Correction of the Exam 2011-2012
Algorithms and Complexity of Constraint Satisfaction Problems

MPRI 2.31.1

29 November 2011

Exercise 1

R is a relation of arity ≥ 3. Denote by $e_i = (e_i^1, \ldots, e_i^k) \in R$ a vector from the relation R, where

$$e_i^j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

for each $j = 1, \ldots, k$.

Complexity analysis:

- R is neither 0-valid nor 1-valid by definition.
- R is not Horn since it is not closed under conjunction: $e_1 \wedge e_2 = 0 \cdots 0 \notin R$.
- R is not dual Horn, since it is not closed under disjuction: $e_1 \vee e_2 \vee \cdots \vee e_k = 1 \ldots 1 \notin R$.
- R is not bijunctive since it is not closed under majority: $\mathrm{maj}(e_1, e_2, e_3) = 0 \cdots 0 \notin R$.
- R is not affine since it is not closed under affinity, but it needs a little bit more effort to show. Suppose that R is affine. Then $a_3 = \mathrm{aff}(e_1, e_2, e_3) = e_1 + e_2 + e_3 \pmod 2 = (1, 1, 1, 0, \ldots, 0) \in R$. By definition, also the vector $a_2 = (1, 1, 0, \ldots, 0)$ belongs to R. However, $\mathrm{aff}(a_2, e_1, e_2) = 0 \cdots 0 \notin R$. Hence, R cannot be closed under affinity and therefore R is not affine.
- The not-all-equal relation $nae = \{001, 010, 100, 011, 101, 110\}$ satifies the definition, hence $nae \in \langle R \rangle$.
- By Shaefer's Dichotomy Theorem, $\textsc{csp}(R)$ is NP-complete, since $\textsc{csp}(nae)$ is NP-complete.

Exercise 2

There are several ways to solve this exercise, but the easiest one is to transform the formulas to clausal form.

$$\begin{array}{lclcl} R(x,y,z) & = & [x \rightarrow (y \rightarrow z)] & = & [\neg x \vee \neg y \vee z] \\ K(x,y) & = & [(x \rightarrow y) \rightarrow x] & = & [x \wedge (x \vee \neg y)] \\ T(x,y,z) & = & [(x \rightarrow z) \rightarrow (y \rightarrow z)] & = & [x \vee \neg y \vee z] \end{array}$$

$K(x,y)$ and $T(x,y,z)$ are dual Horn, but $R(x,y,z)$ is not dual Horn — this is visible already from the formulas, but it can be tested by closure properties when we convert the relations to sets of Boolean vectors. Hence $R(x,y,z)$ cannot be constructed from $K(x,y)$ and $T(x,y,z)$.

Exercise 3

There are several possibilities to solve this exercise.

Since S is Horn, we have both $[x] \in \langle S \rangle$ and $[\neg x] \in S$, since $[x]$ and $[\neg x]$ are Horn relations. Hence we can use *pinning* to solve our problem.

In the first step, compute the (minimal) model m_1 of the formula φ by mean of the Horn algorithm presented during the cours. If $m_1 \notin \{m, m', m''\}$, answer YES. Otherwise, let $zv(\varphi)$ be the variables set to 0 in the last step of the Horn algorithm.

In the second step, set $\varphi' = \varphi \wedge x$ for each $x \in zv(\varphi)$. Repeat the same algorithm as in the first step. This will produce another model m_2 or give a negative answer. If the answer is negative, choose another variable $x \in zv(\varphi)$. If there are none, answer NO. Otherwise test m_2 with respect to m, m', and m''. Construct $zv(\varphi')$.

In the third step, set $\varphi'' = \varphi' \wedge x'$ for each $x' \in zv(\varphi')$. Repeat the same algorithm as in the second step.

In the fourth step, set $\varphi''' = \varphi'' \wedge x''$ for each $x'' \in zv(\varphi'')$. Repeat the same algorithm once more and test the obtained model.

In the last step, set $\varphi_* = \varphi''' \wedge x_*$ for each $x_* \in zv(\varphi''')$ and test if φ_* is satisfiable. If YES, then there exists yet another model.

There exists another (very clever) solution presented by Aurélie Lagoutte. We first introduce three new variables z_m, $z_{m'}$, and $z_{m''}$ with the meaning that the found vector is equal to m, m', and m'', respectively. We construct a new literal l_m^i for each model m and every coordinate $i \in \{1, \ldots, k\}$ as follows:

$$l_m^i = \begin{cases} x_i & \text{if } m[i] = 0, \\ \neg x_i & \text{if } m[i] = 1. \end{cases}$$

We create the literals l_m^i, $l_{m'}^i$, and $l_{m''}^i$ for each coordinate $i \in \{1, \ldots k\}$.

We create a new formula φ' from φ by adding the following Horn clauses for each $i \in \{1, \ldots, k\}$:

$$\neg z_{m'} \vee \neg z_{m''} \vee l_m^i, \qquad \neg z_m \vee \neg z_{m''} \vee l_{m'}^i, \qquad \neg z_{m'} \vee \neg z_m \vee l_{m''}^i,$$

and

$$z_m \rightarrow \neg l_m^i, \qquad z_{m'} \rightarrow \neg l_{m'}^i, \qquad z_{m''} \rightarrow \neg l_{m''}^i.$$

Since S is a set of Horn clauses, we have all relations $[c] \in \langle S \rangle$ for each constructed Horn clause c. The Horn formula φ' is satisfiable if and only if there exists yet another model for φ.