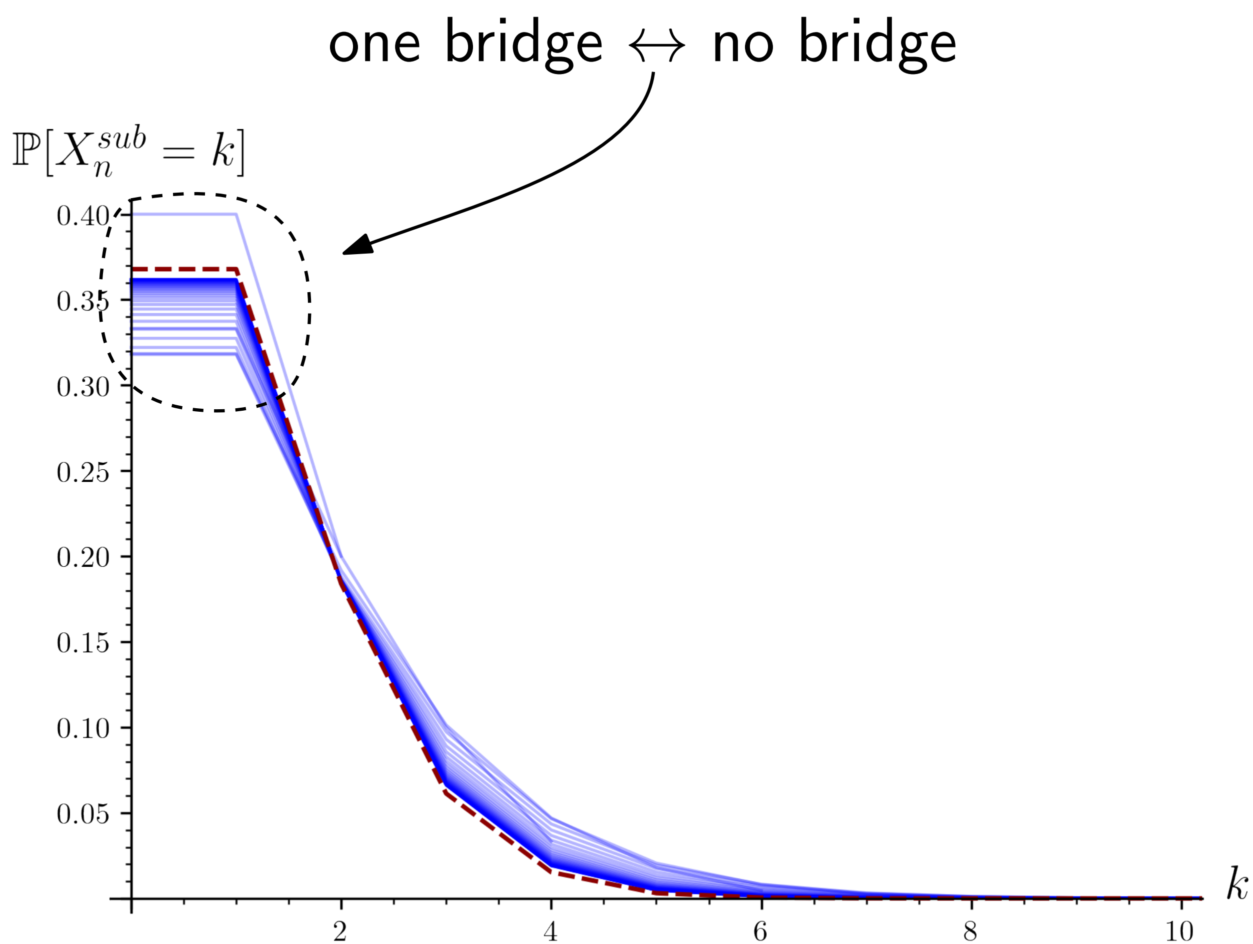

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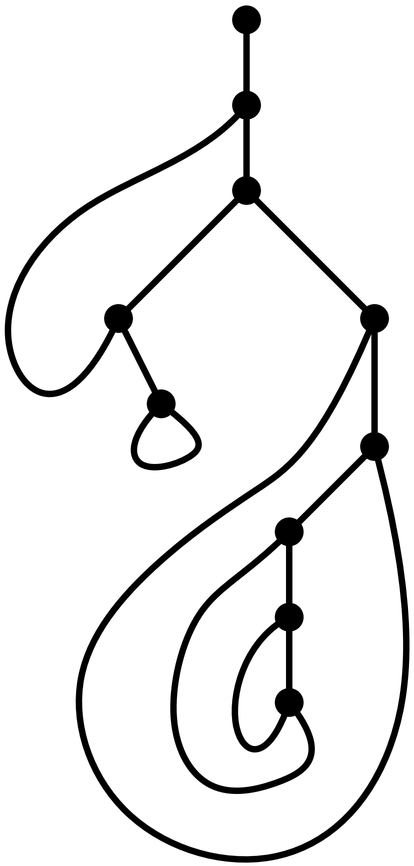
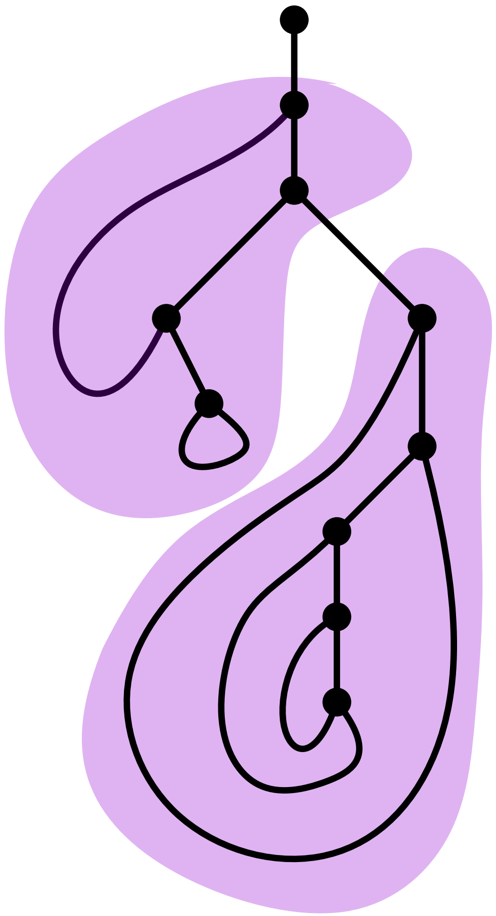
β-reduction as map rewriting



$\lambda x.(\lambda y.y(\lambda z.\lambda w.zw))((\lambda u.u)x)$

β-reduction as map rewriting



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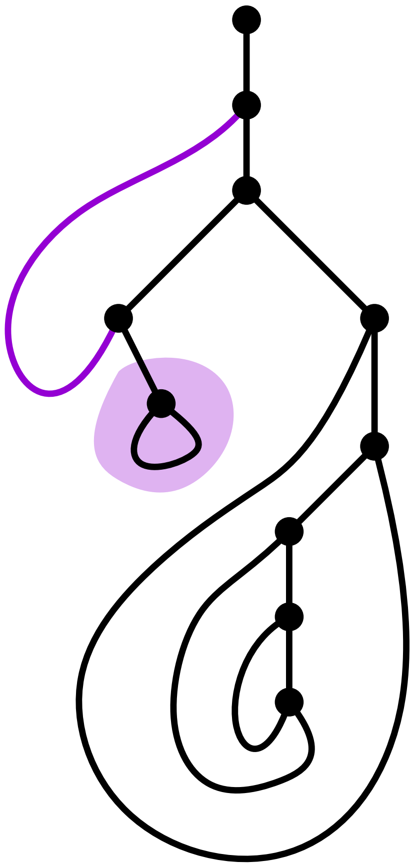

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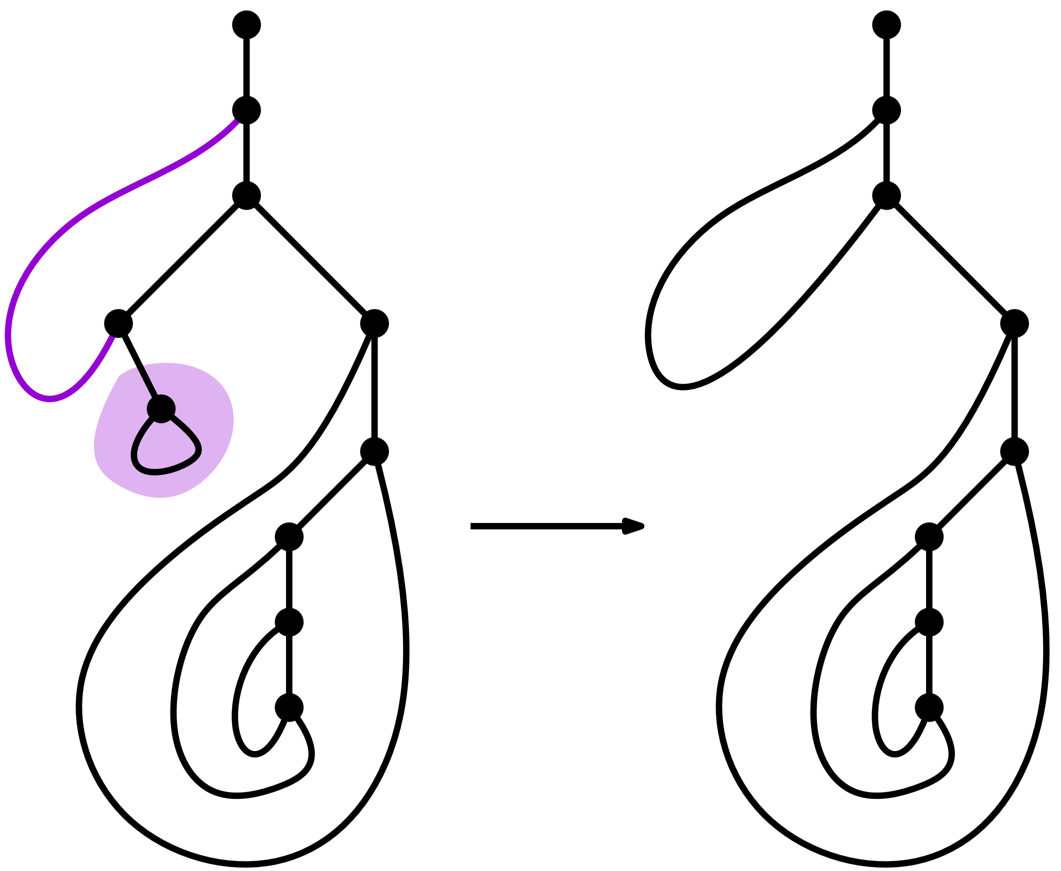

$$\lambda x.(\lambda y.y(\lambda z.\lambda w.zw))((\lambda u.u)x)$$

$$\longrightarrow \lambda x.((\lambda y.y(\lambda z.\lambda w.zw))x)$$

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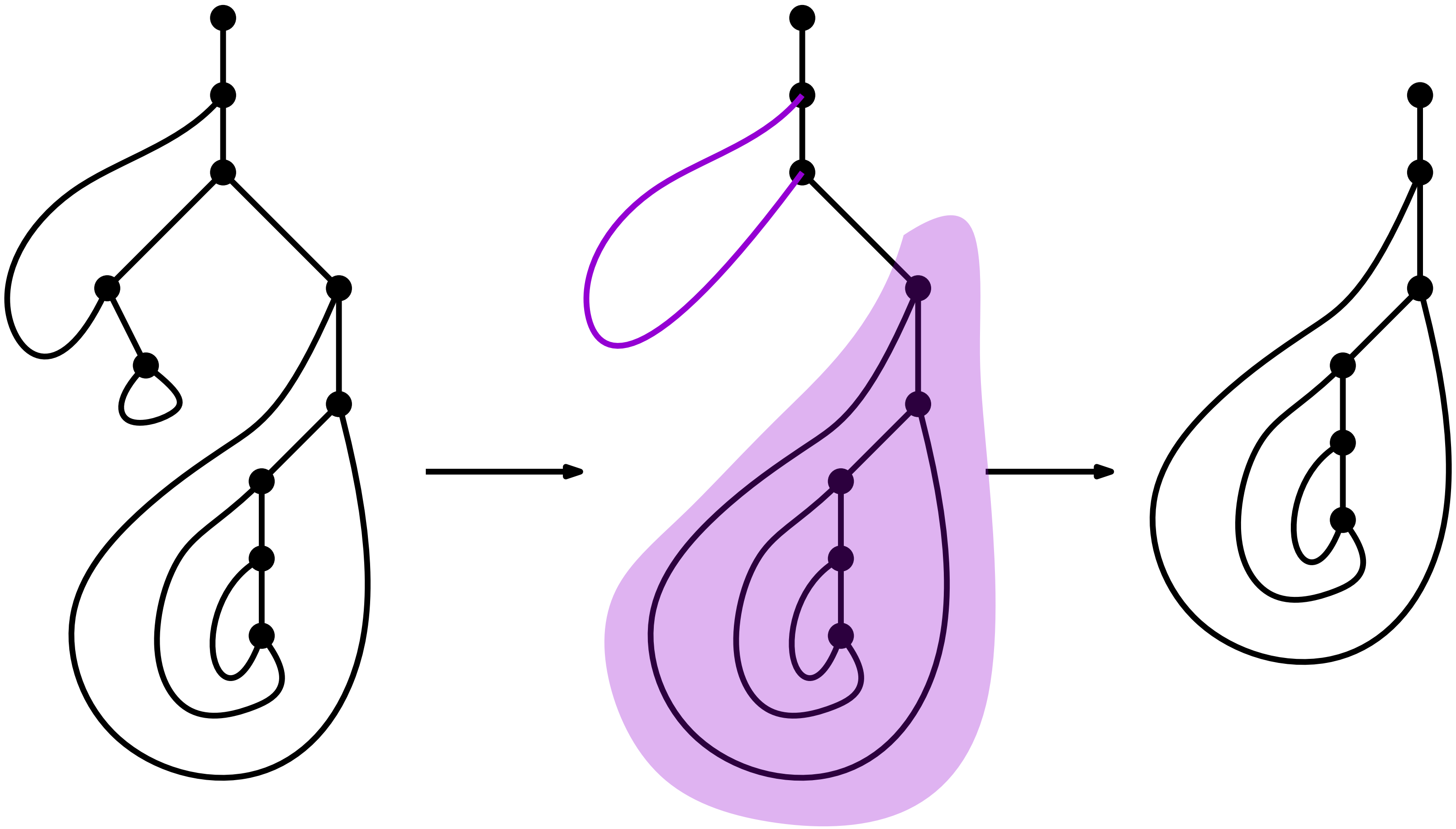

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$$[z^n](A \cdot B) = \sum_{k=n_0}^{n} a_k b_{n-k}$$

first order asymptotics given by $k = n_0$ and $k = n$

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crucial ingredient: coefficients are growing *rapidly*

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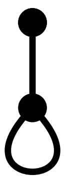
no redex



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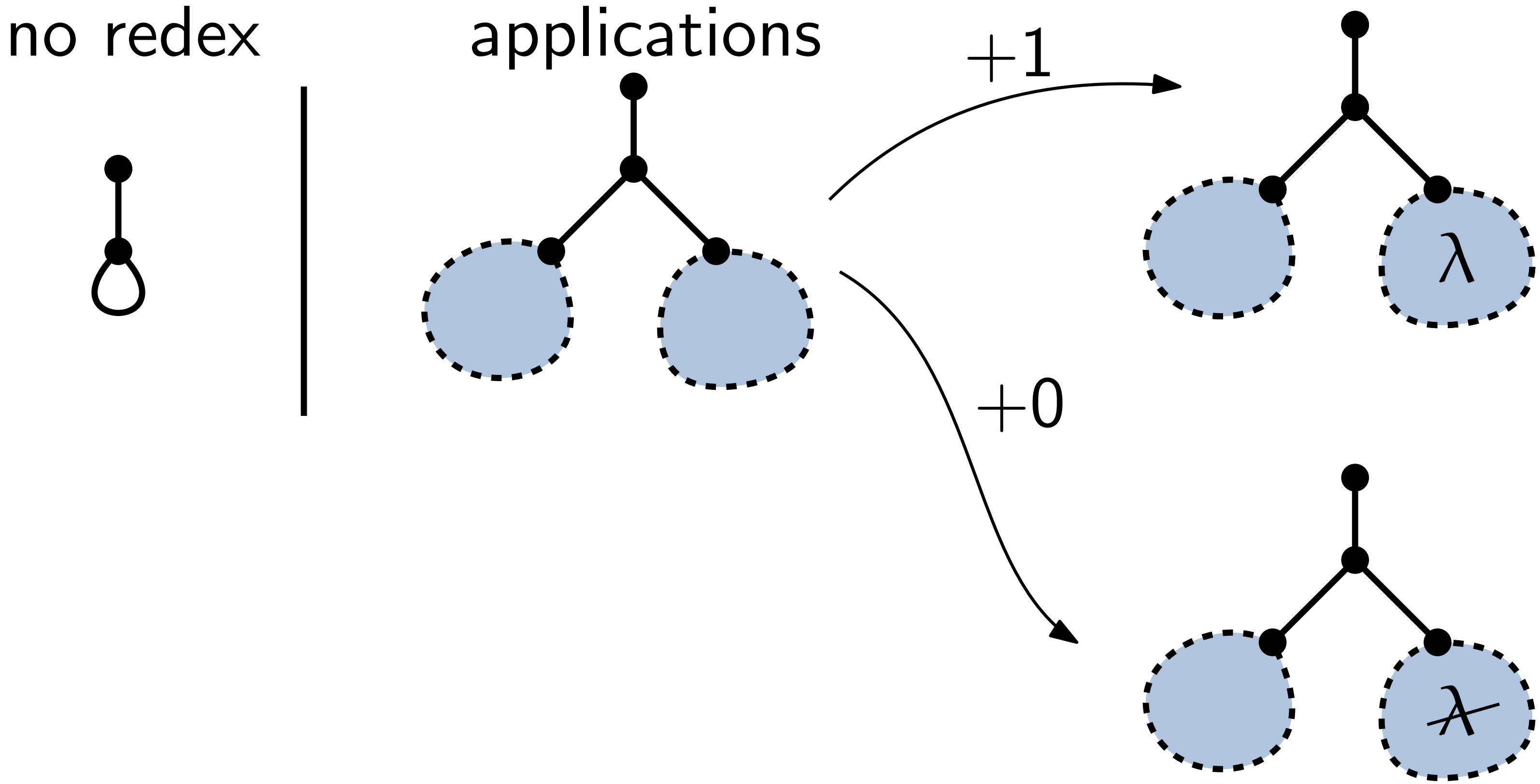

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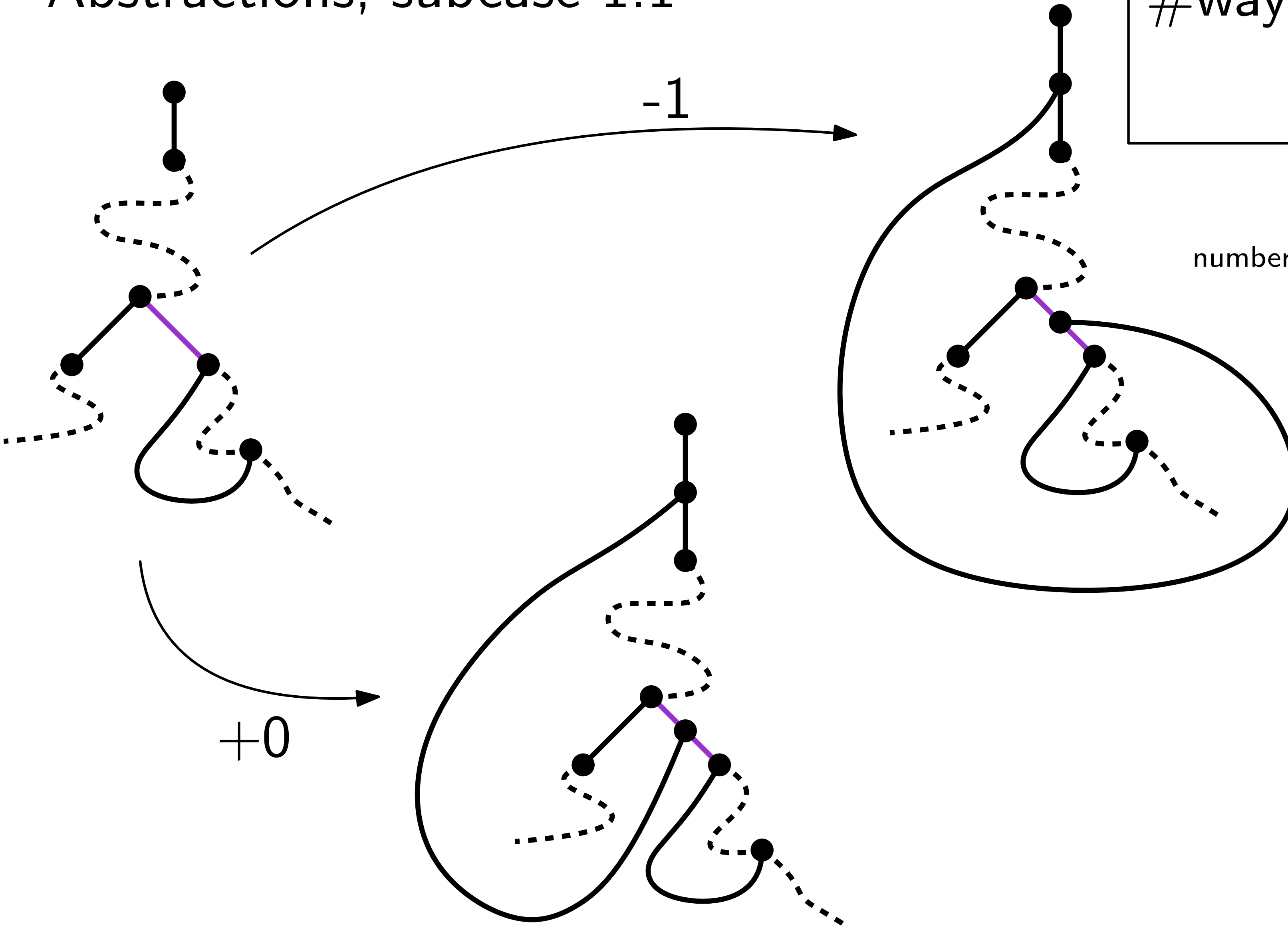
Abstractions, subcase 1.1



#ways to do this

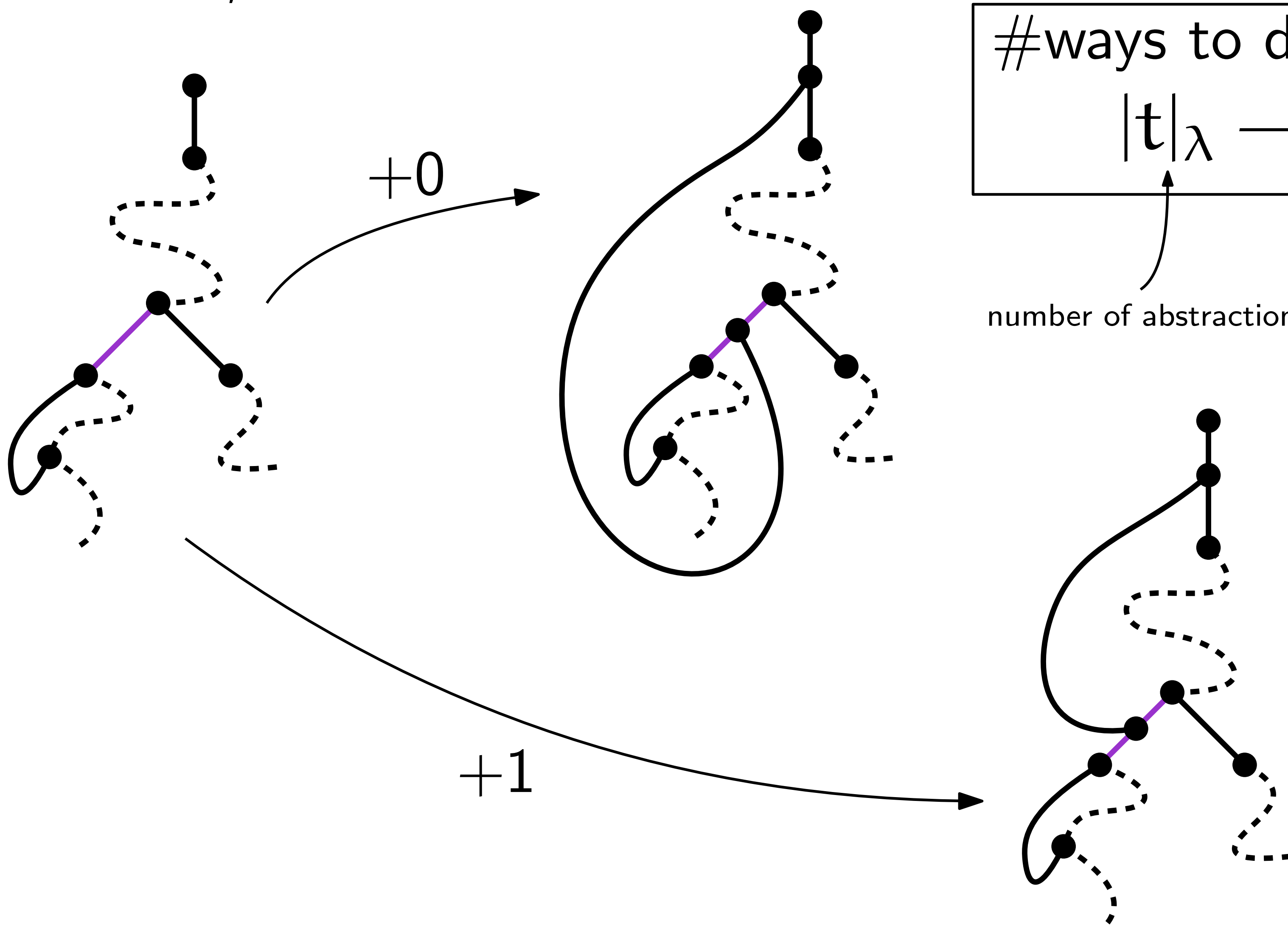
$|t|_\beta$

number of redices in t

Mean number of β-redices in closed terms

- Tracking redices during the decomposition

Abstractions, subcase 1.2



#ways to do this

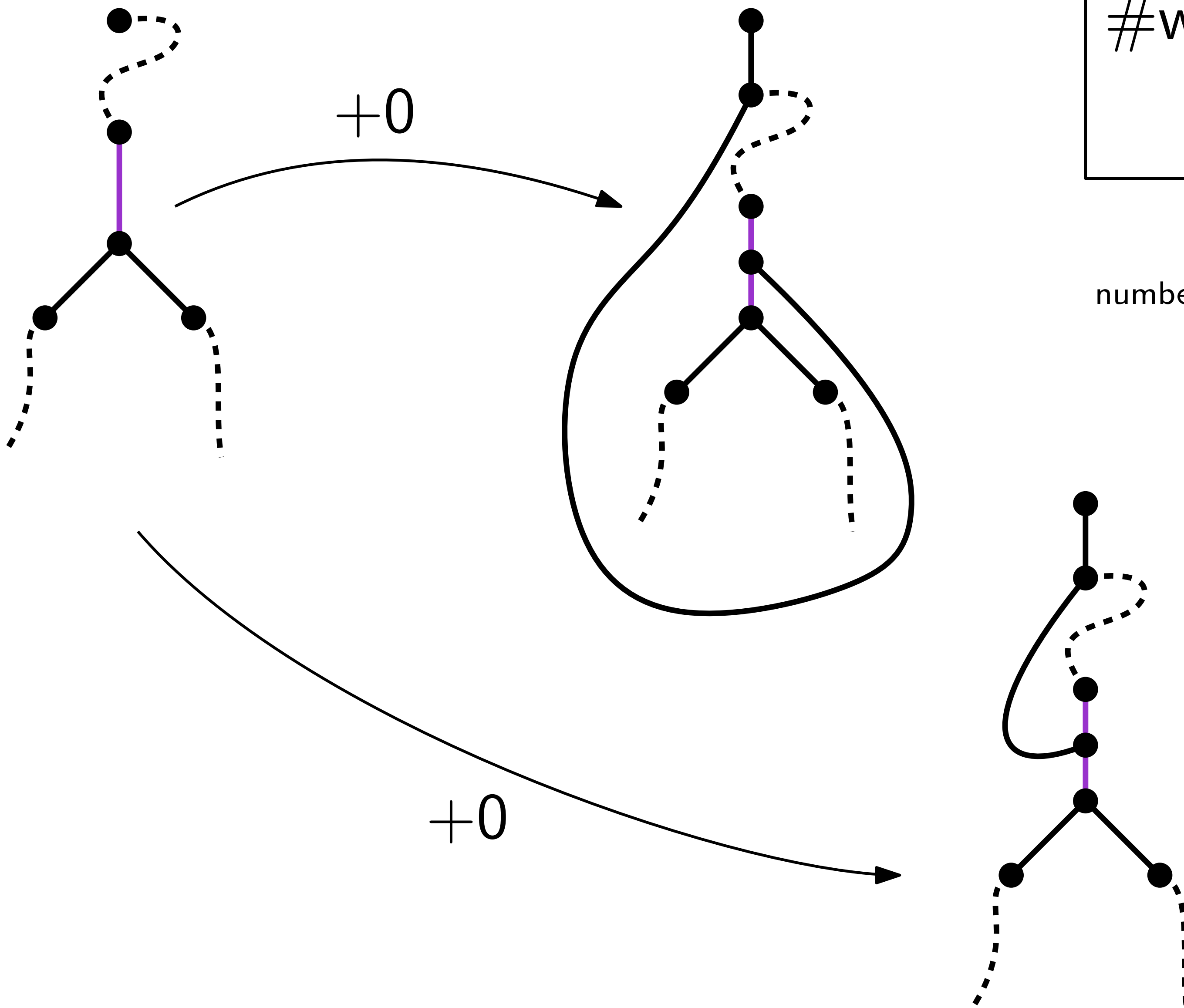
$$|t|_\lambda - |t|_\beta$$

number of abstractions in t

Mean number of β-redices in closed terms

- Tracking redices during the decomposition

Abstractions, subcase 1.3



#ways to do this

$$|\mathrm{t}| - |\mathrm{t}|_\lambda$$

number of subterms in t = size of t

Mean number of β-redices in closed terms

- Building the specification of the OGF

- $|t|_\lambda = \frac{|t|+1}{3}, |t| - |t|_\lambda = \frac{2|t|-1}{3}$
- $r\partial_r T_0 = \sum_{t \in T_0} |t|_\beta z^{|t|} r^{|t|_\beta}$
- $\frac{z\partial_z T_0 + T_0}{3} = \sum_{t \in T_0} \frac{|t|+1}{3} z^{|t|} v^{|t|_\beta}$
- $\frac{2z\partial_z T_0 - T_0}{3} = \sum_{t \in T_0} \frac{2|t|-1}{3} z^{|t|} v^{|t|_\beta}$

Mean number of β-redices in closed terms

- Translating to a differential equation and pumping

$$\mathsf{T}_0 = -z\Big(z^2(r+1)(1+(r-1)z\mathsf{T})(r-1)\partial_r \mathsf{T}_0 - \frac{(1+z(r-1)\mathsf{T})z^3(r+5)\partial_z \mathsf{T}_0}{3} - \frac{z^3(r-1)^2\mathsf{T}_0^2}{3} - \frac{4z^2(r-1)\mathsf{T}_0}{3} - z - \mathsf{T}_0^2\Big)$$

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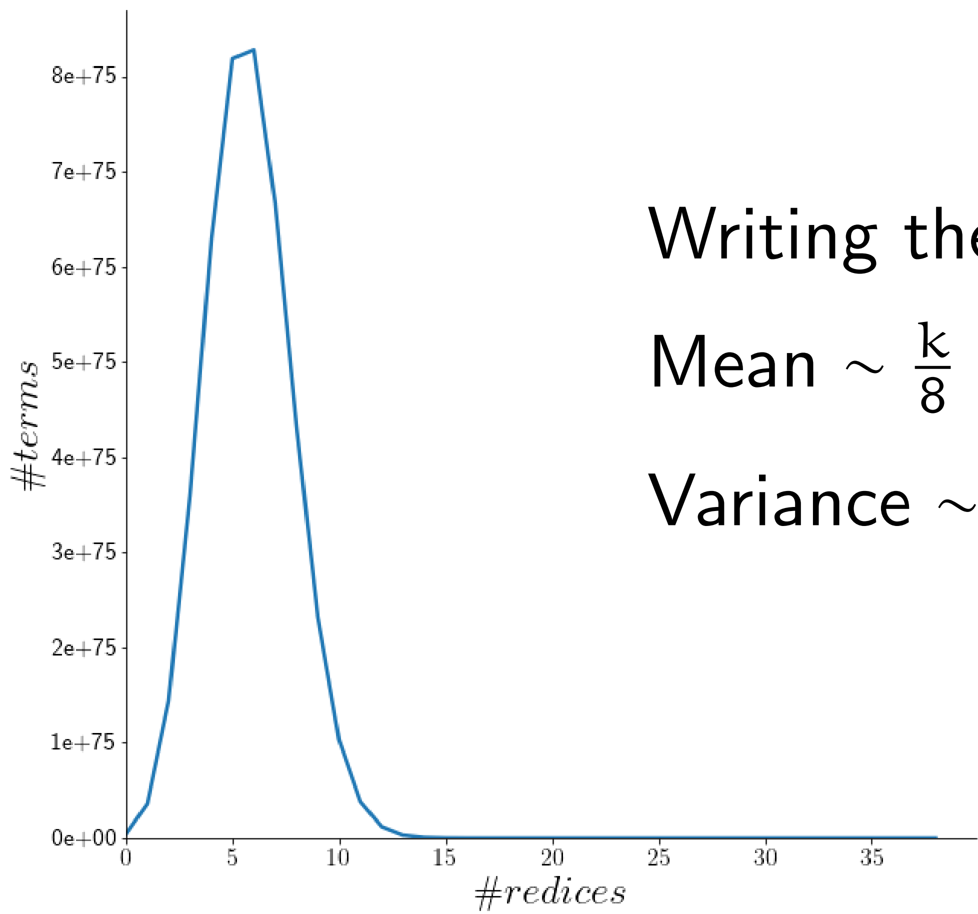
A plot of the dist. of redices for terms/maps of size $n = 119$



Writing the size as $n = 3k + 2$, we have:

Mean $\sim \frac{k}{8}$

Variance $\sim \frac{29k}{320}$

Three special kinds of β-redices

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- Consider the following three families of redices

$$(\lambda x.C[(x\ u)])(\lambda y.t_2) \quad (p_1) \qquad ((\lambda x.\lambda y.t_1)t_2)t_3 \quad (p_2)$$

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- A reduction step applied to any of these leaves the number of redices invariant.
- These are the only patterns with this property.

- Can be used to give a lower bound on number of steps to reach normal form:

$$\#steps \geqslant |t|_\beta + |t|_{p1} + |t|_{p_2} + |t|_{p_3}$$

Enumerating $\mathfrak{p}_1$-patterns

- Tracking the creation/destruction of patterns during the recursive decomposition:

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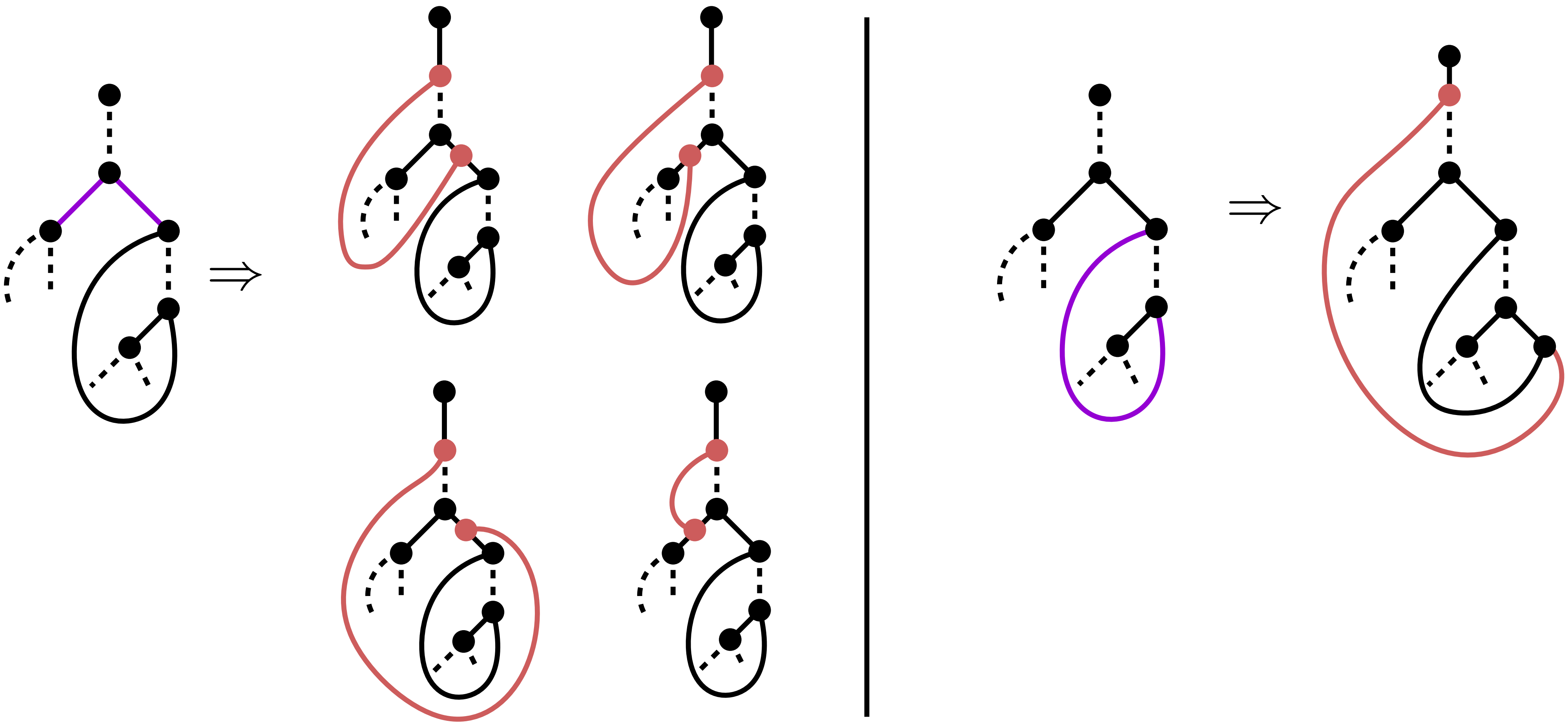
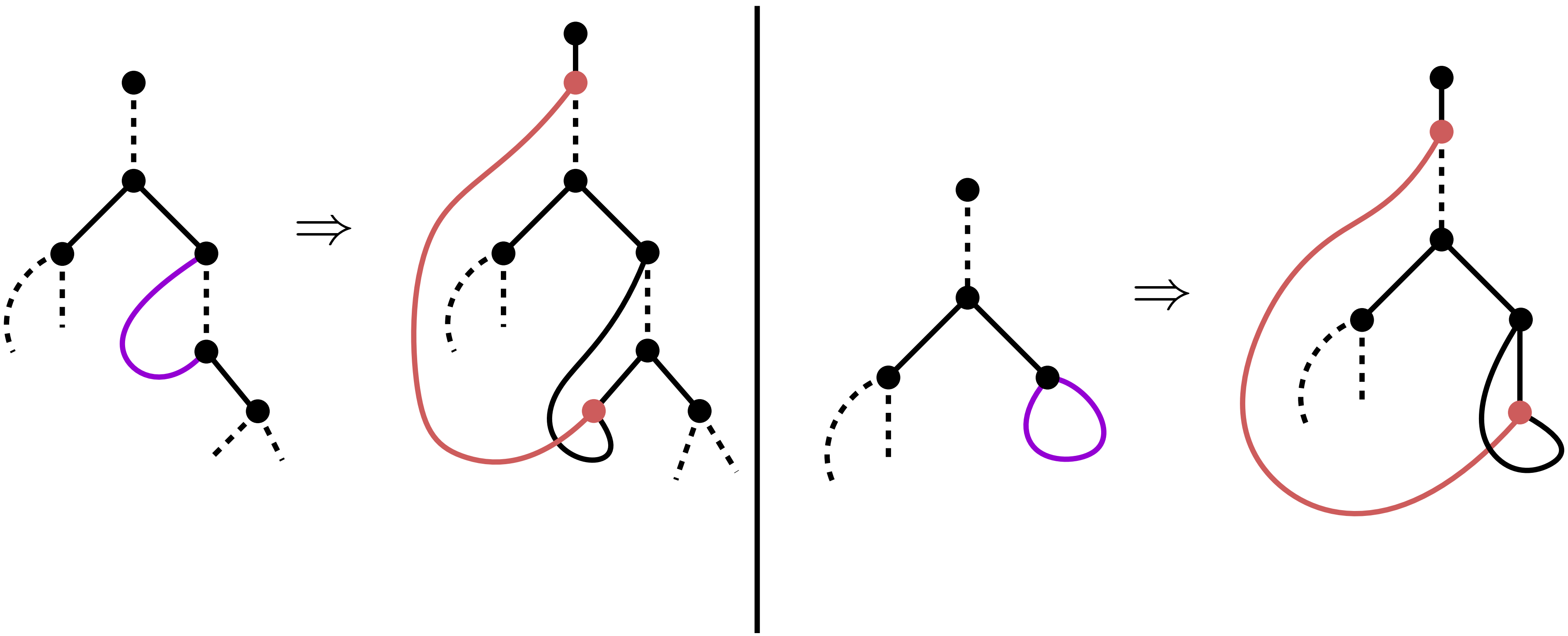
  Cuts destroying a $\mathfrak{p}_1$-pattern:



Enumerating p_1-patterns

- Tracking the creation/destruction of patterns during the recursive decomposition:

  Cuts creating a p_1-pattern:



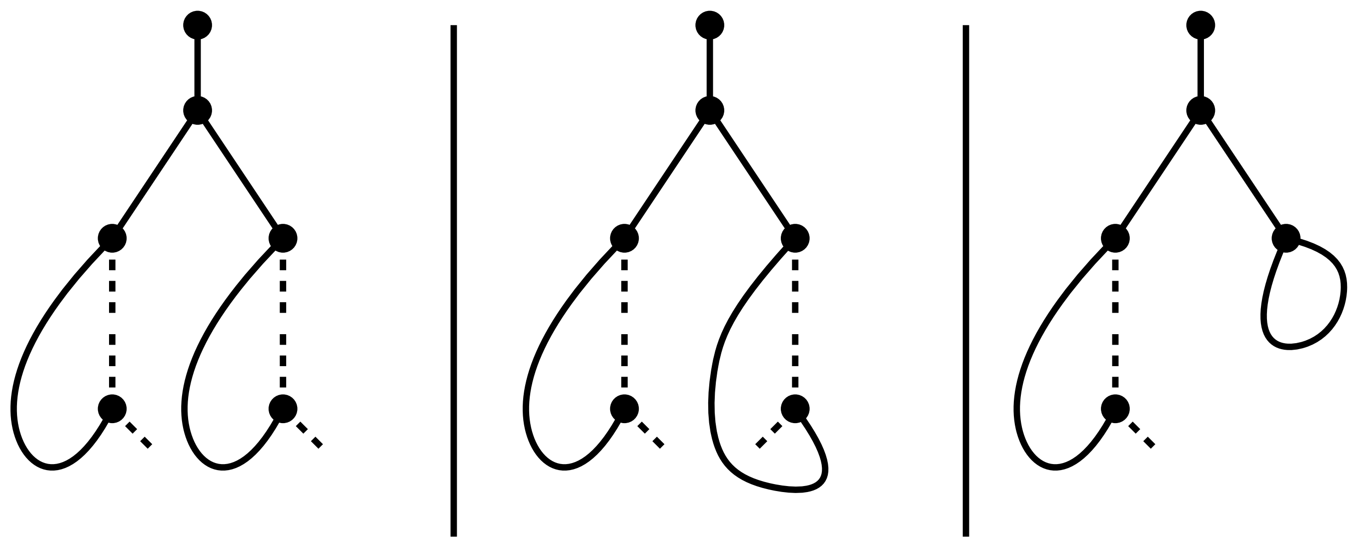
Thus we also need to keep track of:

$$C_1[\lambda x.C_2[(t_1\ x)]])(\lambda y.t_2)] \qquad C_1[(\lambda x.x)(\lambda y.t_2)]$$

Enumerating p_1-patterns

- Tracking the creation/destruction of patterns during the recursive decomposition:

  Applications creating p_1 and auxilliary patterns:



Thus, for an app. of the form $(l_1\ \lambda y.t_1)$ we need to consider how l_1 was formed.

Enumerating p_1-patterns

- Thus we have the following equations:

$$L = \Lambda + A$$

$$\Lambda = z^2 + 2z^4 S_z + (v - u + 4(1-u))z^3 S_u + (u - v + 4(1-v))z^3 S_v$$

$$A = zS^2 + (u-1)z(z^4 S_z + (v - u + 2(1-u))z^3 S_u + 2(1-v)z^3 S_v) \cdot \Lambda$$
$$+(v-1)z(z^2 + z^4 S_z + (u - v + 2(1-v))z^3 S_u + 2(1-u)z^3 S_u) \cdot \Lambda$$

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$$+(v-1)z(z^2 + z^4 S_z + (u - v + 2(1-v))z^3 S_u + 2(1-u)z^3 S_u) \cdot \Lambda$$

- Extracting the mean:

$$\partial_u S|_{u=1,v=1}$$
$$= \left(2zS\partial_u S + 2z^4 \partial_{z,u} S + z^7 \partial_z S + 2z^9 (\partial_z S)^2 - 5z^3 \partial_u S + z^3 \partial_v S\right)|_{u=1,v=1}$$

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$$\begin{aligned} A = {} & zS^2 + (u-1)z(z^4 S_z + (v - u + 2(1-u)) z^3 S_u + 2(1-v) z^3 S_v) \cdot \Lambda \\ & + (v-1)z(z^2 + z^4 S_z + (u - v + 2(1-v)) z^3 S_u + 2(1-u) z^3 S_u) \cdot \Lambda \end{aligned}$$

- Extracting the mean:

bijection needed!

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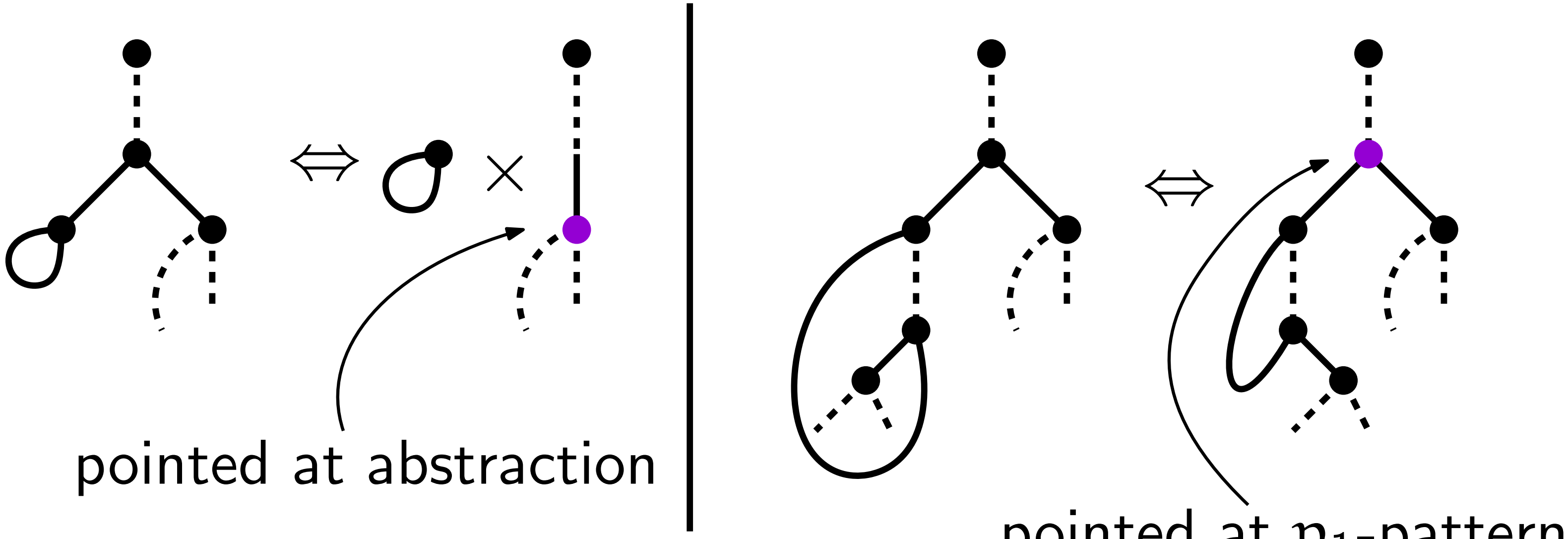

Enumerating $\mathfrak{p}_1$-patterns

- Finally we obtain a mean number of occurences:

$$\mathbb{E}[X_{\mathfrak{p}_1}] \sim \frac{1}{6}$$

Enumerating $\mathfrak{p}_1$-patterns and $\mathfrak{p}_2$-patterns

- Finally we obtain a mean number of occurences:

$$\mathbb{E}[X_{\mathfrak{p}_1}] \sim \frac{1}{6}$$

- Analogously, we have a mean number of occurences for $\mathfrak{p}_2$:

$$\mathbb{E}[X_{\mathfrak{p}_2}] \sim \frac{1}{48}$$

Both are asymptotically constant in expectation!

Enumerating p_3-patterns

- As before, we'll also need to enumerate auxilliary patterns:

$$(\lambda x.\lambda y.t_1)$$

$$(\lambda x.\lambda y.t_1)\ t_2$$

$$(\lambda x.\lambda y.t_1)\ t_2\ t_3 \quad (p_3)$$

Enumerating p_3-patterns

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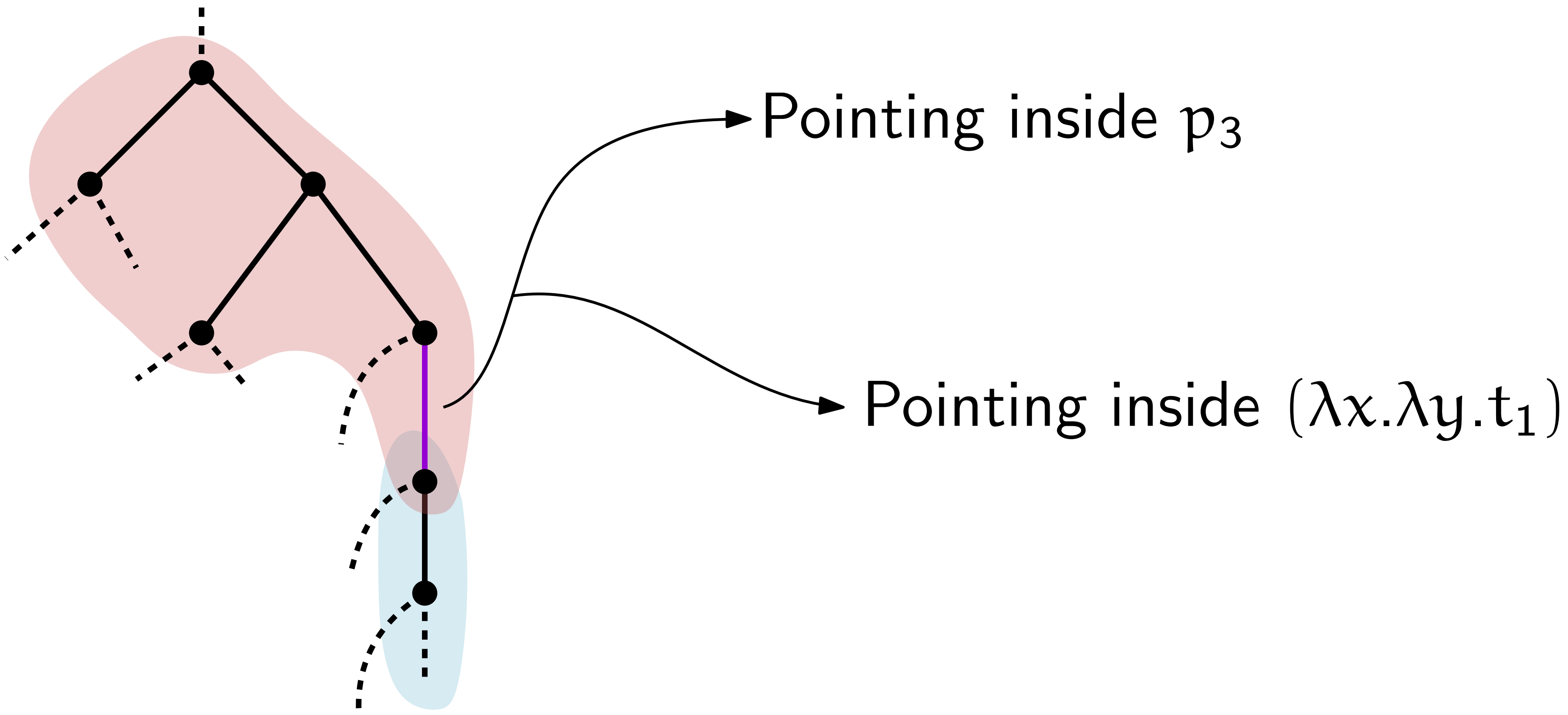
$$(\lambda x.\lambda y.t_1) \qquad (\lambda x.\lambda y.t_1)\ t_2 \qquad (\lambda x.\lambda y.t_1)\ t_2\ t_3 \quad (p_3)$$

- However we run into a problem:



Enumerating $\mathfrak{p}_3$-patterns

- Generatingfunctionology fails, we revert to more elementary methods:

$$\mathbb{E}(X_{\mathfrak{p}_4}) = \mathbb{E}_{\Lambda_n}(X_{\mathfrak{p}_4}) \cdot \frac{|\Lambda_n|}{|L_n|} + \mathbb{E}_{A_n}(X_{\mathfrak{p}_4}) \cdot \frac{|A_n|}{|L_n|}$$

Enumerating $\mathfrak{p}_3$-patterns

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Enumerating p_3-patterns

- Generatingfunctionology fails, we revert to more elementary methods:

asymptotically negligible

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Magic: linear over *families* of all possible abstractions created via cuts from a fixed term!

$$\overline{X}_n = 2n\overline{X}_{n-3} - 10\overline{X}_{n-3} + 2\overline{Y}_{n-3}$$
$$\overline{Y}_n = 2n\overline{Y'}_{n-3} - 6\overline{Y'}_{n-3} + \overline{Z'}_{n-3}$$

where: $\overline{X}_n$ is the sum of $X_{p_4,n}$ over families of abs.,

$\overline{Y}_n$ is the same for the pattern $(\lambda x.\lambda y.t_1)\ t_2$, and

$\overline{Y'}_n$ is the same for $Y'_n = Y_n - X_n$

Enumerating p_3-patterns

- Finally, using the asymptotic mean for Z_n, counting occurences of the $\lambda x.\lambda y.t_1$ pattern, we have:

$$\mathbb{E}[X_n] = \frac{n}{240}$$

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- Therefore, for the number W_n of steps required to reduce a term of size $n = 3k + 2$ to its β-normal form, we have:

$$\mathbb{E}[X_n] \geqslant \frac{11k}{80}$$

which is quite close to Noam's conjecture of $\mathbb{E}[W_n] = \frac{k}{7}$!

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Thank you for your patience!

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