Harrop: a new tool in the kitchen of intuitionistic logic

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Background: proof-theoretic kitchen

Background: structural proof theory

What are we talking about?

- Proof theory:
  - Given some *ingredients* (axioms) and *tools* (inference rules)
  - What can we *cook* (prove) in the proof system?

$$\vdash (A \to B) \to A \to B$$

- **Structural** proof theory: how do recipes look like?

$$\frac{A \to B \vdash A \to B}{\vdash (A \to B) \to A \to B}$$

Backgound: natural deduction, intuitionistic propositional logic

Natural deduction: a very well known set of tools to cook proofs in intuitionistic propositional logic (**IPC**)

They look like this:

$$\wedge\text{-I: } \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \wedge\text{-E } \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \vee\text{-I } \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

$$\vee\text{-E } \frac{\Gamma \vdash A \vee B \qquad \Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma \vdash C}$$

That is: pairs of introduction and elimination tools for each connective

Background: proof reductions

A central notion in structural proof theory: **proof reduction**
(aka "normalization", "cut elimination")

Say we have a recipe ending with

$$\dfrac{\dfrac{\overset{\Pi_1}{A \vdash B}}{\vdash A \to B}\ \rightarrow\text{-I} \qquad \overset{\Pi_2}{\vdash A}}{\vdash B}\ \rightarrow\text{-E}$$

There is a *detour*! Preparing $A \to B$ first is not necessary

$$\dfrac{\dfrac{\overset{\Pi_1}{A \vdash B}}{\vdash A \to B} \qquad \overset{\Pi_2}{\vdash A}}{\vdash B} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\Pi_2 \\ \vdash A \\ \Pi_1\end{matrix}}{\vdash B}$$

Background: proof reductions

Why do we like Intuitionistic Logic?

- Disjunction property: if $\vdash A \vee B$ is provable, we know which of the two is provable
- Curry-Howard: proofs correspond to idealized functional programs

Why Natural Deduction?

- Transform directly axioms in elimination rules
- Easily get Curry-Howard terms

Intro: Admissibility in propositional intuitionistic logic

Basic definitions

Definition (Admissible and derivable rules)

A rule $A \,/\, B$ is admissible if whenever $\vdash A$ is provable, then $\vdash B$ is provable. It is derivable if $\vdash A \to B$ is provable

Definition (Structural completeness)

A logic is structurally complete if all admissible rules are derivable

So: an admissible rule is a tool we can actually do without

If it is derivable: we can describe *how* to do without, with a recipe inside the logic!

In a structurally complete logic: we always have recipes to explain how to avoid using admissible tools

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Note

Classical logic is structurally complete. Just think of truth tables!

Technical remark

Rules here refer just to formulas, not to Natural Deduction judgements!

Thus: different from *cut/weakening admissibility*

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Theorem (Harrop 1960)
Intuitionistic propositional logic is not structurally complete

Proof.
Counterexample: $\neg B \to (A_1 \vee A_2)\,/\,(\neg B \to A_1) \vee (\neg B \to A_2)$ is admissible but not derivable □

We study: admissible but non-derivable "principles" (axioms/rules)

A bit of history

- Friedman (1975): Are the admissible rules of IPC countable?
- Rybakov (1984) answered positively; De Jongh and Visser conjectured a *basis* for them
- Iemhoff (2001) proved the conjecture semantically
- Less known: Rozière (1992) independently obtained the same result proof-theoretically

Visser's basis

Theorem (Iemhoff 2001, Rozière 1992)
All admissible and non derivable rules are obtained by the usual intuitionistic rules and the following rules

$$V_n : \quad (B_i \to C_i)_{i=1\ldots n} \to A_1 \vee A_2 / \begin{cases} \bigvee_{j=1}^{n} ((B_i \to C_i)_{i=1\ldots n} \to B_j) \\ \vee \\ ((B_i \to C_i)_{i=1\ldots n} \to A_1) \\ \vee \\ ((B_i \to C_i)_{i=1\ldots n} \to A_2) \end{cases}$$

Our plan: proof reductions and admissible rules

Problem
Whenever we have a recipe for A, there is one for B. We can't have a recipe for $A \to B$

How are the two recipes related?

Looks like we miss some *reductions*!

Idea

- Allow admissible inferences in a Natural Deduction system
- Add reduction rules to make these inferences disappear
- . . . end up with the desired intuitionistic recipe!

Adding tools to the intuitionistic kitchen

Harrop's rule and the Kreisel-Putnam logic

The most famous admissible principle of **IPC**: Harrop's principle

$$(\neg B \to (A_1 \vee A_2)) \to (\neg B \to A_1) \vee (\neg B \to A_2)$$

By adding it to **IPC** we obtain the Kreisel-Putnam logic **KP**

It's a particular case of V_1, with $\perp$ for C:

$$V_1 : ((B \to C) \to (A_1 \vee A_2)) \to (((B \to C) \to A_1) \vee ((B \to C) \to A_2))$$

Trivia
KP was the first non-intuitionistic logic to be shown to have the disjunction property

Harrop's rule and the Kreisel-Putnam logic

The most famous admissible principle of **IPC**: Harrop's principle

$$(\neg B \to (A_1 \vee A_2)) \to (\neg B \to A_1) \vee (\neg B \to A_2)$$

Transform it to a Natural Deduction rule (based on disjunction elimination)

$$\frac{\Gamma, \neg B \vdash A_1 \vee A_2 \qquad \begin{array}{c}\Gamma, \neg B \to A_1 \vdash D \\ \Gamma, \neg B \to A_2 \vdash D\end{array}}{D}$$

Ask a proof of the antecedent of Harrop, eliminate the conclusion

Harrop's rule and the Kreisel-Putnam logic

The most famous admissible principle of **IPC**: Harrop's principle

$$(\neg B \to (A_1 \vee A_2)) \to (\neg B \to A_1) \vee (\neg B \to A_2)$$

The final rule, together with a Curry-Howard term annotation:

$$\frac{\Gamma, x : \neg B \vdash t : A_1 \vee A_2 \qquad \begin{array}{c}\Gamma, y : \neg B \to A_1 \vdash u_1 : D \\ \Gamma, y : \neg B \to A_2 \vdash u_2 : D\end{array}}{\Gamma \vdash \mathtt{hop}[x\,.\,t \mid y\,.\,u_1 \mid y\,.\,u_2] : D}$$

Harrop's rule and the Kreisel-Putnam logic

Let's turn to the reduction rules!

In the Curry-Howard notation:

- *Harrop-inj:* $\texttt{hop}[x.\texttt{inj}_i\, t \mid y.u_1 \mid y.u_2] \mapsto u_i\{\lambda\vec{x}.\, t/y\}$
- *Harrop-app:* $\texttt{hop}[x.\texttt{efq}\,[x\,t] \mid y.u_1 \mid y.u_2] \mapsto u_i\{(\lambda x.\,\texttt{efq}\; x\,t)/y\}$

$$\text{Harrop: } \dfrac{\dfrac{\begin{matrix}\Pi \\ \Gamma, \neg B \vdash A_1\end{matrix}}{\Gamma, \neg B \vdash A_1 \vee A_2}\,\vee\text{-I} \qquad \begin{matrix}\Xi_1 \\ \Gamma, \neg B \to A_1 \vdash D\end{matrix} \qquad \begin{matrix}\Xi_2 \\ \Gamma, \neg B \to A_2 \vdash D\end{matrix}}{\Gamma \vdash D}$$

... reduces to

$$\dfrac{\begin{matrix} \dfrac{\begin{matrix}\Pi \\ \Gamma, \neg B \vdash A_i\end{matrix}}{\Gamma \vdash \neg B \to A_i}\,\to\text{-I} \\ \Xi_i \end{matrix}}{\Gamma \vdash D}$$

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$$\text{Harrop: } \dfrac{\texttt{efq}\;\dfrac{\begin{array}{c}\Pi\\ \Gamma, \neg B \vdash \bot\end{array}}{\Gamma, \neg B \vdash A_1 \vee A_2} \qquad \begin{array}{c}\Xi_1\\ \Gamma, \neg B \to A_1 \vdash D\end{array} \qquad \begin{array}{c}\Xi_2\\ \Gamma, \neg B \to A_2 \vdash D\end{array}}{\Gamma \vdash D}$$

...reduces to

$$\dfrac{\begin{array}{c}\to\text{-I}\;\dfrac{\texttt{efq}\;\dfrac{\begin{array}{c}\Pi\\ \Gamma, \neg B \vdash \bot\end{array}}{\Gamma, \neg B \vdash A_1}}{\Gamma \vdash \neg B \to A_1}\\ \Xi_1\end{array}}{\Gamma \vdash D}$$

Definition (Normal form)
A proof is in normal form if no reduction rule is applicable to it

Definition (Strong Normalization)
A proof system has the Strong Normalization property if all proofs reduce to a normal form, regardless of the strategy

Theorem
Our calculus for **KP** *has the Strong Normalization property*

Harrop's rule and the Kreisel-Putnam logic

Lemma (Classification)

Let $\Gamma_\neg \vdash t\colon \tau$ *for* t *in n.f. and* t *not an exfalso:*

- *If* $\tau = A \to B$, *then* t *is an abstraction or a variable in* $\Gamma_\neg$;
- *If* $\tau = A \vee B$, *then* t *is an injection;*
- *If* $\tau = A \wedge B$, *then* t *is a pair;*
- *If* $\tau = \bot$, *then* $t = x\,v$ *for some* v *and some* $x \in \Gamma_\neg$;

Theorem (Disjunction property)

If $\vdash t\colon A \vee B$, *then there is* t' *such that either* $\vdash t'\colon A$ *or* $\vdash t'\colon B$.

Rozière's logic AD

What if we try to add the full V_1 principle?

Theorem (Rozière, 1992)
In the logic characterized by the axiom V_1, all V_i are derivable and all admissible rules are derivable

Rozière called this logic **AD** and showed that it isn't classical logic. However:

Theorem (Iemhoff, 2001)
The only logic with the disjunction property where all V_n are admissible is **IPC**

Rozière's logic AD

We can as before provide a term assignment for **AD**:

$$\frac{\Gamma, x : B \to C \vdash t : A_1 \vee A_2 \qquad \begin{array}{r} \Gamma, y : (B \to C) \to A_1 \vdash u_1 : D \\ \Gamma, y : (B \to C) \to A_2 \vdash u_2 : D \\ \Gamma, z : (B \to C) \to B \vdash v : D \end{array}}{\Gamma \vdash \mathtt{V}_1 [x \,.\, t \mid y \,.\, u_1 \mid y \,.\, u_2 \mid z \,.\, v] : D}$$

Although it doesn't have the disjunction property, **AD** seems an interesting and not well studied logic.

Rozière posed the problem of finding a functional interpretation for it; we go in this direction by providing a term assignment to proofs

Conclusions

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- Transformed admissible rules into natural deduction rules
- Studied arising logics
- Studied associated Curry-Howard calculus (in the paper)

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But hey! The resulting recipes are in **KP**, **AD**, . . . not **IPC**!

Check out (soon):

M M and A C. “Admissible Tools in the Kitchen of Intuitionistic Logic”. In: *7th International Workshop on Classical Logic & Computation*. 2018

Where we take a different road, sticking to **IPC** and characterizing all the Visser rule

Conclusions

The *logic based on admissible principles* way:

- More in-depth study of **AD**
- Add new admissible principles

The *admissibility* way:

- Port the system for Visser's rules to other (modal) logics
- Study admissible principles of intuitionistic arithmetic (**HA**)
- ...and admissible principles of first-order logic

Thank you

A Curry-Howard system for admissible rules

Idea: explain Visser's basis with Natural Deduction + Curry-Howard

Advantages:

- Axioms can be translated to rules right away
- Simple way to assign lambda terms
- Focus on reduction rules

The rule should have the shape of a disjunction elimination

Natural deduction rules for V_n

Add to a Natural Deduction system a rule for each of the V_n:

$$\frac{\emptyset, (B_i \to C_i)_i \vdash A_1 \vee A_2 \qquad \begin{array}{c}\Gamma, (B_i \to C_i)_i \to A_1 \vdash D \\ \Gamma, (B_i \to C_i)_i \to A_2 \vdash D\end{array} \qquad [\Gamma, (B_i \to C_i)_i \to B_j \vdash D]_{j=1\ldots n}}{\Gamma \vdash D}$$

Idea: a disjunction elimination, parametrized over n implications

Note The context of the main premise is empty. Otherwise we would be able to prove V_n!

Term assignment

Usual terms for **IPC**, plus the new one for the V-rules

$$
\begin{array}{rcl}
t, u, v & ::= & x \mid u\,v \mid \lambda x.\,t \mid \mathtt{efq}\ t \\
& \mid & \mathtt{<}\,u\,\mathtt{,}\ v\mathtt{>} \mid \mathtt{proj}_i\,t \mid \mathtt{inj}_i\,t \\
& \mid & \mathtt{case[}t \mathbin{\mathtt{|}} y\,\mathtt{.}\,u \mathbin{\mathtt{|}} y\,\mathtt{.}\,v\mathtt{]} \\
& \mid & \boxed{\mathtt{V}_n\mathtt{[}\vec{x}\,\mathtt{.}\,t \mathbin{\mathtt{|}} y\,\mathtt{.}\,u_1 \mathbin{\mathtt{|}} y\,\mathtt{.}\,u_2 \mathbin{\mathtt{|}} z\,\mathtt{.}\,\vec{v}\mathtt{]}\ \text{(Visser)}}
\end{array}
$$

$$
\frac{\vec{x} : (B_i \to C_i)^i \vdash t : A_1 \vee A_2 \qquad \begin{array}{c} \Gamma, y : (B_i \to C_i)^i \to A_1 \vdash u_1 : D \\ \Gamma, y : (B_i \to C_i)^i \to A_2 \vdash u_2 : D \\ [\Gamma, z : (B_i \to C_i)^i \to B_j \vdash v_j : D]^{j=1\dots n} \end{array}}{\mathtt{V}_n\mathtt{[}\vec{x}\,\mathtt{.}\,t \mathbin{\mathtt{|}} y\,\mathtt{.}\,u_1 \mathbin{\mathtt{|}} y\,\mathtt{.}\,u_2 \mathbin{\mathtt{|}} z\,\mathtt{.}\,\vec{v}\mathtt{]} : D}
$$

Reduction rules

Evaluation contexts for **IPC**:

$$W \quad ::= \quad [\cdot] \mid W\,t \mid t\,W \mid \mathtt{efq}\ W$$
$$\mid \quad \mathtt{proj}_i\, W \mid \mathtt{case}[W \mid - \mid -]$$

Evaluation contexts for $\mathtt{V}_n$: structural closure of the reduction rules

The usual rules for **IPC**, plus:

- *Visser-inj:* $\mathtt{V}_n[\vec{x}.\mathtt{inj}_i\, t \mid y\,.\,u_1 \mid y\,.\,u_2 \mid z\,.\,\vec{v}] \quad \mapsto \quad u_i\{\lambda\vec{x}.\,t/y\}\ (i = 1, 2)$
- *Visser-app:* $\mathtt{V}_n[\vec{x}\,.\,W[x_j\, t] \mid y\,.\,u_1 \mid y\,.\,u_2 \mid z\,.\,\vec{v}] \quad \mapsto \quad v_j\{\lambda\vec{x}.\,t/z\}\ (j = 1 \ldots n)$

The reduction rules tell us:

- One of the disjuncts is proved directly, or
- A proof for an B_j was provided, to be used on a V-hypothesis

This provides a succint explanation of what admissible rules can do

The context is empty, so all the hypotheses are Visser-hypotheses, and we can move the terms around

Subject reduction and termination are easy results!

Arithmetic

Theorem (De Jongh)
The propositional formulas whose arithmetical instances are provable in **HA** *are the theorems of* **IPC**

There is ongoing work to relate admissibility in **IPC** and **HA** through provability logics

We believe our approach can be extended to such cases

For example: the arithmetical *Independence of premises* $(\neg P \to \exists x\, A(x)) \to \exists x\, (\neg P \to A(x))$ can be interpretated with a rule resembling ours

First-order logic

The situation in first-order logic seems much more complicated

However there are some well-behaved examples: Markov's Principle; Constant Domains

These principles give rise to the class of *Herbrand-constructive logics* (Aschieri & M.): whenever $\exists x\, A(x)$ is provable, there are $t_1, \ldots t_n$ such that $A(t_1) \vee \ldots A(t_n)$ is provable