Admissible tools in the kitchen of intuitionistic logic

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Intro: Admissibility in propositional intuitionistic logic

Basic definitions

Definition (Admissible and derivable rules)
A rule φ/ψ is admissible if whenever $\vdash \varphi$ is provable, then $\vdash \psi$ is provable. It is derivable if $\vdash \varphi \to \psi$ is provable

Definition (Structural completeness)
A logic is structurally complete if all admissible rules are derivable

Note!
Classical logic is structurally complete.

Different from *cut/weakening admissibility*!

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Theorem (Harrop 1960)
Intuitionistic propositional logic is not structurally complete

Proof.
Counterexample: $\neg\alpha \to (\gamma_1 \vee \gamma_2)/(\neg\alpha \to \gamma_1) \vee (\neg\alpha \to \gamma_2)$ is admissible but not derivable □

We are interested in: admissible but non-derivable "principles"

A bit of history

- Friedman (1975) posed the question of whether the admissible rules of IPC are countable
- Rybakov (1984) answered positively; De Jongh and Visser conjectured a *basis* for them
- Iemhoff 2001 finally proved the conjecture with semantic methods
- Less known: Rozière 1993 independently obtained the same result with proof theoretic techniques

Visser's basis

Theorem (Rozière 1993, Iemhoff 2001)
All admissible and non derivable rules are obtained by the usual intuitionistic rules and the following rules

$$V_n : \quad (\alpha_i \to \beta_i)_{i=1\ldots n} \to \gamma \vee \delta / \begin{cases} \bigvee_{j=1}^{n}((\alpha_i \to \beta_i)_{i=1\ldots n} \to \alpha_j) \\ \vee \\ ((\alpha_i \to \beta_i)_{i=1\ldots n} \to \gamma) \\ \vee \\ ((\alpha_i \to \beta_i)_{i=1\ldots n} \to \delta) \end{cases}$$

Visser's basis

Visser's basis is important not only for **IPC**:

Theorem (Iemhoff 2005)
If the rules of Visser's basis are admissible in a logic, they form a basis for the admissible rules of that logic

This has been applied to modal logics: Gödel Logic, Gödel-Dummet Logic. . .

A Curry-Howard system for admissible rules

Program: “Natural Deduction + Curry-Howard for everything”

Advantages:

- Axioms can be translated to rules right away
- Simple way to assign lambda terms
- Focus on reduction rules

Possible downsides

- Ugly rules/terms
- Difficult termination proofs

Natural deduction rules for V_n

Ad to a Natural Deduction system a rule for each of the V_n:

$$\frac{\emptyset, (\alpha_i \to \beta_i)^i \vdash \gamma_1 \vee \gamma_2 \qquad \begin{array}{c}\Gamma, (\alpha_i \to \beta_i)^i \to \gamma_1 \vdash \psi \\ \Gamma, (\alpha_i \to \beta_i)^i \to \gamma_2 \vdash \psi\end{array} \qquad [\Gamma, (\alpha_i \to \beta_i)^i \to \alpha_j \vdash \psi]^{j=1\dots n}}{\Gamma \vdash \psi}$$

Idea: a disjunction elimination, parametrized over n implications

Note The context of the main premise is empty. Otherwise we would be able to prove V_n!

Term assignment

Usual terms for **IPC**, plus the new one for the V-rules

$$
\begin{array}{rcl}
t, u, v & ::= & x \mid u\,v \mid \lambda x.\,t \mid \texttt{efq}\,t \\
& \mid & \langle u, v\rangle \mid \texttt{proj}_i\,t \mid \texttt{inj}_i\,t \\
& \mid & \texttt{case}\,[t \parallel y\,.\,u \mid y\,.\,v] \\
& \mid & \boxed{\texttt{V}_n\,[\vec{x}\,.\,t \parallel y\,.\,u_1 \mid y\,.\,u_2 \parallel z\,.\,\vec{v}]\ \text{(Visser)}}
\end{array}
$$

$$
\frac{\vec{x} : (\alpha_i \to \beta_i)^i \vdash t : \gamma_1 \vee \gamma_2 \qquad \begin{array}{c} \Gamma, y : (\alpha_i \to \beta_i)^i \to \gamma_1 \vdash u_1 : \psi \\ \Gamma, y : (\alpha_i \to \beta_i)^i \to \gamma_2 \vdash u_2 : \psi \\ [\Gamma, z : (\alpha_i \to \beta_i)^i \to \alpha_j \vdash v_j : \psi]^{j=1\ldots n} \end{array}}{\texttt{V}_n\,[\vec{x}\,.\,t \parallel y\,.\,u_1 \mid y\,.\,u_2 \parallel z\,.\,\vec{v}] : \psi}
$$

Reduction rules

Evaluation contexts for **IPC**:

$$\begin{array}{rcl} H & ::= & [\cdot] \mid H\,t \mid \mathtt{efq}\,H \\ & \mid & \mathtt{proj}_i\,H \mid \mathtt{case}\,[H \parallel - \mid -] \end{array}$$

Evaluation contexts for $\mathtt{V}_n$:

$$\begin{array}{rcl} E & ::= & [\cdot] \mid H[E] \\ & \mid & \mathtt{V}_n\,[\vec{x}\,.\,E \parallel - \mid - \parallel \cdots] \end{array}$$

The usual rules for **IPC**, plus:

- *Visser-inj:* $\mathtt{V}_n\,[\vec{x}\,.\,\mathtt{inj}_i\,t \parallel y\,.\,u_1 \mid y\,.\,u_2 \parallel z\,.\,\vec{v}\,] \quad \mapsto \quad u_i\{\lambda\vec{x}.\,t/y\}\ (i = 1, 2)$
- *Visser-app:* $\mathtt{V}_n\,[\vec{x}\,.\,H[x_j\,t] \parallel y\,.\,u_1 \mid y\,.\,u_2 \parallel z\,.\,\vec{v}\,] \quad \mapsto \quad v_j\{\lambda\vec{x}.\,t/z\}\ (j = 1 \ldots n)$

The reduction rules tell us:

- Either one of the disjuncts is proved directly, or
- A witness for an α_j was provided, to be used on an hypothesis

This provides a succint explanation of what admissible rules can do

The context is empty, so all the hypotheses are Visser-hypotheses, and we can move the terms around

Subject reduction and termination are easy results!

Logics characterized by admissible principles

Logics characterized by admissible principles

By lifting the restriction on the context, we can prove the axioms inside the logic

We obtain Curry-Howard systems for the intermediate logics characterized by admissible principles

Harrop's rule and the Kreisel-Putnam logic

The most famous admissible principle of **IPC**: Harrop's rule

$$(\neg\alpha \to (\gamma_1 \vee \gamma_2)) \to (\neg\alpha \to \gamma_1) \vee (\neg\alpha \to \gamma_2)$$

By adding it to **IPC** we obtain the Kreisel-Putnam logic **KP** (trivia: the first non-intuitionistic logic to be shown to have the disjunction property)

This is just a particular case of the rule V1, with $\bot$ for β:

$$\dfrac{\Gamma, \alpha \to \bot \vdash \gamma_1 \vee \gamma_2 \qquad \begin{array}{c} \Gamma, (\alpha \to \bot) \to \gamma_1 \vdash \psi \\ \Gamma, (\alpha \to \bot) \to \gamma_2 \vdash \psi \\ \Gamma, (\alpha \to \bot) \to \alpha \vdash \psi \end{array}}{\psi}$$

Harrop's rule and the Kreisel-Putnam logic

The terms for Harrop's rule are a simplified version of $\mathtt{V}_1$:

$$\frac{\Gamma, x : \neg\alpha \vdash t : \gamma_1 \vee \gamma_2 \qquad \begin{array}{c}\Gamma, y : \neg\alpha \to \gamma_1 \vdash u_1 : \psi \\ \Gamma, y : \neg\alpha \to \gamma_2 \vdash u_2 : \psi\end{array}}{\Gamma \vdash \mathtt{hop}[x\,.\,t \parallel y\,.\,u_1 \mid y\,.\,u_2] : \psi}$$

In particular, we omit the third disjunct (it is trivial)

Harrop's rule and the Kreisel-Putnam logic

Similarly, the reduction rules become

- *Harrop-inj:* $\mathtt{hop}[x\,.\,\mathtt{inj}_i\,t \parallel y\,.\,u_1 \mid y\,.\,u_2] \quad \mapsto u_i\{\lambda\vec{x}.\,t/y\}$
- *Harrop-app:* $\mathtt{hop}[x\,.\,H[x\,t] \parallel y\,.\,u_1 \mid y\,.\,u_2] \quad \mapsto u_i\{(\lambda x.\,\mathtt{efq}\,x\,t)/y\}$

Note The app case looks different: there is no v term, but we know that any use of Harrop hypotheses must lead to a contradiction; thus conclude on either of u_i

Harrop's rule and the Kreisel-Putnam logic

Lemma (Classification)

Let $\Gamma_\neg \vdash t : \tau$ for t in n.f. and t not an exfalso:

- *If $\tau = \varphi \to \psi$, then t is an abstraction or a variable in $\Gamma_\neg$;*
- *If $\tau = \varphi \vee \psi$, then t is an injection;*
- *If $\tau = \varphi \wedge \psi$, then t is a pair;*
- *If $\tau = \bot$, then $t = x\,v$ for some v and some $x \in \Gamma_\neg$;*

Theorem (Disjunction property)

If $\vdash t : \varphi \vee \psi$, then there is t' such that either $\vdash t' : \varphi$ or $\vdash t' : \psi$.

Rozière's logic AD

What if we try to add the full V_1 principle?

Theorem (Rozière 1993)
In the logic characterized by the axiom V_1, all V_i are derivable and all admissible rules are derivable

Rozière called this logic AD and showed that it isn't classical logic. However:

Theorem (Iemhoff 2001)
The only logic with the disjunction property where all V_n are admissible is **IPC**

Rozière's logic AD

We can as before provide a term assignment for AD:

$$\frac{\Gamma, x : \alpha \to \beta \vdash t : \gamma_1 \vee \gamma_2 \qquad \begin{array}{c}\Gamma, y : (\alpha \to \beta) \to \gamma_1 \vdash u_1 : \psi \\ \Gamma, y : (\alpha \to \beta) \to \gamma_2 \vdash u_2 : \psi \\ \Gamma, z : (\alpha \to \beta) \to \alpha \vdash v : \psi\end{array}}{\Gamma \vdash \mathtt{V}_1\,[x \,.\, t \parallel y \,.\, u_1 \mid y \,.\, u_2 \parallel z \,.\, v] : \psi}$$

Although it doesn't have the disjunction property, AD seems an interesting and not well studied logic.

Rozière posed the problem of finding a functional interpretation for it; we go in this direction by providing a term assignment to proofs

Future work

Future work

The *admissibility* way:

- Port the system for Visser's rules to other (modal) logics

The *logic based on admissible principles* way:

- More in-depth study of AD
- Study admissible principles of intuitionistic arithmetic (**HA**)
- ... and admissible principles of first-order logic

Theorem (De Jongh)
The propositional formulas whose arithmetical instances are provable in **HA** *are the theorems of* **IPC**

There is ongoing work to relate admissibility in **IPC** and **HA** through provability logics

We believe our approach can be extended to such cases

For example: the arithmetical *Independence of premises* $(\neg P \to \exists x\, A(x)) \to \exists x\, (\neg P \to A(x))$ can be interpretated with a rule resembling ours

First-order logic

The situation in first-order logic seems much more complicated

However there are some well-behaved examples: Markov's Principle; Constant Domains

These principles give rise to the class of *Herbrand-constructive logics* (Aschieri & M.): whenever $\exists x\, A(x)$ is provable, there are $t_1, \ldots t_n$ such that $A(t_1) \vee \ldots A(t_n)$ is provable

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