Algorithmes et Complexité des **Problèmes de Satisfaction de Contraintes** (travaux dirigés n° 7)

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Exercise 1 (examen 2008-2009)

Show how the following Boolean relations can be implemented by the given set of Boolean relations or prove that the implementation is impossible :

- Can $[x \not\equiv y]$ (exclusive or) be implemented by conjunction $[x \wedge y]$ and implication $[x \to y]$?
- Can $[x \not\equiv y]$ (exclusive or) be implemented by conjunction $[x \wedge y]$, implication $[x \to y]$, and the relation $R = \{01\}$?
- Can $[x \to y]$ (implication) be implemented by disjunction $[x \vee y]$ and exclusive or $[x \not\equiv y]$?

Solution to Exercise 1

- We have $[x \wedge y] = \{11\}$, $[x \to y] = \{00, 01, 11\}$, and $[x \not\equiv y] = \{01, 10\}$. Both relations $[x \wedge y]$ and $[x \to y]$ are 1-valid, i.e. closed under the constant function $f(x) = 1$, and each constraint built by a primitive positive formula from these two atomic constraints is also 1-valid, but $[x \not\equiv y]$ is *not* 1-valid, therefore there is no primitive positive definition of $[x \not\equiv y]$ by $[x \wedge y]$ and $[x \to y]$. Using the closure of both $[x \wedge y]$ and $[x \to y]$ under conjunction or disjunction, whereas $[x \not\equiv y]$ is *neither* closed under conjunction, *nor* under disjunction, also leads to the required proof.
- The relations $[x \wedge y] = \{11\}$, $[x \to y] = \{00, 01, 11\}$, and $R(x, y) = \{01\}$ are closed under conjunction (resp. disjunction), therefore each constraint built by primitive positive formula from these three atomic constraints is also closed under conjunction (resp. disjunction). However, $[x \not\equiv y] = \{01, 10\}$ is *not* closed under conjunction (*nor* under disjunction), therefore there is no primitive positive definition of $[x \not\equiv y]$ by $[x \wedge y]$, $[x \to y]$, and $R = \{01\}$.
- We have $(x \to y) = \exists z\, (z \vee y) \wedge (z \not\equiv x)$, since $(x \to y)$ is equivalent to $(\neg x \vee y)$.

Exercise 2 (examen 2008-2009)

Consider the following sets of relations S and show (with a proof) whether $\text{CSP}(S)$ is polynomial-time decidable or NP-complete. If $\text{CSP}(S)$ is polynomial-time decidable, determine the class of relations S (Horn, dual Horn, bijunctive, affine, 0- or 1-valid, or their combination).

- $S = \{R_1, R_2\}$ where $R_1 = [x \to (y \to z)]$ and $R_2 = [(x \to y) \to z]$.
- $S = \{R_1, R_2\}$ where $R_1 = \{0011, 0101, 0110, 1001, 1010, 1100\}$ and $R_2 = \{0001, 0011, 0111\}$.
- $S = \{R_1, R_2\}$ where
  $R_1 = \{0001, 0011, 0101, 0111, 1001, 1011, 1101\}$ and
  $R_2 = \{0010, 0100, 0110, 1000, 1010, 1100\}$.

Solution to Exercise 2

- Both relations $R_1 = [x \to (y \to z)]$ and $R_2 = [(x \to y) \to z]$ are 1-valid, therefore also S is 1-valid and $\textsc{csp}(S)$ is in P.
  ATTENTION : the relation R_2 is not 0-valid.
- We have $nae(x, y, z) = \exists u\, R_1(x, y, z, u)$, therefore is $\textsc{csp}(S)$ NP-complete. Alternatively, we can have 1-in-3$(x, y, z) = \exists u \exists v\, R_1(x, y, z, u) \wedge R_2(v, v, v, u)$, what also proves $\textsc{csp}(S)$ to be NP-complete. Of course, testing whether both relations R_1 and R_2 are closed under conjunction, disjunction, majority, and minority, plus showing that they are neither 0- nor 1-valid, gives an equivalent but much more complicated proof.
- We have $nae(x, y, z) = \exists u\, R_2(x, y, z, u)$ what proves $\textsc{csp}(S)$ to be NP-complete. Alternatively, testing whether R_1 and R_2 are closed under conjunction, disjunction, majority, affinity, and the constant 0 and 1, gives an alternative but much more complicated proof.

Exercise 3 (examen 2009-2010)

Show how the following Boolean relations can be implemented by the given set of Boolean relations or prove that the implementation is impossible :

- Can $[x \vee y]$ (disjunction) be implemented by conjunction $[x \wedge y]$ and exclusive or $[x \not\equiv y]$?
- Can $[x \rightarrow (y \rightarrow z)]$ (right double implication) be implemented by the relation $[(x \rightarrow y) \vee z]$ and exclusive or $[x \not\equiv y]$?

Solution to Exercise 3

- The relations $[x \wedge y]$ and $[x \not\equiv y]$ are both affine, but $[x \vee y]$ is not affine, therefore there is no primitive positive definition (i.e. implemented by conjunction, variable identification, and existential quantification) of the relation $[x \vee y]$ over $S = \{[x \wedge y], [x \not\equiv y]\}$. This means $[x \vee y] \notin \langle S \rangle$.
- We write both relations containing the implication in clausal form. Hence we need to show that there exists a primitive positive definition of the relation $[\neg x \vee \neg y \vee z]$ over $[\neg x \vee y \vee z]$ and $[x \not\equiv y]$. The solution is

$$\begin{aligned}[x \rightarrow (y \rightarrow z)] = [\neg x \vee \neg y \vee z] &= \exists v\, [\neg x \vee v \vee z] \wedge [v \not\equiv y] \\ &= \exists v\, [(x \rightarrow v) \vee z] \wedge [v \not\equiv y].\end{aligned}$$

There exists also another, more straightforward solution

$$[x \rightarrow (y \rightarrow z)] = [(y \rightarrow z) \vee \neg x] = \exists v\, [(y \rightarrow z) \vee v] \wedge [v \not\equiv x].$$

Exercise 4 (examen 2009-2010)

Consider the following sets of relations S and show (with a proof) whether $\textsc{csp}(S)$ is polynomial-time decidable or NP-complete. If $\textsc{csp}(S)$ is polynomial-time decidable, determine the class of relations S (Horn, dual Horn, bijunctive, affine, 0- or 1-valid, or their combination).

- $S = \{R_1, R_2, R_3\}$ where $R_1 = [(x \rightarrow y) \rightarrow \neg z]$, $R_2 = [(x \vee y) \rightarrow z]$, and $R_3 = [x \wedge y]$.
- $S = \{R_1, R_2\}$ where $R_1 = \{0011, 0110, 1001, 1010, 1100\}$ and $R_2 = \{0001, 0011, 0111\}$.

Solution to Exercise 4

- Transform the relations R_1 and R_2 to clausal form, obtaining $R_1 = [(x \vee \neg z) \wedge (\neg y \vee \neg z)]$ and $R_2 = [(\neg x \vee z) \wedge (\neg y \vee z)]$. Relation R_3 is already in clausal form. By inspection, we see that the set of relations S is Horn and bijunctive. Moreover, R_3 is not 0-valid since $[x \wedge y] = \{11\}$, R_1 is neither dual Horn since the clausal form of $(x \to y) \to \neg z$ contains the clause $\neg y \vee \neg z$ (this can be also checked by computing the models of $[(x \to y) \to \neg z]$ and testing the closure under disjunction), nor affine since R_1 contains 5 vectors which is not a power of 2. Hence S is both Horn and bijunctive, and therefore $\textsc{csp}(S)$ is polynomial-time decidable following Schaefer's Dichotomy Theorem.
- By inspection, we see that R_1 is neither 0-valid, nor 1-valid. Moreover, R_1 is not Horn, since $0011 \wedge 1100 = 0000 \notin R_1$, nor dual Horn since $0011 \vee 1100 = 1111 \notin R_1$, nor affine since R_1 has 5 vectors which is not a power of 2, nor bijunctive since $\mathrm{maj}(1001, 1010, 1100) = 1000 \notin R_1$. The relation R_2 does not play a significant role. Hence $\textsc{csp}(S)$ is NP-complete following Schaefer's Dichotomy Theorem.

Exercice 5 (cours 3)

Nous avons des relations

$$\begin{array}{rclrcl} or_0 & = & [x \vee y \vee z], & or_3 & = & [\neg x \vee \neg y \vee \neg z], \\ bor_0 & = & [x \vee y], & bor_2 & = & [\neg x \vee \neg y], \\ \text{1-in-3} & = & \{100, 010, 001\}, & nae & = & \{0,1\}^3 \smallsetminus \{000, 111\}. \end{array}$$

Lesquelles des implantations suivantes sont-elles vrai ? Construisez les implantations si elle sont correctes.

1. 1-in-3 $\in \langle or_0, or_3 \rangle$
2. $or_0 \in \langle \text{1-in-3} \rangle$ et $or_3 \in \langle \text{1-in-3} \rangle$
3. $or_0 \in \langle bor_0, bor_2 \rangle$ et $or_3 \in \langle bor_0, bor_2 \rangle$
4. $nae \in \langle \text{1-in-3} \rangle$
5. 1-in-3 $\in \langle nae \rangle$

Solution de l'Exercice 5 (1)

Pour 1-in-3 $\in \langle or_0, or_3 \rangle$:

$$
\begin{array}{lcl}
neq(x,y) & = & or_0(x,y,y) \wedge or_3(x,y,y) \\
\text{1-in-3}(x,y,z) & = & \exists v_1 \exists v_2 \exists v_3 \\
 & & or_0(x,y,z) \wedge or_3(x,y,z) \\
 & \wedge & or_3(v_1,y,z) \wedge neq(x,u_1) \\
 & \wedge & or_3(x,u_2,z) \wedge neq(y,u_2) \\
 & \wedge & or_3(x,y,u_3) \wedge neq(z,u_3)
\end{array}
$$

Solution de l'Exercice 5 (2)

Pour $or_0 \in \langle$1-in-3$\rangle$ et $or_3 \in \langle$1-in-3$\rangle$

$$
\begin{array}{lcl}
[\text{1-in-3}(x_T, x_F, x_F)] & = & \{100\} \\
[\text{1-in-3}(x, y, 0)] & = & \{01, 10\} \\
neq(x,y) & = & \exists x_T \exists x_F \\
 & & \text{1-in-3}(x, y, x_F) \wedge \text{1-in-3}(x_T, x_F, x_F) \\
or_0(x_1, x_2, x_3) & = & \exists y_2 \exists y_3 \exists z_1 \exists z_2 \exists z_3 \exists z_4 \exists z_5 \\
 & & \text{1-in-3}(x_1, z_1, z_2) \wedge \text{1-in-3}(y_2, z_1, z_3) \\
 & \wedge & \text{1-in-3}(y_3, z_2, z_4) \wedge \text{1-in-3}(z_2, z_3, z_5) \\
 & \wedge & neq(x_2, y_2) \wedge neq(x_3, y_3) \\
or_3(x_1, x_2, x_3) & = & \exists y_1 \exists y_2 \exists y_3 \\
 & & neq(x_1, y_1) \wedge neq(x_2, y_2) \wedge neq(x_3, y_3) \\
 & \wedge & or_0(y_1, y_2, y_3)
\end{array}
$$

Solution de l'Exercice 5 (3)

Pour : $or_0 \in \langle bor_0, bor_2 \rangle$ et $or_3 \in \langle bor_0, bor_2 \rangle$:

- impossible car $\langle bor_0, bor_2 \rangle$ est bijunctive (visible par définition), mais ni or_0 ni or_3 ne le sont pas
- $\mathrm{maj}(001, 010, 100) = 000 \notin or_0$
- $\mathrm{maj}(011, 101, 110) = 111 \notin or_3$

Solution de l'Exercice 5 (4)

Pour $nae \in \langle$1-in-3$\rangle$:

- $nae = or_0(x, y, z) \land or_3(x, y, z)$
- utiliser les résultats du point 2

Solution de l'Exercice 5 (5)

Pour 1-in-3 $\in \langle nae \rangle$:

- impossible, car nae est complementive (fermée par $\neg$), mais 1-in-3 ne l'est pas
- $001 \in$ 1-in-3, $\neg(001) = 110$, $110 \notin$ 1-in-3